$L^2$ Geometry of the Symplectomorphism Group

James Benn

University of Notre Dame

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The Exponential Map on $\mathcal{D}_0^s(M)$

1. Introduction
2. Fredholm Properties of the Exponential Map

Conjugate Points

1. Existence of Conjugate Points
2. Multiplicity of Conjugate Points
Outline

1. The Exponential Map on $D^s_\omega(M)$
   - Introduction
   - Fredholm Properties of the Exponential Map

2. Conjugate Points
   - Existence of Conjugate Points
   - Multiplicity of Conjugate Points
$M$ is a closed, orientable Symplectic manifold of dimension $2n$, with Symplectic form $\omega$, Riemannian metric $g$ and almost complex structure $J$ satisfying

\[ J^2 = -I \]

\[ g(J\cdot, J\cdot) = g(\cdot, \cdot) \]

\[ g(\cdot, J\cdot) = \omega(\cdot, \cdot) \]
Let $\mathcal{D}_\omega^s$ denote the group of all Sobolev $H^s$ diffeomorphisms of $M$ preserving the Symplectic form $\omega$.

If $s > n + 1$ then $\mathcal{D}_\omega^s$ becomes an infinite dimensional manifold whose tangent space at the identity is given by

$$T_e \mathcal{D}_\omega^s = \{ J\nabla F + h : F \in H_0^{s+1}(M), h \text{ harmonic} \}.$$ 

Using right translations, the $L^2$ inner product on vector fields

$$(u \circ \eta, v \circ \eta)_{L^2} = \int_M g(u, v) \circ \eta \, d\mu$$

defines a right-invariant metric on the group, which induces a smooth right invariant Levi-Civita connection and curvature tensor.
The Geodesic Equation on $D^s_\omega$

- A curve $\eta(t)$ in $D^s_\omega$ is a geodesic of the $L^2$ metric if and only if the vector field $v = \dot{\eta}(t) \circ \eta^{-1}(t)$ satisfies the Symplectic Euler equations on $M$

$$\partial_t v + \nabla_v v = \omega^\# \delta \nabla^{-1} d \omega^\flat (\nabla_v v)$$

$$\mathcal{L}_v \omega = 0, \quad v(0) = v_o$$

- When $M$ is two dimensional the Symplectic Euler Equations coincide with the 2D Euler equations of incompressible hydrodynamics
The Exponential Map on $\mathcal{D}_\omega^s(M)$
Conjugate Points

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Global Existence of Geodesics

Theorem

[Ebin 2012] Solutions to the geodesic equation of the $L^2$ metric on $\mathcal{D}_\omega^s(M)$ are unique and exist for all time.

Consequently, the $L^2$ metric admits an exponential map which is defined on the whole tangent space $T_e \mathcal{D}_\omega^s$

$$\exp_e : T_e \mathcal{D}_\omega^s \to \mathcal{D}_\omega^s$$

$$v_o \mapsto \exp_e(tv_o) = \eta(t)$$

where $\eta(t)$ is a geodesic of the $L^2$ metric issuing from the identity in the direction $v_o$. 
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Conjugate Points

**Definition**

Let \( \eta(t) \) be a geodesic of the \( L^2 \) metric in \( \mathcal{D}_\omega^s \) with initial velocity \( v_\circ \). A point \( \eta(t^*) \) \( (t^* > 0) \) is conjugate to the point \( \eta(0) \) if the linear map \( D\exp(t^*v_\circ) \) fails to be an isomorphism.

- In contrast with finite dimensional geometry a linear map between infinite dimensional spaces, with empty kernel, may not be an isomorphism.
- Following Grosman [Gro], \( \eta(t^*) \) is *monoconjugate* to \( \eta(0) = e \) if \( D\exp(t^*v_\circ) \) fails to be injective and *epiconjugate* if \( D\exp(t^*v_\circ) \) fails to be surjective.
The Exponential Map on $\mathcal{D}_\mu(M)$
Conjugate Points

Introduction
Fredholm Properties of the Exponential Map

Singularities of $\exp_e$

- In Ebin-Misiolek-Preston ([E-M-P]), singularities of the exponential map (i.e. conjugate points) were studied in the context of the Euler equations of Hydrodynamics. Their results were

**Theorem**

For $M^2$ a compact two-dimensional manifold without boundary, the exponential map of the $L^2$ metric on $\mathcal{D}_\mu(M^2)$ is a non-linear Fredholm map of index zero.

For $M^3$ a compact three dimensional manifold, the exponential map of the $L^2$ metric is NOT a Fredholm map of index zero on $\mathcal{D}_\mu(M^3)$.

- In three dimensions mono conjugate points accumulate and converge to an epiconjugate point; Preston ([P2]) has shown that this is a typical pathology.
The Exponential Map on $\mathcal{D}_{\omega}^s(M)$

**Theorem**

[B, 2014] Let $M$ be a closed, orientable Symplectic manifold of dimension $2n$. Then, the $L^2$ exponential map on $\mathcal{D}_{\omega}^s(M)$ is a non-linear Fredholm map of index zero.

**Corollary**

Monoconjugate and epiconjugate points coincide in $\mathcal{D}_{\omega}^s(M)$, are isolated and of finite multiplicity along finite geodesic segments.
Proof Sketch

The Jacobi equation along \( \eta(t) = \exp_e(tu_0) \in D^s_\omega \) is given by

\[
\frac{D^2}{dt^2} J + R^\omega(J, \dot{\eta})\dot{\eta} = 0
\] (1)

with initial conditions

\[
J(0) = u_0, \quad J'(0) = w_0,
\] (2)

where \( R^\omega \) is the smooth, right-invariant Riemann curvature tensor of the \( L^2 \) metric.

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If $J$ is the Jacobi field along $\eta$ with initial conditions

\[ J(0) = 0, \quad J'(0) = w_0, \]

then

\[ \Phi(t)w_0 := D \exp_e(tu_0)tw_0 = J(t) \]

defines a family of bounded linear operators from $T_e \mathcal{D}^s_\omega$ to $T_{\eta(t)} \mathcal{D}^s_\omega$. 
Using the Jacobi equation we find that

\[ \Phi(t) = D\eta(t) \cdot \left[ \int_0^t \text{Ad}_{\eta^{-1}(s)} \text{Ad}_{\eta^{-1}(s)}^* \, ds \right. \]

\[ \left. - \int_0^t \text{Ad}_{\eta^{-1}(s)} \text{Ad}_{\eta^{-1}(s)}^* K_{\nu_o} dR_{\eta^{-1}(s)} \Phi(s) \, ds \right] \]

where \(\text{Ad}_\eta X = D\eta \cdot X \circ \eta^{-1}\) is the usual push-forward of vector fields, \(\text{Ad}_\eta^*\) its formal \(L^2\) adjoint, \(K_{\nu_o}(w) = \text{ad}_w^* \nu_o\) with \(\text{ad}_w^*\) the formal \(L^2\) adjoint of \(-\mathcal{L}_w\) - the usual Lie derivative, and \(dR_{\eta^{-1}(t)}\) the differential of composition on the right with \(\eta^{-1}(t)\).
Proof Sketch

- In brief, invertibility of $\Omega(t) = \int_0^t \text{Ad}_{\eta^{-1}(s)}\text{Ad}_{\eta^{-1}(s)}^* \, ds$ on $T_e D_s^\omega$, follows from the estimates

  $$C(t) \|w_0\|_{L^2} \leq \|\Omega(t)w_0\|_{L^2}$$

  and

  $$C(t) \|w_0\|_{H^s} \leq \|\Omega(t)w_0\|_{H^s} + K \|w_0\|_{H^{s-1}}.$$

- To see that $K_{v_0}$ is compact, we compute it explicitly and find that for any $w \in T_e D_s^\omega$

  $$K_{v_0} w = \text{ad}_w^* v_0 = J\nabla \triangle^{-1} g(w, \nabla \star (dg^b(v_0) \wedge \omega^{n-1})).$$
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The Isometry Subgroup

- Let $M$ be a closed Symplectic manifold with Symplectic form and Riemannian metric $g$. The isometry group, denoted by $Iso(M)$, consists of those diffeomorphisms satisfying

$$\eta^*g = g.$$ 

- Every isometry of $M$ is contained in $D_\omega$.

- Every Killing vector field generates a stationary solution to the Symplectic Euler equations.
Examples of Conjugate points in $D^s_\omega$

Let $M$ be the complex projective plane $\mathbb{CP}^2$ with the Fubini-Study metric:

$$h_{ij} = h(\partial_i, \bar{\partial}_j) = \frac{(1 + |z|^2)\delta_{ij} - \bar{z}_i z_j}{(1 + |z|^2)^2}$$

where $z = (z_1, z_2, z_3)$ is a point in $\mathbb{CP}^2$, $|z|^2 = z_1^2 + z_2^2 + z_3^2$. The isometry group of $\mathbb{CP}^2$ is given by $PU(3)$, the projective unitary group. $PU(3)$ is given by the quotient of the unitary group, $U(3)$, by its center, $U(1)$, embedded as scalars. Thus, in terms of matrices, $PU(3)$ consists of complex $3 \times 3$ matrices whose center consists of elements of the form $e^{i\theta}I$. Elements of $PU(3)$ correspond to equivalence classes of unitary matrices, where two matrices $A$ and $B$ are equivalent if $A = e^{i\theta}I \times B$ and we write $A \equiv B$. 

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Consider the following 2-parameter variation of isometries acting on $\mathbb{CP}^2$

$$\gamma(s,t) =$$

$$\begin{bmatrix}
  i & 0 & 0 \\
  0 & i \cos s & \sin s \\
  0 & \sin s & i \cos s
\end{bmatrix}
\begin{bmatrix}
  i \cos t & \sin t & 0 \\
  \sin t & i \cos t & 0 \\
  0 & 0 & i
\end{bmatrix}
\begin{bmatrix}
  -i & 0 & 0 \\
  0 & -i \cos s & \sin s \\
  0 & \sin s & -i \cos s
\end{bmatrix}$$

Notice that $\gamma(s,0) = il \equiv l$ since $il$ corresponds to a rotation by $90^\circ$ in each coordinate, i.e. is an element of $U(1)$ embedded as scalars.
Examples of conjugate points in $D^S_\omega$

Compute

$$\dot{\gamma}(s, t) \circ (\gamma(s, t))^{-1} = \begin{bmatrix} 0 & -i \cos s & \sin s \\ -i \cos s & 0 & 0 \\ -\sin s & 0 & 0 \end{bmatrix} = V(s, t)$$

and $V(s, t)$ satisfies the Symplectic Euler equation for each $s$.

The variation field of the family of geodesics $\gamma(s, t)$ is

$$Y(t) = \frac{d}{ds} (\gamma(s, t))|_{s=0} = \begin{bmatrix} 0 & 0 & i \sin t \\ 0 & 0 & 1 - \cos t \\ -i \sin t & \cos t - 1 & 0 \end{bmatrix}$$

which clearly vanishes for $t = 0$ and $t = 2\pi$. That is, $\gamma(2\pi)$ is conjugate to $e = \gamma(0)$. 

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Conjugate Points on $D^s_\omega$

**Theorem**

[B, 2014] Conjugate points exist on $D^s_\omega(\mathbb{C}P^n)$ for all $n \geq 2$. 
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Theorem

[B, 2014] Every geodesic of the $L^2$ metric which lies in $\text{Iso}(M)$ and which is of length greater than $\pi r$ (for some positive constant $r$) has conjugate points, all of which have even multiplicity.
Proof Idea

- Restrict the $L^2$ metric to the finite dimensional subgroup $Iso(M)$ and its finite dimensional Lie (sub) algebra $T_eIso(M) \subset T_eD_\omega$.
- The $L^2$ metric becomes bi-invariant and the Riemann curvature tensor becomes
  \[
  R(w, u)v = -\frac{1}{4} \operatorname{ad}_v \circ \operatorname{ad}_u w \quad u, v, w \in T_eIso(M) \tag{5}
  \]
- Using (5) we can show that the sectional curvature of the plane $\sigma$ spanned by any two unit vectors in $T_eIso(M)$ is positive and bounded away from zero.
- The first statement then follows from the classical Theorem of Bonnet and Myers.
Proof Idea

- When $\eta(t)$ consists of isometries, $\text{Ad}_{\eta}(t) = \text{Ad}_{\eta^{-1}}(t)$ and the Jacobi equation reduces to
  \[
  \partial_t (\text{Ad}_{\eta^{-1}}(t)y(t)) = -K_{v_0}(\text{Ad}_{\eta^{-1}}(t)y(t)) + w_0
  \]
  where $y = J \circ \eta^{-1}$, $J(0) = 0$, $J'(0) = w_0$.

- The operator $K_{v_0}$ is skew self-adjoint in the $L^2$ metric and compact.

- Thus we have a complete orthonormal basis of $T_e \mathcal{D}^0_\omega$ consisting of eigenvectors of $K_{v_0}$. 
Proof Idea

Letting \( u(t) = \text{Ad}_{\eta^{-1}(t)} y(t) \), and using the orthonormal basis of \( K_{\nu_o} \), our equation becomes

\[
\sum_j \partial_t u^j(t) \varphi_j = - \sum_j \lambda_j u^j(t) \varphi_j + \sum_j w_o^j \varphi_j
\]

or

\[
\partial_t u^j(t) = - \lambda_j u^j(t) + w_o^j
\]

with solution

\[
u^j(t) = \frac{1 - e^{\lambda_j t}}{\lambda_j} w_o^j
\]

since \( u_j(0) = y_j(0) = 0 \).
Proof Idea

- Now $0 = u(t^*) = \text{Ad}_{\eta^{-1}(t^*)}y(t^*)$ if and only if $y(t^*) = 0$.
- So, $\eta(t^*)$ is conjugate to $\eta(0)$ if and only if
  \[ 0 = u_i(t^*) = \frac{1 - e^{\lambda_i t^*}}{\lambda_i} w_o^i, \]
i.e. if and only if $K_v$ has an eigenvector with eigenvalue
  $\lambda_i = \frac{2\pi i}{t^*} k, \ k \in \mathbb{Z}/\{0\}$.
- Since complex eigenvalues come in conjugate pairs the multiplicity of every conjugate point is even:
  \[ u_1(t) = \frac{1 - e^{\lambda_1 t}}{\Re(\lambda_1)} w_o^1 \quad u_2(t) = \frac{1 - e^{-\lambda_1 t}}{-\Re(\lambda_1)} w_o^2 \]

D. G. Ebin, *Geodesics on the Symplectomorphism Group*, GAFA, Published Online (2012)


