Stochastic Solitons in Computational Anatomy

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Organization of the talk

1. Review: Momentum in images

2. Peakon momentum maps in 1D and 2D

3. Statistical models

4. Fokker-Planck equations for peakons and pulsons

5. Numerical experiments for stochastic landmark motion

6. Stratonovich and Itô Stochastic Euler-Poincaré equations

7. Summary
Review: Dynamics of ‘shapes’ $C^1(S^1, \mathbb{R}^2)$
Most problems in CA can be formulated as finding the time-dependent deformation map $\varphi_t : M \to M$ with minimal geodesic cost, defined by

$$\text{Cost}(t \mapsto \varphi_t) = \int_0^1 \ell(u_t) \, dt = \frac{1}{2} \int_0^1 \| u_t \|^2_{\mathcal{X}(M)} \, dt,$$

with $\frac{d\varphi_t}{dt} = u_t \circ \varphi_t$

under the constraint that the map $\varphi_t$ carries a template $I_0$ at $t = 0$ to the target $I_1$ at $t = 1$ and $\| \cdot \|_{\mathcal{X}(M)}$ is a given Riemannian metric.

The variable $u_t \in \mathcal{X}(M)$ is called the (Eulerian) velocity, and

$$m_t := \frac{\delta \ell}{\delta u_t} = Lu_t \quad \text{for} \quad \| u_t \|^2_{\mathcal{X}(M)} = \langle Lu_t, u_t \rangle$$

is called the momentum for $L^2$ pairing $\langle \cdot, \cdot \rangle$ and positive symmetric operator $L : \mathcal{X}(M) \to \mathcal{X}(M)^*$. 
Clebsch approach \Rightarrow \text{landmark momentum dynamics}

Consider Hamilton's principle for a Lagrangian $\ell(u) : \mathcal{X}(M) \to \mathbb{R}$. Constrain HP by the action of vector fields $u \in \mathcal{X}(M)$ as

$$\dot{q}(t) = u(q, t) \quad \text{for} \quad q \in M.$$  

For $N$ landmarks, $q(t) = \{ q_a(t), a = 1, 2, \ldots, N \}$, we take HP as

$$0 = \delta S = \delta \int_0^1 \left( \ell(u) + \sum_{a=1}^{N} \left< p, \dot{q} - u(q(t)) \right> \right) dt$$

Stationarity of HP leads to the following equations of motion [3],

$$\dot{q} = u(q, t), \quad \dot{p} = - \frac{du^T}{dq} \cdot p, \quad m(x, t) := \frac{\delta \ell}{\delta u} = \sum_{a=1}^{N} p \delta(x - q).$$

The 1st two eqns imply that the momentum $m(x, t)$ evolves by the EPDiff equation,

$$\partial_t m = - \text{ad}_u^* m = - u \cdot \nabla m - (\nabla u)^T m - m(\text{div} u).$$
Peakons: \( m(x, t) = p \delta(x - q) \), embeddings \( C^1(\mathbb{Z}, \mathbb{R}) \)

When \( \ell(u) = \frac{1}{2} \| u \|_{H^1}^2 = \frac{1}{2} \int u^2 + u_x^2 \, dx \) for \( M = \mathbb{R} \), then \( m = u - u_{xx} \) and \( \partial_t m = -(um)_x - mu_x \) produces the 1D solitons called ‘peakons’.

Singular peakon (landmark) solutions emerge from smooth initial conditions and form a finite dimensional solution set for EPDiff(\( H^1 \)).
Here, we have the EPDiff equation with \( u \in \mathcal{X}(\mathbb{R}^2) \), \( \ell : \mathcal{X}(\mathbb{R}^2) \to \mathbb{R} \)

\[
0 = \delta S = \delta \int \ell(u) + \sum_{n} \langle p, \dot{q} - u(q, t) \rangle dt, \quad \implies \quad \frac{d}{dt} \frac{\delta \ell}{\delta u} + \text{ad}^*_u \frac{\delta \ell}{\delta u} = 0.
\]
Definition: Momentum maps and their evolution

Smooth invertible maps $\varphi_t$ act on the symplectic manifold $T^*M$ by flows of cotangent lifts of vector fields $u_t = \frac{d\varphi_t}{dt} \circ \varphi_t^{-1} \in \mathfrak{X}$ acting on $M$.

The associated momap $m : T^*M \to \mathfrak{X}^*$ evolves via the EPDiff eqn

$$\frac{dm}{dt} = - \text{ad}^*_u m \quad \text{for} \quad m = \frac{\delta \ell}{\delta u} \in \mathfrak{X}^*$$
Momentum maps for $C^1$ embeddings $C^1(S^1, \mathbb{R}^2)$

Embeddings $C^1(S^1, \mathbb{R}^2)$ admit a dual pair of momaps

Momaps recast processes of shape change and reparametrization as: Right & Left group reductions by $\text{Diff}(S^1)$ and $\text{Diff}(\mathbb{R}^2)$, respectively, of the canonical Hamiltonian motion on $T^* C^1(S^1, \mathbb{R}^2)$.

Embedding phase space $T^* C^1(S^1, \mathbb{R}^2)$

Left (changes shape) $J_{\text{Sing}}$  
Right (preserves shape) $J_S$

(Changes shape) $\mathcal{X}(\mathbb{R}^2)^*$  
(Reparameterizes) $\mathcal{X}(S^1)^*$

DDH and JE Marsden
Momentum maps and measure valued solutions of the Euler-Poincaré equations for the diffeomorphism group
These momaps help *quantify* differences in shapes

... via geodesics along *diffeomorphisms* that map one shape to another; e.g. when “shape” is defined as a closed planar curve, that is, as a $C^1(S^1, \mathbb{R}^2)$ embedding.
Left and right transformations of $q \in C^1(S^1, \mathbb{R}^2)$

- The Right & Left actions of Lie groups $\text{Diff}_+(S^1)$ & $\text{Diff}_+(\mathbb{R}^2)$, respectively, on $q \in C^1(S^1, \mathbb{R}^2)$ **commute** with each other.

- The **Left action** of $\text{Diff}_+(\mathbb{R}^2)$ on $q \in C^1(S^1, \mathbb{R}^2)$:

  $$ q \mapsto L_g q = g \circ q, \quad g \in \text{Diff}_+(\mathbb{R}^2), $$

  **changes shape**: it transforms between inequivalent curves.

- The **Right action** of $\text{Diff}_+(S^1)$ on $q \in (S^1, \mathbb{R}^2)$:

  $$ q \mapsto R_\eta q = q \circ \eta, \quad \eta \in \text{Diff}_+(S^1), $$

  **preserves shape**, which defines an equivalence class of curves. Namely, the ‘shape space’ of $C^1$ embeddings modulo relabelling,

  $$ C^1(S^1, \mathbb{R}^2)/\text{Diff}_+(S^1) $$
Left action (motion) has a **singular momentum map**

MM for left action of $\text{Diff}_+(\mathbb{R}^2)$ on $q \in C^1(S^1, \mathbb{R}^2)$ is **singular** [3]

$$m = Lu(x) = \int_{S^1} p(s) \delta(x - q(s)) \, ds =: J_{\text{Sing}}(q, p).$$

- This MM obeys the EP equation and is preserved by right action.
- The paths $q(s)$ and their canonical momenta $p(s)$ are governed by the canonical Hamiltonian equations for $H = \frac{1}{2} \langle m, K \ast m \rangle$

$$\dot{q}(s) = u(q(s)) \quad \text{and} \quad \dot{p}(s) = - \frac{\partial u^T}{\partial q} \cdot p(s)$$

The reduced Lagrangian in Hamilton’s principle is

$$\ell(u) = \frac{1}{2} \int u \cdot Lu \, d^2x$$

The momentum is $m = \frac{\delta \ell}{\delta u} = Lu$ and the velocity is $u = K \ast m$, where $K$ is the Green’s function for the momentum operator $L$. 
The momap for right action is *conserved*

The momentum map for right action (reparametrization) is *conserved*

The cotangent-lifted mapping for the right action (reparameterizing) is

\[
(p, q) \mapsto R(p, q; \eta) = \left( q \circ \eta, \frac{p \circ \eta}{\partial \eta / \partial s} \right).
\]

Conservation of the momentum map for right action allows us to set

\[
J_S(q_t, p_t) = -p_t \cdot dq_t(s) = \left( -p_t \cdot \frac{\partial q_t}{\partial s} \right) ds = 0,
\]

Hence we may take \( p_t \) as normal to the planar curve \( q_t \in C^1(S^1, \mathbb{R}^2) \).
The processes of shape change and reparametrization may be recast as evolution of left and right momentum maps, $J_{Sing}$ and $J_S$.

\[ J_{Sing} \text{ evolves by EPDiff, and } J_S \text{ is conserved.} \]

In the case of landmarks, momentum $m_t(x)$ is characterized by a set of $N$ vectors, $p_t$, for a matching problem with $N$ landmarks, $q_t$.

Likewise in the case of $C^1$ embeddings parameterised by $s \in S^1$, there is no redundancy in the $q_t(s), p_t(s)$ representation of time-dependent deformations governed by $m_t = J_{Sing}(q_t, p_t)$ supported on $C^1(S^1, \mathbb{R}^2)$.

Next: Statistical models
Statistical models
Under evolution by EPDiff, statistical models for deformations become statistical models for \((q_t, p_t)\); with the advantage of being easier to build, sample and estimate on a linear space.

**Figure:** Here are three deformations of a disc produced by EPDiff for random momentum initial conditions, given by uncorrelated noise on its initially circular boundary. Figures courtesy of [5].
Trouvé and Vialard [7] studied perturbations of the geodesic equations by adding a random force to the landmark momentum equation, intended to represent a stochastic growth model:

\[
\dot{q}(s) = u(q(s)) \quad \text{and} \quad \dot{p}(s) = -\frac{\partial u^T}{\partial q} \cdot p(s) + \sigma B(t)
\]

As we shall see, these are *stochastic canonical Hamiltonian equations* in the sense of Bismut [1] and Lazáro-Camí and Ortega [6].

**Figure:** A simulation from [7] showing Kunita flow with 40 points on the unit circle on the left of the figure. The z axis (blue arrow) represents the time.
The perturbation considered in [7] applies a stochastic Brownian force (rate of change of momentum) on the particles, rather than making the particle paths stochastic.

\[
\dot{q}(s, t) = u(q(s, t)) \quad \text{and} \quad \dot{p}(s, t) = -\frac{\partial u}{\partial q}^T \cdot p(s, t) + \sigma B(t)
\]

With this stochastic forcing, Trouvé and Vialard [7] proved that:

1. The solutions for these landmark dynamics do not blow up.
2. In infinite dimensions, the solutions are also defined for all times.
3. Simple additive noise in the momentum equation is general enough to account for correlations between points on the curve during landmark evolution under stochastic forcing.
Stochastic canonical Hamiltonian equations (SHEs)

To see that the Trouv´e and Vialard [7] equations

\[ \dot{q}(s, t) = u(q(s, t)) \quad \text{and} \quad \dot{p}(s, t) = -\frac{\partial u^T}{\partial q} \cdot p(s, t) + \sigma B(t) \]

are \textit{stochastic canonical Hamiltonian equations}, we introduce the following Hamilton–Pontryagin variational principle.

\[
S(u, p, q) = \int \ell(u_t) dt + \int \left\langle p, \frac{d}{dt} q - u_t(q) \right\rangle dt - \int_{s,t} \sum_i h_i(p, q) \circ dW_i(t)
\]

Lebesque integrals in time \( t \)

Stratonovich integral

\[
\delta u : \quad m := \frac{\delta \ell}{\delta u} = \int_s \sum p \delta(x - q) \Rightarrow u(x, t) = K \ast m = \int_s \sum p K(x, q)
\]

\[
\delta p : \quad dq = u_t(q) dt + \sum_i \{q, h_i\} \circ dW_i(t)
\]

\[
\delta q : \quad dp = -\left( \frac{\partial u_t}{\partial q} \right) \cdot p dt + \sum_i \{p, h_i\} \circ dW_i(t)
\]
Substituting the momentum map relation for landmarks

\[ u(q_a) = K \ast \sum_{b=1}^{N} p_b \delta(q_a - q_b) = \sum_{b=1}^{N} p_b K(q_a, q_b) \]

into the \( \delta p \) and \( \delta q \) equations from Hamilton’s principle allows us to recognise them as canonical stochastic Hamilton equations

\[
\begin{align*}
\frac{dq}{dt} &= \frac{\partial h}{\partial p} dt \\
\frac{dp}{dt} &= -\frac{\partial h}{\partial q} dt
\end{align*}
\]

with

\[
 h(q, p) dt = \frac{1}{2} \sum_{a,b=1}^{N} p_a p_b K(q_a, q_b) dt + \sum_{i=1}^{M} h_i(q, p) \circ dW_i(t)
\]

The stochastic landmark dynamics of Trouvé and Vialard [7] will be recovered when we replace \( h_i(q, p) \circ dW_i(t) \rightarrow -\sigma q_i B_i(t) \) for Itô noise.
Itô version of SHEs

In the Itô version of stochastic canonical Hamiltonian equations, the noise terms have zero mean, but additional drift terms arise. These drift terms are double canonical Poisson brackets (diffusive).

\[
\begin{align*}
\delta u : \quad m & := \frac{\delta \ell}{\delta u} = \int_s \sum p \delta(x - q) \Rightarrow u(x, t) = K * m = \int_s \sum p K(x, q) \\
\delta p : \quad dq &= u_t(q) dt + \sum_i \{q, h_i\} dW_i(t) + \frac{1}{2} \sum_i \{q, h_i, h_i\} dt \\
\quad \text{Itô Noise for } q & \quad \text{Itô Drift for } q \\
\delta q : \quad dp &= -\left(\frac{\partial u_t}{\partial q}\right) \cdot p dt + \sum_i \{p, h_i\} dW_i(t) + \frac{1}{2} \sum_i \{p, h_i, h_i\} dt \\
\quad \text{Itô Noise for } p & \quad \text{Itô Drift for } p
\end{align*}
\]

The stochastic landmark dynamics of Trouvé and Vialard [7] is recovered when we choose \( h_i(p, q) = -\sigma q_i \), with \( i = 1, 2, 3 \), for \( q \in \mathbb{R}^3 \).
Fokker-Planck equations for peakons and pulsons

Fokker-Planck equations for peakons and pulsons
A stochastic process $X_t$ can be described with the help of a transition density function $\rho(t, x; \bar{x})$ which represents the probability density that the process, initially in the state $\bar{x}$, will reach the state $x$ at time $t$. For the general $N$-dimensional Itô process

$$dX_t^i = a_i(X_t) \, dt + \sum_{m=1}^{M} b_{im}(X_t) \, dW_t^m$$

the transition density function satisfies the Fokker-Planck advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \{a_i(x) \rho\} - \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} \{d_{ij}(x) \rho\} = 0,$$

where $d = bb^T$. 
Consider $N = 1$ pulson subject to one-dimensional (i.e., $M = 1$) Wiener process, with the stochastic potential $h(q, p) = \beta p$, where $\beta$ is a nonnegative real parameter. The stochastic Hamiltonian equations are (Ito and Stratonovich are equivalent here)

\[
\begin{align*}
    dq &= p \, dt + \beta \, dW_t, \\
    dp &= 0,
\end{align*}
\]

so the corresponding Fokker-Planck advection-diffusion equation is

\[
\frac{\partial \rho}{\partial t} + p \frac{\partial \rho}{\partial q} - \frac{1}{2} \beta^2 \frac{\partial^2 \rho}{\partial q^2} = 0
\]

with the initial condition $\rho(0, q, p; \bar{q}, \bar{p}) = \delta(q - \bar{q})\delta(p - \bar{p})$. 
The Fokker-Planck advection-diffusion equation

\[ \frac{\partial \rho}{\partial t} + p \frac{\partial \rho}{\partial q} - \frac{1}{2} \beta^2 \frac{\partial^2 \rho}{\partial q^2} = 0 \]

is easily solved with the help of the fundamental solution for the heat equation, and the solution yields

\[ \rho_\beta(t, q, p; \bar{q}, \bar{p}) = \frac{1}{\beta \sqrt{2\pi t}} e^{-\frac{(q-\bar{q}-pt)^2}{2\beta^2 t}} \delta(p - \bar{p}). \]

This solution means that the pulson/peakon retains its initial momentum/height \( \bar{p} \). The position has a Gaussian distribution which widens with time, and whose maximum is advected with velocity \( \bar{p} \).
Fokker-Planck solution—two pulsons

Consider \( N = 2 \) pulsons subject to two-dimensional (i.e., \( M = 2 \)) Wiener process, with the stochastic potentials \( h_1(q, p) = \beta_1 p_1 \) and \( h_2(q, p) = \beta_2 p_2 \), where \( q = (q_1, q_2) \), \( p = (p_1, p_2) \). The Fokker-Planck advection-diffusion equation for two pulsons is

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_1} \left[ a_1(q, p) \rho \right] + \frac{\partial}{\partial q_2} \left[ a_2(q, p) \rho \right] + \frac{\partial}{\partial p_1} \left[ a_3(q, p) \rho \right] + \frac{\partial}{\partial p_2} \left[ a_4(q, p) \rho \right] - \frac{1}{2} \beta_1^2 \frac{\partial^2 \rho}{\partial q_1^2} - \frac{1}{2} \beta_2^2 \frac{\partial^2 \rho}{\partial q_2^2} = 0
\]

with the initial condition

\[
\rho(0, q, p; \bar{q}, \bar{p}) = \delta(q_1 - \bar{q}_1)\delta(p_1 - \bar{p}_1) + \delta(q_2 - \bar{q}_2)\delta(p_2 - \bar{p}_2),
\]

where

\[
\begin{align*}
a_1(q, p) &= p_1 + p_2 G(q_1 - q_2), & a_3(q, p) &= -p_1 p_2 G'(q_1 - q_2), \\
a_2(q, p) &= p_2 + p_1 G(q_1 - q_2), & a_4(q, p) &= p_1 p_2 G'(q_1 - q_2).
\end{align*}
\]
Despite its relatively simple structure, F-P for two pulsons does not appear to be solvable analytically. Nevertheless, one may verify that the function

\[
\rho(t, q_1, q_2, p_1, p_2; \bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2) = \rho_{\beta_1}(t, q_1, p_1; \bar{q}_1, \bar{p}_1) + \rho_{\beta_2}(t, q_2, p_2; \bar{q}_2, \bar{p}_2),
\]

satisfies the F-P for two pulsons asymptotically as \( q_1 - q_2 \to \pm \infty \), assuming the Green’s function and its derivative both decay in that limit.

This simple observation shows that stochastic pulsons should behave like individual particles when they are far from each other, just as in the deterministic case.
Numerical experiments for stochastic landmark motion
Numerical experiments for stochastic landmark motion

The action functional for the phase-space Hamilton’s principle is

\[ S[q(t), p(t)] = \int_0^T \left( \sum_{a=1}^N p_a \dot{q}_a - H(q, p) \right) dt - \int_0^T \sum_{i=1}^M h_i(q, p) \circ dW_i(t), \]

with \( H(q, p) = \frac{1}{2} \sum_{a,b=1}^N p_a p_b K(q_a, q_b) \) for \( N \) landmarks. We take \( N = 2, q \in \mathbb{R} \), and \( h_1(q, p) = \beta p_2 \), for noise in the \( q_2 \) equation only.

The corresponding discrete action functional is

\[
S_d = \sum_{k=0}^{K-1} \left( \sum_{a=1}^N p_a^k \frac{q_{a}^{k+1} - q_a^k}{\Delta t} - H(q^{k+1}, p^k) \right) \Delta t - \sum_{k=0}^{K-1} \sum_{i=1}^M \frac{h_i(q^k, p^k) + h_i(q^{k+1}, p^{k+1})}{2} \Delta W_i^k
\]
Overtaking two-body collisions of pulsons & peakons

1. $N = 2$ Green’s functions for pulsons and peakons:
   - $K(q_1 - q_2) = e^{-(q_1 - q_2)^2}$, (pulsons)
   - $K(q_1 - q_2) = e^{-2|q_1 - q_2|}$, (peakons).

2. $M = 1$ dimensional noise with the potential $h_1(q, p) = \beta p_2$

3. Initial conditions:
   - $\bar{q}_1 = 0, \quad \bar{p}_1 = 8, \quad \bar{q}_2 = 10, \quad \bar{p}_2 = 1$,
   - $\bar{q}_1 = 0, \quad \bar{p}_1 = 4, \quad \bar{q}_2 = 10, \quad \bar{p}_2 = 1$,
   - $\bar{q}_1 = 0, \quad \bar{p}_1 = 2, \quad \bar{q}_2 = 10, \quad \bar{p}_2 = 1$,
   - $\bar{q}_1 = 0, \quad \bar{p}_1 = 1, \quad \bar{q}_2 = 10, \quad \bar{p}_2 = 1$,
Deterministic v Sample paths for Gaussian pulsons, with $\bar{\rho}_1 = 4$ and $\beta = 4$. Noise makes a big difference!
Figure: Example numerical sample paths for Gaussian pulsons for the simulations with $\bar{\rho}_1 = 4$ and $\beta = 4$. 
Mean paths—Gaussian pulsons, \( \bar{p}_1 = 4 \), vary \( \beta \)

Figure: Numerical mean paths for Gaussian pulsons for the simulations with \( \bar{p}_1 = 4 \). Results for three example choices of the parameter \( \beta \) are presented: \( \beta = 1.5 \) (top), \( \beta = 2.5 \) (middle), and \( \beta = 4.5 \) (bottom).
Sample paths—Gaussian pulsons, $\bar{\rho}_1 = 1$ and $\beta = 5$

Figure: Example numerical sample paths for Gaussian pulsons for the simulations with $\bar{\rho}_1 = 1$ and $\beta = 5$. 
Mean paths—Gaussian pulsons, $\bar{p}_1 = 1$, vary $\beta$

**Figure:** Numerical mean paths for Gaussian pulsons for the simulations with $\bar{p}_1 = 1$. Results for three example choices of the parameter $\beta$ are presented: $\beta = 0.5$ (top), $\beta = 1$ (middle), and $\beta = 2$ (bottom).
Probability of crossing, $\bar{p}_1 = 4$, vary $\beta$

**Figure:** Numerical probability density $\rho$ of the distance $\Delta q(t) = q_2(t) - q_1(t)$ at time $t = 100$ for Gaussian pulsions for the simulations with $\bar{p}_1 = 4$. Results for three example choices of the parameter $\beta$ are presented: $\beta = 1.5$ (*top*), $\beta = 2.5$ (*middle*), and $\beta = 4.5$ (*bottom*).
Figure: The probability of crossing, that is, the probability that $q_2(t) < q_1(t)$ at time $t = 100$, as a function of the parameter $\beta$ for Gaussian pulsons (top) and peakons (bottom).
Figure: Example probability density of the first crossing time $T_c$ for Gaussian pulsors for the simulations with $\bar{p}_1 = 4$ and $\beta = 2.5$. More precisely, this is the conditional probability density given that $T_c < \infty$, i.e., assuming the pulsors do cross, the integral $\int_a^b \rho(\tau) \, d\tau$ yields the probability that the first crossing occurs at time $T_c \in [a, b]$. 
First crossing time

Figure: The mean first crossing time $E(T_c)$ as a function of the parameter $\beta$ for Gaussian pulsons (top) and peakons (bottom). More precisely, this is the conditional expectation $E(T_c | T_c < \infty)$ given that the pulsons do cross (i.e., $T_c < \infty$).
The range of the stochastic effects between the two pulsons may be screened, for example, by applying the stochastic potential

\[ h_1(q, p) = \beta p_2 \exp \left( - \frac{(q_2 - q_1)^2}{\gamma^2} \right). \]

The parameter \( \beta \) adjusts the noise intensity, just as before. The parameter \( \gamma \) controls the range of the stochastic effects. Noise contributes now in both the momentum and position equations.
Sample paths with screened noise

Figure: Examples of numerical paths for Gaussian pulsons for the simulations with the initial conditions $\bar{q}_1 = 0$, $\bar{p}_1 = 4$, $\bar{q}_2 = 10$, and $\bar{p}_2 = 1$, and the screened stochastic potential (40) with the parameters $\beta = 4$ and $\gamma^2 = 4$. 

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Stochastic Stratonovich and Itô Euler-Poincaré equations
(in general, not just for landmarks)
The diamond operation defines momentum maps

**Definition (The diamond operation)**

On a manifold $M$, the diamond operation $(\diamond) : \mathcal{T}^* V \to \mathcal{X}^*$ is defined for a vector space $V$ with $(q, p) \in \mathcal{T}^* V$ and vector field $\xi \in \mathcal{X}$ is given in terms of the Lie-derivative operation $\mathcal{L}_{\xi}$ by

$$J^\xi(q, p) = \langle J(q, p), \xi \rangle_{\mathcal{X}} := \langle p, -\mathcal{L}_{\xi} q \rangle_V = \langle p \diamond q, \xi \rangle_{\mathcal{X}}$$

for the pairings $\langle \cdot, \cdot \rangle_V : \mathcal{T}^* V \times TV \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_{\mathcal{X}} : \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ with $p \diamond q \in \mathcal{X}^*$. 
Stratonovich stochastic variational principle [2]

Theorem (Stratonovich Stochastic Euler-Poincaré equations)

The action for the stochastic variational principle \( \delta S = 0 \) is,

\[
S(u, p, q) = \int \left( \ell(u, q) + \left\langle p, \frac{dq}{dt} + \mathcal{L}_u q \right\rangle \right) dt + \int \sum_i \left\langle p \diamond q, \xi_i(x) \right\rangle_x \circ dW_i(t).
\]

This leads to the following Stratonovich form of the stochastic Euler–Poincaré (SEP) equations

\[
dm + \mathcal{L}_{dx_t} m - \frac{\delta \ell}{\delta q} \diamond q dt = 0, \quad dq = -\mathcal{L}_{dx_t} q, \quad dp = \frac{\delta \ell}{\delta q} dt + \mathcal{T}_{dx_t} p,
\]

where \( dx_t = u(x, t) dt - \sum_i \xi_i(x) \circ dW_i(t) \in \mathfrak{X} \)

is the Stratonovich stochastic vector field and

\[
m := \frac{\delta \ell}{\delta u} = p \diamond q \in \mathfrak{X}^*
\]

is the left momentum map, which evolves. The right momentum map is still conserved.
Itô form of the stochastic Euler-Poincaré equations \[2\]

**Corollary (Itô form)**

Upon transforming to the Itô representation, the stochastic Euler-Poincaré equations take the following form with \( m = p \odot q \),

\[
\begin{align*}
dm + \mathcal{L}_{\dot{x}_t} m \, dt - \frac{\delta \ell}{\delta q} \odot q \, dt &= \frac{1}{2} \sum_j \mathcal{L}_{\xi_j(x)} \left( \mathcal{L}_{\xi_j(x)} m \right) \, dt =: \Delta_{\text{Lie}} m \, dt, \\
dq + \mathcal{L}_{\dot{x}_t} q \, dt &= \frac{1}{2} \sum_j \mathcal{L}_{\xi_j(x)} \left( \mathcal{L}_{\xi_j(x)} q \right) \, dt =: \Delta_{\text{Lie}} q \, dt, \\
dp - \mathcal{L}^T_{\dot{x}_t} p \, dt - \frac{\delta \ell}{\delta q} \, dt &= -\frac{1}{2} \sum_j \mathcal{L}^T_{\xi_j(x)} \left( \mathcal{L}^T_{\xi_j(x)} p \right) \, dt =: \Delta_{\text{Lie}} p \, dt,
\end{align*}
\]

with **Itô stochastic vector field**, \( d\hat{x}_t = u(x, t) \, dt - \sum_i \xi_i(x) \, dW_i(t) \).

**Note:** The right momentum map is conserved for Stratonovich, but not for Itô, because of \( \Delta_{\text{Lie}} \) terms. Consequently, the peakon momentum need not be normal to level sets of image data \( (q) \) on the Itô path \( d\hat{x}_t \).
Our usual thinking about landmark dynamics is based on dual pairs of momentum maps.

Previous work on stochastic landmark dynamics was based on additive noise in either the momentum initial condition [5], or the momentum equation of motion [7].

The latter is a stochastic Hamilton equation (SHE), and the entire theory of SHEs can be brought to bear on landmark dynamics.

For SHEs, placing an additive stochastic process in the landmark momentum equation does not cause topology change in the order of landmarks on the real line, but it does in the position equation.

A change in the order of EPDiff landmarks on 2D contours would correspond to “penetration” of one contour through the other.

The use of SHEs in the development of stochastic growth models is just beginning!
Thanks for listening!
References


A Trouvé, FX Vialard [2012]. Shape splines and stochastic shape evolutions: a second order point of view. *Quart. Appl. Math* 70, 21925110.1090/S0033-569X-2012-01250-4