No-signaling-in-time as a condition for macrorealism: the case of neutrino oscillations

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We study necessary and sufficient conditions for macrorealism (known as no-signaling-in-time and arrow-of-time conditions) in the context of neutrino flavor transitions, within both the plane wave description and the wave packet approach. We then compare the outcome of the above investigation with the implication of various formulations of Leggett–Garg inequalities. In particular, we show that the fulfillment of the addressed conditions for macrorealism in neutrino oscillations implies the fulfillment of Leggett–Garg inequalities, whereas the converse is not true. Finally, in the framework of wave packet approach, we also prove that, for distances longer than the coherence length, the no-signaling-in-time condition is always violated whilst Leggett–Garg inequalities are not.

I. INTRODUCTION

Neutrino mixing and oscillations represent the main indications of physics beyond the Standard Model [1– 5]. Among the multifaceted aspects of the above phenomenon, in recent years the quantum informational properties of mixed flavor states have been widely investigated [6–15]. An important achievement along this direction is the characterization of the intrinsic quantum nature of neutrino oscillations, which has been probed with the data available from the MINOS experiment by means of the Leggett–Garg inequalities (LGIs) [16].

Loosely speaking, LGIs are typically regarded as the temporal analogues of Bell inequalities; whilst the latter quantify the quantumness of a given system via spatially-separated tests (thus dealing with quantum nonlocality), the former rely on the notion of macroscopic coherence based upon temporal auto-correlation functions [17–21]. Indeed, LGIs are closely related to the concept of macro-realism, an intuitive view of our classical macroscopic world according to which measurements do not perturb the state of the probed system and reveal a pre-existing, observable quantity.

Because of their relevance, LGIs have been extensively employed in experimental verifications [16, 22–25]. On the same footing, in the last decades systems revealing phenomena of mixing and flavor oscillations have become the subject of an emergent exploration dealing with classicality and macroscopic superpositions [14, 26–39]. As a matter of fact, it is no coincidence that neutrinos provide a promising probe for testing the validity of LGIs, since their flavor oscillations exhibit quantum coherence even after the particles have traveled macroscopic distances [14, 34–37].

Despite the pivotal role covered by LGIs, experiments centered around macrorealism reveal a more complex structure if compared with tests based upon local realism [40]. The crucial difference lies in the fact that, whilst Bell inequalities are both necessary and sufficient conditions for local realism [41], the fulfillment of LGIs is not in a one-to-one correspondence with macrorealism. Indeed, the validity of the standard LGIs and their variants such as the Wigner form of LGIs (WLGIs) [42] turns out not to be sufficient for macrorealism [40, 43, 44]. For this reason, it is essential to introduce another set of conditions for macrorealism which would be both necessary and sufficient; such a set is given by a combination of *no-signaling-in-time* (NSIT) (which is an alternative necessary condition for macrorealism [19, 43]) and arrowof-time (AoT) conditions [43, 45]. Being equalities for joint probabilities rather than inequalities, these requirements are more suitable to be interpreted as quantum witnesses.

In this paper, we study the NSIT and AoT conditions in the case of two-flavor neutrino oscillations. We find that, while AoT conditions are always trivially satisfied, neutrino oscillations always violate NSIT excluding an integer set of isolated points. However, if a wave-packet treatment is considered and the measurements are performed at sufficiently large intervals of time (corresponding to distances longer than the coherence length), the NSIT conditions are always violated. This fact confirms that, even after the occurrence of wave-packet decoherence, neutrinos still retain their intrinsic quantum nature, thereby preventing a macrorealistic interpretation of flavor transitions even at late times. In conjunction with that, we also compare the validity of LGIs (WLGIs) with the validity of NSIT and AoT conditions; in so doing, we find that LGIs (WLGIs) are never violated when NSIT and AoT are not, and that for large-time intervals all the LGIs (WLGIs) are fulfilled.

The remainder of the paper is organized as follows: in

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Section II, we review the notion of macrorealism and the related quantifiers we will employ to support our reasoning (namely, LGIs, WLGIs and NSIT/AoT). In Section III, we provide the necessary tools to investigate neutrino oscillations and analyze the ensuing NSIT conditions in the two-flavor approximation; with these results, we then establish a comparison with the predictions stemming from LGIs and WLGIs. Finally, Section IV contains conclusions and future perspectives.

II. MACROREALISM AND NSIT CONDITIONS

According to our daily experience, we do not observe macroscopic objects around us being in two different positions at the same time. Furthermore, a motionless object with a net vanishing force acting on it stays at all times in a given place which can be determined by simply looking at it. *Macrorealism* aims at formalizing this knowledge by relying on the following basic assumptions¹:

- *macrorealism per se*: given a set of available macroscopically distinct states, a macroscopic object is in one of them at any given time;
- *non-invasive measurability*: it is possible in principle to determine the state of the macroscopic object without affecting either its state or its dynamical evolution.

Similarly to the celebrated Bell inequalities in the framework of local realism, one can derive a set of inequalities (known as LGIs) that have to be satisfied by any physical system abiding by the above macrorealistic prescriptions. To show this in a simple case, let us consider a system with a dichotomous macroscopic observable O with associated values ± 1 which is consecutively measured N times by an observer at fixed time points $\{t_0, t_1, ..., t_{N-1}\}$. Assuming for simplicity N = 3 (*i.e.*, three measurements at times t_0, t_1, t_2), the measurement statistics with respect to the 2-time correlation functions $C_{ij} = \langle O(t_i)O(t_j) \rangle$ has to satisfy the LGIs [20, 44]

$$\mathcal{L}_1(t_0, t_1, t_2) = 1 + C_{01} + C_{12} + C_{02} \ge 0, \qquad (1)$$

$$\mathcal{L}_2(t_0, t_1, t_2) = 1 - C_{01} - C_{12} + C_{02} \ge 0, \qquad (2)$$

$$\mathcal{L}_3(t_0, t_1, t_2) = 1 + C_{01} - C_{12} - C_{02} \ge 0, \qquad (3)$$

$$\mathcal{L}_4(t_0, t_1, t_2) = 1 - C_{01} - C_{12} - C_{02} \ge 0, \qquad (4)$$

if macrorealism holds true. Hence, as for Bell inequalities, these relations can be used to explore the quantumness of a system and the existence of macroscopic superpositions. Indeed, in quantum mechanics the LGIs (1)-(4) can be (and are) violated, in particular by the systems coherently oscillating between the states on which $O = \pm 1$, respectively [20].

The Leggett–Garg inequalities (1)-(4) can then be regarded as the temporal counterpart of the Bell inequalities, and just like the latter they are not unique. As a matter of fact, alternative forms of LGIs can be found by focusing solely on the joint probabilities $P(O_i, O_j)$ of finding outcomes O_i and O_j after measuring O at times t_i and t_j , respectively (instead of evaluating the functions C_{ij} [21, 42]). Indeed, macrorealism entails the existence of an overall joint probability distribution $P(O_0, O_1, O_2)$ of definite outcomes at all measurement times t_0, t_1, t_2 . Thus, the two-time probabilities $P(O_i, O_i)$ can be straightforwardly calculated as marginals of the overall joint probability distribution. The requirement of positivity $P(O_0, O_1, O_2) \geq 0$ demands specific constraints on $P(O_i, O_j)$; the shape of such constraints can be summarized in the so-called WL-GIs [35]

$$\mathcal{W}_1(t_0, t_1, t_2) = P(O_1, O_2) - P(-O_0, O_1) - P(O_0, O_2) \le 0,$$
(5)

$$\mathcal{W}_2(t_0, t_1, t_2) = P(O_0, O_2) - P(O_0, -O_1) - P(O_1, O_2) \le 0,$$
(6)

$$W_3(t_0, t_1, t_2) = P(O_0, O_1) - P(O_1, -O_2) - P(O_0, O_2) \le 0.$$
⁽⁷⁾

As it occurs for LGIs (1)-(4), WLGIs (5)-(7) can be violated by quantum mechanical probabilities.

Interestingly, it has been pointed out that all forms of LGIs represent only a necessary (but not a sufficient) condition for macrorealism, which can still be violated even if LGIs are satisfied [40, 43]. This raises the need to seek alternative conditions that could signal a quantum behavior for the cases in which LGIs provide an incomplete description. A necessary and sufficient condition is given by a set of equalities [43] consisting of two classes that constrain signaling from past to future (known as nosignaling-in-time conditions, or NSIT) and from future to past (known as arrow-of-time conditions, or AoT). In the case N = 3 (the measurements considered in the present

¹ Often, a third extra condition of *induction* is considered [20], which states that the outcome of a measurement on a system cannot be affected by what will/will not be measured on it later.

work), one can identify three NSIT conditions

NSIT⁽¹⁾:
$$P(O_2) = \sum_{O_1} P(O_1, O_2),$$
 (8)

NSIT⁽²⁾:
$$P(O_0, O_2) = \sum_{O_1} P(O_0, O_1, O_2),$$
 (9)

NSIT⁽³⁾:
$$P(O_1, O_2) = \sum_{O_0} P(O_0, O_1, O_2),$$
 (10)

and three AoT conditions

AoT⁽¹⁾:
$$P(O_0, O_1) = \sum_{O_2} P(O_0, O_1, O_2),$$
 (11)

AoT⁽²⁾:
$$P(O_0) = \sum_{O_1} P(O_0, O_1),$$
 (12)

AoT⁽³⁾:
$$P(O_1) = \sum_{O_2} P(O_1, O_2).$$
 (13)

Remarkably, it can be proved that NSIT conditions imply all possible forms of LGIs.

In the following, we apply the notions introduced above in the context of neutrino flavor transitions to compare the different conditions for macrorealism.

III. MACROREALISM IN NEUTRINO OSCILLATIONS

A. Phenomenology of neutrino oscillations

Neutrinos provide a paradigmatic example of mixed particles, whose physical (flavor) states distinguishable in a weak process do not coincide with the (mass) eigenstates of their Hamiltonian, which propagate with frequencies that depend on the corresponding masses. In the relativistic regime, neutrino mass eigenstates evolve according to

$$|\nu_j(t)\rangle = e^{-iE_j t} |\nu_j(0)\rangle, \qquad (14)$$

$$E_j = \sqrt{p^2 + m_j^2} \approx E + \frac{m_j}{2E}, \qquad (15)$$

where the masses m_j are taken to be much smaller than their momentum and E = p is the energy of a massless neutrino. On the other hand, flavor states are welldescribed as superpositions of the mass eigenstates [1, 2]

$$|\nu_{\sigma}(t)\rangle = \sum_{j} U^{*}_{\sigma j} |\nu_{j}(t)\rangle, \qquad (16)$$

with coefficients given by the elements $U_{\sigma j}$ of the mixing matrix U. The non-equivalence of physical flavor states and mass eigenstates of the particle Hamiltonian ascribed to the mixing phenomenon is responsible for the oscillation between distinct flavor states. If a neutrino is produced in a weak process at time t = 0 with a given flavor σ , it evolves into a superposition of flavor states at t > 0 in such a way that the probability of detecting another flavor ρ is

$$P_{\sigma \to \rho}(t) = |\langle \nu_{\rho}(t) | \nu_{\sigma}(0)|^{2}$$
$$= \sum_{j,k} U_{\rho j} U_{\sigma k} U_{\rho k}^{*} U_{\sigma j}^{*} \exp\left(-i\frac{\Delta m_{jk}^{2}}{2E}t\right) ,(17)$$

where $\Delta m_{jk}^2 \equiv m_j^2 - m_k^2$. In particular, for the two-flavor case (a typical approximation that successfully describes many experiments with good accuracy [46]), the mixing matrix is given by

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad (18)$$

with θ being the *mixing angle*. Under these assumptions, the flavor oscillation probability is given by the Pontecorvo formula

$$P_{\sigma \to \rho}(t) = \sin^2(2\theta) \, \sin^2\left(\frac{\Delta m^2}{2E}t\right) \,, \quad \sigma \neq \rho \,, \quad (19)$$

$$P_{\sigma \to \sigma}(t) = 1 - P_{\sigma \to \rho}(t) , \qquad (20)$$

and $\Delta m^2 \equiv \Delta m_{12}^2$. In light of these features, flavor neutrinos resemble the behavior of two-level systems such as spin-1/2 states and polarized photons; hence, they are naturally liable to be studied in the framework of macro-realism.

Note that, in the scenario described so far, mass eigenstates possess a definite momentum p; therefore, they are considered as propagating plane waves. Nevertheless, the above picture still manages to fit most of neutrino physics phenomenology that is probed in actual experiments. However, a more realistic investigation of neutrinos requires a treatment of mass eigenstates in terms of wave packets. To this aim, let us now consider a neutrino propagating along the x-direction

$$\left|\nu_{\sigma}(x,t)\right\rangle = \sum_{j} U_{\sigma j}^{*} \psi_{j}(t,x) \left|\nu_{j}\right\rangle, \qquad (21)$$

where the wave packets $\psi_j(t, x)$ can be chosen as being Gaussian functions [47]

$$\psi_j(t,x) = \left(\sqrt{2\pi}\sigma_x\right)^{-\frac{1}{2}} e^{i(px-E_jt)} e^{-\frac{(x-v_jt)^2}{4\sigma_x^2}}$$

Here, p is the average momentum of the wave packet², while σ_x is the spatial spreading and

$$v_j = \frac{p}{E_j} \approx 1 - \frac{m_j^2}{2E^2},$$
 (22)

² To keep our considerations simple and without loss of generality, we can impose the same average momentum for all mass eigenstates $|\nu_j\rangle$.

where v_j are the group velocities. The flavor oscillation formula is thus given by

$$P_{\sigma \to \rho}(t,x) = \left(\sqrt{2\pi}\sigma_x\right)^{-1} \sum_{j,k} U_{\rho j} U_{\sigma k} U_{\rho k}^* U_{\sigma j}^* e^{-i\frac{\Delta m_{jk}^2}{2E}t} \times e^{-\frac{(x-v_jt)^2}{4\sigma_x^2} - \frac{(x-v_kt)^2}{4\sigma_x^2}}.$$
 (23)

In neutrino experiments, there is no direct access to time measurements, but the distance between the source and the detector is known. Therefore, concerning neutrino phenomenological studies, time is typically superseded by space. As our aim is to test macrorealism involving measurements taken at different times, a reverse conversion of space into time is mandatory. This procedure does not affect the oscillation formula, which essentially remains the same because of the interchangeability between time and space in the relativistic regime [47].

Now, we can integrate (23) over x and normalize it in order to obtain a consistent probabilistic description (*i.e.*, $\sum_{\sigma} P_{\sigma \to \rho}(t) = 1$). Eventually, one obtains the following oscillation formula:

$$P_{\sigma \to \rho}(t) = \sum_{j,k} U_{\rho j} U_{\sigma k} U_{\rho k}^* U_{\sigma j}^* \exp\left(-i\frac{\Delta m_{jk}^2}{2E}t\right) \\ \times \exp\left(-\frac{(\Delta m_{jk}^2)^2 t^2}{32E^4\sigma_x^2}\right).$$
(24)

The exponential damping factor is responsible for the relative spread of mass-neutrino wave packets and, in turn, for the decoherence mechanism which averages the oscillations on long-time intervals (distances). Therefore, it is possible to identify a characteristic space/time scale at which the decoherence occurs, namely the so-called *coherence length*

$$L_{jk}^{coh} = \frac{4\sqrt{2}E^2}{\left|\Delta m_{jk}^2\right|} \sigma_x \,. \tag{25}$$

Finally, by specializing Eq. (24) for the two-flavor case, the oscillation formula reads

$$P_{\sigma \to \rho}(t) = \frac{\sin^2(2\theta)}{2} \left(1 - e^{-\left(\frac{t}{L^{coh}}\right)^2} \cos\left(\frac{\Delta m^2}{E}t\right) \right),$$
(26)
with $L^{coh} = \frac{4\sqrt{2}E^2}{|\Delta m^2|} \sigma_x.$

We are now ready to introduce the necessary and sufficient conditions for macrorealism in neutrino oscillations. For this purpose, both the plane-wave and the wavepacket description of two-flavor Dirac neutrinos will be considered.

B. Necessary and sufficient NSIT/AoT conditions for macrorealism in neutrino oscillations

In order to test macrorealism in neutrino oscillations using the combined NSIT/AoT conditions (8)–(13), we choose neutrino flavor to be the macroscopic dichotomous observable O(t). Since we work within the two-flavor approximation (where the flavor can be either electronic e or muonic μ), we define it as $O(t) = |\nu_e(t)\rangle\langle\nu_e(t)| - |\nu_\mu(t)\rangle\langle\nu_\mu(t)|$, which thus represents a dichotomous variable with values ± 1 corresponding to e- and μ -neutrino flavors, respectively. The ensuing joint probabilities in the NSIT/AoT conditions (8)–(13) for the measurement outcomes can be straightforwardly rewritten in terms of flavor oscillating probabilities using the conditional probability rule

$$P(O_i, O_j) = P(O_i)P(O_j|O_i) = P_{O_0 \to O_i}(t_i)P_{O_i \to O_j}(t_j - t_i).$$
(27)

Without loss of generality, we assume that an electronic neutrino is produced at time $t_0 = 0$ and its flavor is subsequently measured at $t_1 = t$ and $t_2 = 2t$. When the measurement outcomes O_i are fixed, we assume $O_0 = +1 \equiv e, O_1 = -1 \equiv \mu$, and $O_2 = -1 \equiv \mu$. Therefore, the full set of NSIT/AoT conditions in neutrino oscillations is

$$\left. \begin{array}{l} P_{e \to \mu}(2t) = P_{e \to e}(t)P_{e \to \mu}(t) + P_{e \to \mu}(t)P_{\mu \to \mu}(t) \\ P_{e \to e}(0)P_{e \to \mu}(2t) = P_{e \to e}(0)P_{e \to e}(t)P_{e \to \mu}(t) + P_{e \to e}(0)P_{e \to \mu}(t)P_{\mu \to \mu}(t) \\ P_{e \to \mu}(t)P_{\mu \to \mu}(t) = P_{e \to e}(0)P_{e \to \mu}(t)P_{\mu \to \mu}(t) + P_{e \to \mu}(0)P_{e \to \mu}(t)P_{\mu \to \mu}(t) \\ P_{e \to e}(0)P_{e \to \mu}(t) = P_{e \to e}(0)P_{e \to \mu}(t)P_{\mu \to e}(t) + P_{e \to e}(0)P_{e \to \mu}(t)P_{\mu \to \mu}(t) \\ P_{e \to e}(0) = P_{e \to e}(0)P_{e \to e}(t) + P_{e \to e}(0)P_{e \to \mu}(t) \\ P_{e \to \mu}(t) = P_{e \to \mu}(t)P_{\mu \to e}(t) + P_{e \to \mu}(t)P_{\mu \to \mu}(t) \end{array} \right\} \quad \text{AoT}$$

Interestingly, by suitably manipulating the AoT conditions, one ends up with three relations which are identically satisfied at all times³, that is

AoT⁽¹⁾: 1 =
$$P_{\mu \to e}(t) + P_{\mu \to \mu}(t)$$
, (28)

AoT⁽²⁾: 1 =
$$P_{e \to e}(t) + P_{e \to \mu}(t)$$
, (29)

AoT⁽³⁾: 1 =
$$P_{\mu \to e}(t) + P_{\mu \to \mu}(t)$$
. (30)

This is somewhat expected, because the AoT conditions are usually satisfied in standard quantum mechanics [40]. Thus, for neutrino oscillations, AoT conditions can be safely neglected, thereby leaving the NSIT conditions as the relevant ones. Accounting for the symmetry of flavor oscillation probabilities under exchange of flavors, *i.e.*, $P_{e\to e}(t) = P_{\mu\to\mu}(t)$ and $P_{e\to\mu}(t) = P_{\mu\to e}(t)$, the NSIT are then given by

$$\text{NSIT}^{(1)}: P_{e \to \mu}(2t) = 2P_{e \to \mu}(t)P_{e \to e}(t), \qquad (31)$$

NSIT⁽²⁾:
$$P_{e \to \mu}(2t) = 2P_{e \to \mu}(t)P_{e \to e}(t)$$
, (32)

$$\text{NSIT}^{(3)}: \quad P_{e \to \mu}(t) P_{\mu \to \mu}(t) = P_{e \to \mu}(t) P_{\mu \to \mu}(t) (33)$$

It is straightforward to check that $NSIT^{(3)}$ is a trivial relation, whilst $NSIT^{(1)}$ and $NSIT^{(2)}$ coincide. Consequently, macrorealism in neutrino oscillations can be witnessed by a single necessary and sufficient NSIT condition:

$$\mathcal{N}(t) \equiv P_{e \to \mu}(2t) - 2P_{e \to \mu}(t) P_{e \to e}(t) = 0. \quad (34)$$

The function \mathcal{N} is plotted in Fig. 1 as a function of time for both the plane-wave and the Gaussian wave-packet flavor oscillation probabilities (19) and (26), respectively. It is worth stressing that, for the plane-wave description, the NSIT condition (34) is periodically fulfilled in isolated points. On the other hand, in the realistic wave-packet scenario, Eq. (34) is fulfilled only for fewer values of the time with respect to the previous case; this occurs because the behavior of $\mathcal{N}(t)$ is similar to the plane-wave result only for small t (*i.e.*, when the damping exponential is still close to unity). Nevertheless, it is crucial to observe that, for large t, $\mathcal{N}(t)$ approaches a constant value which in general is different from zero, thereby preventing flavor transitions to be interpreted in a macrorealistic way. This occurs because, at late times, one can check that

$$\lim_{t \to +\infty} \mathcal{N}(t) = -\frac{\sin^2(4\theta)}{8}, \qquad (35)$$

which is identically zero only for integer multiples of the maximal mixing angle $\pi/4$.

The above picture can be easily explained in quantum informational terms. Indeed, if neutrinos with different



FIG. 1. $\mathcal{N}(t)$ for the plane-wave (red) and the Gaussian wavepacket (blue) approach as a function of time expressed in eV^{-1} . The values used to generate the plot have been taken from the MINOS experiment [50], with $\sin^2 \theta = 0.314$, $\Delta m^2 =$ $7.92 \times 10^{-5} \mathrm{eV}^2$, $E = 10 \mathrm{GeV}$ and $\sigma_x = 0.5 \mathrm{GeV}^{-1}$.

flavors are viewed as being qubits of a two-level system [6–9], it can be shown that, despite the decoherence due to the wave-packet spreading, the amount of quantum correlations shared by the qubits is always non-vanishing, thus allowing for the constant presence of a signature of quantum behavior [48, 49]. In turn, this fact entails that, regardless of the distance traveled and of the wave-packet separation, for realistic values of the mixing angle (such as the ones used in Fig. 1 [50]) under no circumstances the phenomenon of flavor transition is compatible with macrorealism.

Before concluding this section, an important remark has to be made: the obtained results related to the NSIT/AoT conditions for macrorealism in neutrino oscillations are independent of the choice of the initial condition (namely, the neutrino flavor at t = 0) and the values of the outcomes O_0 , O_1 , and O_2 . In fact, by following the same steps as above, one can easily prove that any arbitrary choice for O_0 , O_1 , and O_2 leads to the same necessary and sufficient condition (34). This statement further corroborates the reliability of neutrino oscillations as a suitable instrument to investigate macrorealism.

C. Comparison of NSIT/AoT with other conditions for macrorealism

In order to compare the condition (34) of macrorealism in neutrino oscillations with the predictions obtained with LGIs, we have to adapt the latter to the problem at hand. To this aim, we first investigate the LGIs in their standard formulations (1)-(4), which require the evaluation of the correlation functions in terms of flavor oscil-

³ Note that this occurrence might not be true when considering the three-flavor scenario because of the presence of a non-vanishing CP-violating phase.

lation probabilities:

$$C_{ij} = \langle O(t_i)O(t_j) \rangle$$

= $P_{e \to e}(t_i) \Big(P_{e \to e}(t_j - t_i) - P_{e \to \mu}(t_j - t_i) \Big)$
+ $P_{e \to \mu}(t_i) \Big(P_{\mu \to \mu}(t_j - t_i) - P_{\mu \to e}(t_j - t_i) \Big).$ (36)

Bearing this in mind, we have

$$\begin{split} C_{01} &= P_{e \to e}(0) \left(P_{e \to e}(t) - P_{e \to \mu}(t) \right) \\ &+ P_{e \to \mu}(0) \left(P_{\mu \to \mu}(t) - P_{\mu \to e}(t) \right), \\ C_{12} &= P_{e \to e}(t) \left(P_{e \to e}(t) - P_{e \to \mu}(t) \right) \\ &+ P_{e \to \mu}(t) \left(P_{\mu \to \mu}(t) - P_{\mu \to e}(t) \right), \\ C_{02} &= P_{e \to e}(0) \left(P_{e \to e}(2t) - P_{e \to \mu}(2t) \right) \\ &+ P_{e \to \mu}(0) \left(P_{\mu \to \mu}(2t) - P_{\mu \to e}(2t) \right). \end{split}$$

Finally, invoking the symmetry of flavor oscillation probabilities under the exchange of flavor subscripts, it is immediate to verify that

$$C_{01} = P_{e \to e}(t) - P_{e \to \mu}(t), \qquad (37)$$

$$C_{12} = P_{e \to e}(t) - P_{e \to \mu}(t),$$
 (38)

$$C_{02} = P_{e \to e}(2t) - P_{e \to \mu}(2t).$$
(39)

Now, plugging the found correlation functions (37)–(39) into the definitions (1)-(4), we reach the expression of LGIs in the framework of neutrino oscillations, that is

$$\mathcal{L}_1(t) = 2P_{e \to e}(t) + 2P_{e \to e}(2t) - 2P_{e \to \mu}(t) \ge 0, \, (40)$$

$$\mathcal{L}_{2}(t) = 2P_{e \to \mu}(t) - P_{e \to \mu}(2t) \ge 0, \qquad (41)$$

$$\mathcal{L}_3(t) = 2P_{e \to e}(2t) \ge 0, \qquad (42)$$

$$\mathcal{L}_4(t) = 2P_{e \to \mu}(t) + 2P_{e \to \mu}(2t) - 2P_{e \to e}(t) \ge 0.$$
(43)

It is evident that Eq. (42) is trivially satisfied. However, it is interesting to compare the other Leggett–Garg conditions with the NSIT (34). A plot of the above functions together with $\mathcal{N}(t)$ for the flavor oscillation probability (26) in the wave-packet description is displayed in Fig. 2. It can be immediately observed that the entire set of LGIs (40)-(43) is always fulfilled at late times, whilst $\mathcal{N}(t) \neq 0$. This divergence in the predictions could have been foreseen, since the NSIT (34) is a necessary and sufficient condition for macrorealism, but the LGIs (40)-(43) are not.

Turning the attention on the Wigner formulation of LGIs (5)-(7), we observe that they are already cast in terms of probabilities of measurement outcomes, and hence the identification with flavor oscillation probabilities turns out to be more natural. Indeed, we obtain

$$W_1(t) = P_{e \to e}(t) P_{\mu \to e}(t) - P_{\mu \to e}(2t) \leq 0, \quad (44)$$

$$\mathcal{W}_2(t) = P_{e \to \mu}^2(t) - P_{e \to e}(2t) \le 0, \qquad (45)$$

$$\mathcal{W}_3(t) = P_{e \to e}(t) P_{\mu \to e}(t) - P_{\mu \to e}(2t) \le 0, \quad (46)$$



FIG. 2. $\mathcal{N}(t)$ (blue) vs $\mathcal{L}_1(t)$ (brown), $\mathcal{L}_2(t)$ (red), and $\mathcal{L}_4(t)$ (black) as functions of time expressed in eV^{-1} . The former witnesses violation of the NSIT condition whenever it is not equal to zero, while the latter witness violation of LGIs whenever they are negative. It is worth highlighting that, for large t, all $\mathcal{L}_j(t)$ are always non-negative while $\mathcal{N}(t)$ differs from zero. The values used to generate the plot have been taken from the MINOS experiment [50], with $\sin^2 \theta = 0.314$, $\Delta m^2 = 7.92 \times 10^{-5} \text{ eV}^2$, E = 10 GeV and $\sigma_x = 0.5 \text{ GeV}^{-1}$.

from which we deduce that the first and the last conditions coincide, thus leading to two non-trivial inequalities for neutrino oscillations. In Fig. 3, we compare the relevant WLGIs with the NSIT condition (34). As in the case of the standard LGIs (40)–(43), the WLGIs (44)– (46) are satisfied for large t, where according to the NSIT condition no macrorealistic interpretation can be alleged, thereby confirming the previous considerations. In fact, the obtained results for both formulations of LGIs reveal that a macrorealistic description is not necessarily valid in the regime where such inequalities are satisfied.

IV. CONCLUSIONS

In this paper, we have provided a preliminary analysis of necessary and sufficient conditions for macrorealism in neutrino flavor transitions. In particular, we have unambiguously found that the set of necessary and sufficient NSIT/AoT conditions derived in Ref. [43] reduces to a single, non-trivial NSIT relation for macrorealism which can be potentially probed in two-flavor neutrino experiments. Moreover, concerning the wave-packet approach, we have seen that the effect of decoherence for long detection times/distances allows for a net deviation from a macrorealistic interpretation, thereby unambiguously attributing a quantum nature to the phenomenon of neutrino oscillations. For this reason, neutrinos can never be described in a macrorealistic way, even when quantum coherence is apparently degraded because of the wave packet spreading.

Additionally, we have compared the aforementioned



FIG. 3. $\mathcal{N}(t)$ (blue) vs $\mathcal{W}_1(t)$ (red, first plot) and $\mathcal{W}_2(t)$ (red, second plot) as functions of time expressed in eV^{-1} . The former witnesses violation of the NSIT condition whenever it is not equal to zero, while the latter witness violation of WLGIs whenever they are positive. Note that for large t, all $\mathcal{W}_j(t)$ are always non-positive while $\mathcal{N}(t)$ differs from zero. The values used to generate the plot have been taken from the MINOS experiment [50], with $\sin^2 \theta = 0.314$, $\Delta m^2 = 7.92 \times 10^{-5} \ \mathrm{eV}^2$, $E = 10 \ \mathrm{GeV}$ and $\sigma_x = 0.5 \ \mathrm{GeV}^{-1}$.

NSIT condition for macrorealism with the LGIs in their standard and Wigner formulations. In both scenarios, we have discovered that, as long as the NSIT requirement is met, the LGIs are satisfied. However, at late times, we have shown that the LGIs are not faithful quantifiers of the macrorealistic description, since they are fulfilled whilst the NSIT condition is always violated.

Our research paves the way toward a more accurate study of macrorealism for neutrino flavor transitions. Although the phenomenology of neutrino oscillations can be effectively studied in the framework of quantum mechanics (QM), a proper treatment of neutrinos demands the application of quantum field theory (QFT) due to their relativistic nature [5]. As a preliminary analysis along this direction, in Ref. [14] violations of the WLGIs in neutrino oscillations have been compared in the context of QM and QFT. Interestingly, it turns out that QFT violates the WLGIs more frequently than QM, which is in agreement with the results obtained for the Bell tests of local realism within the general framework of algebraic QFT [51–53]. As a further evidence for the same occurrence, it has been proven that even vacuum correlations in QFT can lead to a maximal quantum violation of Bell inequalities [13]. Therefore, both studies seem to indicate that QFT is *less classical* than QM, and these results has to be reviewed by means of the NSIT/AoT conditions for macrorealism.

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