Universal Quantum Computation via Superposed Orders of Single-Qubit Gates

Kyrylo Simonov, Marcello Caleffi*, *Senior Member, IEEE*, Jessica Illiano, Angela Sara Cacciapuoti, *Senior Member, IEEE*

***Abstract*—Superposed orders of quantum channels have already been proved – both theoretically and experimentally – to enable unparalleled opportunities in the quantum communication domain. As a matter of fact, superposition of orders can be exploited within the quantum computing domain as well, by relaxing the (traditional) assumption underlying quantum computation about applying gates in a well-defined causal order. In this context, we address a fundamental question arising with quantum computing: whether superposed orders of single-qubit gates can enable universal quantum computation. As shown in this paper, the answer to this key question is a definitive "*yes*". Indeed, we prove that any two-qubit controlled quantum gate can be *deterministically* realized, including the so-called Barenco gate that alone enables universal quantum computation.**

***Index Terms*—Universal Quantum Computing, Quantum Computation, Quantum Gates, Superposed Orders, Quantum Switch**

I. Introduction

During the last ten years, there has been an increasing widespread interest on investigating the advantages arising from the superposition of traversing orders of quantum channels in the quantum communication domain, due to the outstanding possibilities arising from the non-classical propagation of quantum carriers [1]–[5]. More into details, while classical carriers propagate through classical communication paths, quantum channels can be placed in a genuinely quantum setting with no counterpart in classical world [6], [7], giving rise to the concept of *quantum path* [8]–[10]. This non-classical propagation has been experimentally validated [11], [12], and it finds a possible physical realization in the device known as *quantum switch* [13], [14], which enables a coherent superposition of alternative channel configurations. Specifically, by the means of the quantum switch, the quantum carrier can travel through different communication channels in a quantum superposition of different causal orders. This, in turn, makes the order of the communication channels indefinite and it returns disruptive advantages, as instance, for communications through noisy channels [15]–[17]. Furthermore, it has been recently demonstrated that the quantum switch can be exploited for generating different classes of genuine multipartite entangled states starting from separable inputs [18], [19]. Therefore, the engineering of the unconventional quantum propagation phenomena is key in the quantum communication domain.

It must be noted, though, that the possibilities enabled by the superposition of orders of quantum operations can be exploited within the quantum computing domain as well [20], [21]. Specifically, it is possible to relax the (traditional) assumption underlying quantum computation, i.e., quantum gates applied in a well-defined causal order only [22]. By doing this, a novel and more-general computing framework arises, which has been applied to a number of problems, both theoretically [20], [21], [23]–[27] and experimentally [11], [28].

In this context, we address a fundamental question arising with quantum computing, namely, whether superposed orders of quantum operations can constitute a novel paradigm for *universal quantum computing*.

As proved in the following, the answer to this key question is a definitive "yes". Specifically, through the manuscript we show that any controlled quantum gate can be *deterministically* realized via superposition of the traversing orders of single-qubit gates. Notably, this includes the two-qubit gate known as *Barenco gate*, which *alone* enables universal quantum computation [29].

A. Contribution

Our contributions can be summarized as follows:

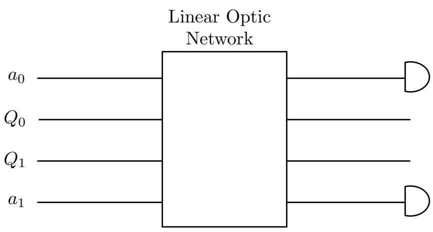
- we provide a general framework for the realization of controlled quantum gates through superposed orders of single-qubit unitaries;
- we prove that this framework enables the *deterministic* realization of any two-qubit controlled gate;
- we specialize the framework for two-qubit controlled gates widely used in quantum computing, including `CNOT`, `CZ`, and the *universal* Barenco gate.

We note that the relevance of our work can be linked to the photonic computing domain. As a matter of fact, photonic quantum gates are either probabilistic or based on pre-shared multipartite-entangled states, as discussed in Section II-A. Conversely, our framework enables deterministic implementation of universal quantum computation, by exploiting only single-qubit gates without the need of pre-sharing large entangled states. And we do hope that this manuscript can fuel further research – both theoretical and experimental – about the powerful setup enabled by superposed orders of quantum gates for photonic quantum computing.

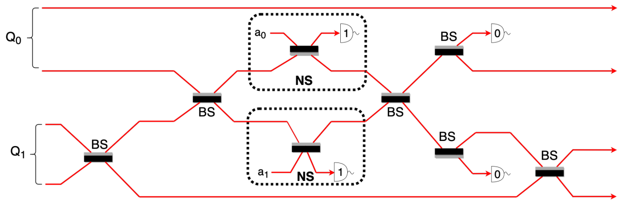
K. Simonov is independent researcher. M. Caleffi, J. Illiano, and A. S. Cacciapuoti are with the www.QuantumInternet.it research group, *FLY: Future Communications Laboratory*, at the Department of Electrical Engineering and Information Technology (DIETI), University of Naples Federico II, Naples, 80125 Italy. E-mail: kyrylo.simonov@univie.ac.at, marcello.caleffi@unina.it, jessica.illiano@unina.it, angelasara.cacciapuoti@unina.it.

M. Caleffi acknowledges PNRR MUR project RESTART-PE00000001, A. S. Cacciapuoti acknowledges PNRR MUR project NQSTI-PE00000023.

*Corresponding author.



(a) Simplified diagram of a CNOT logic gate linear optic implementation where Q_0 and Q_1 represent the control qubit and the target qubit and a_0, a_1 are two ancillary photons.



(b) KLM CNOT scheme with simplified NS gates. Q_0 and Q_1 denote the control qubit and the target qubit, respectively. This scheme assumes the logical qubits to be encoded trough spatial modes (path encoding).

Fig. 1: Non-deterministic CNOT gate via linear optics.

II. Background

In this section we first summarize the main challenges arising with the implementation of controlled operations in optical setups. Then, we provide the reader with a concise guide through the supermap formalism required for modeling superposed orders of quantum gates.

A. Optical Controlled Operations

It is widely recognized by both the industrial and academic communities that light represents the prominent candidate for quantum information carriers. More into details, the advantages arising from photons as quantum information carrier are manifold.

Photons hardly interact with the environment, and they do not require a complex cooling infrastructure or high vacuum chambers. Additionally, they support long-range transmission with low losses through optical fibers channels or waveguides. Also, single photons exhibit multiple degrees of freedom (DoFs) that represent a resource for quantum information encoding. Indeed, photons are commonly known as the physical realization of polarization-encoded qubits. However, photons can also exist in a superposition of time bins or frequency bins. Furthermore, by exploiting multi-mode waveguides, photons also allow path-encoded qubits and high dimensional quantum states such as path-encoded qudits [34].

Unfortunately, regardless the appealing features for quantum communications, photonic quantum technologies still represent a challenge from a quantum computing perspective. Indeed, such technologies suffer from the probabilistic nature of single-photon sources and photon-photon non linear interactions, which realize quantum logic gates.

However, **gate-based** photonic quantum computing can be realized also by exploiting linear optical elements [35]. Specifically, quantum logic operations can be obtained probabilistically through linear optical elements, ancillary photons and post-selection based on the output of single-photon detectors. In this context, particularly challenging is the realization of the logic gate CNOT, whose scheme for optical implementation is represented in Figure 1 and can be summarized as follows. The two input photons (control qubit Q_0 and target qubit Q_1, respectively) and two additional ancilla photons (a_0,a_1) are combined through a linear optic network of Beam Splitters (BS) as pictorially represented in Figure 1a. If both the ancilla qubits are detected at the output of the optical network – specifically, both the detectors signal a single photon detection – then the target qubit has been successfully subjected to the CNOT logic operation. To better understand the nondeterministic nature of the optical implementation of the gate, we consider the well known Knill-Laflamme-Milburn (KLM) CNOT scheme [35], represented in Figure 1b. The basic unit of the scheme is a nonlinear sign-shift gate (NS) which given the input state $|\mu\rangle = a\,|0\rangle + b\,|1\rangle + c\,|2\rangle$ returns the output state $|\mu'\rangle = a\,|0\rangle + b\,|1\rangle - c\,|2\rangle$, where $|0\rangle$, $|1\rangle$ and $|2\rangle$ denote the vacuum state, single photon state and two-photon state, respectively. Consider the qubits encoded through spatial modes (path), the NS gate is obtained through three BS and two number resolving detectors. The NS gate successfully performs an heralded π phase rotation, when exactly one photon is detected on one detector and no photons are detected at the other. This event occurs with probability $1/4$. The KLM CNOT gate is constructed from two NS gates, hence the CNOT success probability is $(1/4)^2 = 1/16$. We represent in Figure 1b the KLM CNOT gate with two simplified NS gates, namely, the NS gates are implemented through one BS and one detector. The result is still an heralded sign shift, where the detection of one photon represents the herald event, however the success probability is slightly decreased (from 0.25 to 0.23) [36].

For overcoming the nondeterministic nature of *gate-based* schemes, the so-called **cluster-state-based** photonic quantum computing has been developed [37]. The key idea is that, in the absence of deterministic two-photon operations, an initial cluster state can be built up offline using non-deterministic interactions. Successively, the computation progresses by manipulating the cluster state via deterministic single-qubit operations through optical elements [37].

Our framework merges both the appealing features of *gate-based* and *cluster-state-based* photonic computing. Specifically, our framework overcomes the limitations of the former schemes, by enabling deterministic computing. And it overcomes also the limitations exhibited by the latter, since it does not require any pre-shared multipartite-entangled state, whose generation requires offline non-deterministic interactions.

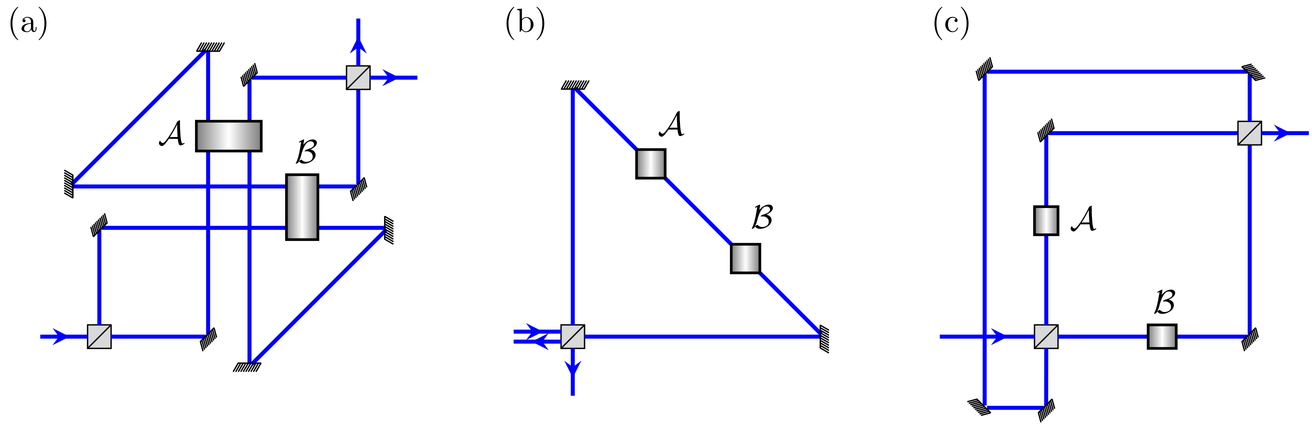


Fig. 2: Schematic diagram of some of experimentally implemented architectures of the photonic quantum switch. (a) An implementation via a Mach-Zehnder geometry, where the target qubit is encoded in polarization of the photon, while the control qubit is mapped into its path degree of freedom using the first beam splitter and coherently recombining the paths $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}$ at the second beam splitter [11], [12], [14], [30], [31]. (b) An implementation via a Sagnac geometry, where the target qubit is encoded in polarization of the photon (as in (a)), whereas a single beam splitter introduces the path degree of freedom as control and completes superposition of causal orders of $\mathcal{A}$ and $\mathcal{B}$ [32]. (c) An implementation via a geometry, where the target qubit is encoded in the path degree of freedom of the photon, while the role of control qubit is played by its polarization [13], [33].

B. Superposed Orders of Quantum Operations via Quantum Switch

While quantum circuit model is one of the most widely used paradigms of quantum computations, a lot of effort has been put to extend it to computations of higher order. Such objects as quantum combs, which can be seen as quantum circuits with open slots for arbitrary quantum gates, have allowed one to solve problems not achievable with the quantum circuit model [38]–[40]. Nevertheless, quantum combs, which put the gates into a certain order on a circuit board, are not the most general computational model of higher order that can be achieved within quantum mechanics. Indeed, it allows for higher-order operations putting quantum gates into configurations – such as the superpositions of causal orders discussed in the following – that cannot be reduced to their well-ordered compositions [7], [20], [21], [40], [41].

Superposition of causal orders of quantum operations can be realized via the quantum switch, i.e., a quantum device already implemented in numerous table-top optical and NMR experiments [11]–[14], [30]–[33], [42]–[45] as schematized in Fig. 2. In what follows, we provide the reader with the mathematical description of the quantum switch.

Given input and output systems $\mathbf{I}$ and $\mathbf{O}$, any quantum operation transforming the former to the latter can be represented by a completely positive trace-preserving (CPTP) map $\mathcal{A} : \mathcal{L}(\mathcal{H}_\mathbf{I}) \to \mathcal{L}(\mathcal{H}_\mathbf{O})$, where $\mathcal{H}_{\mathbf{I}/\mathbf{O}}$ denotes Hilbert space of system $\mathbf{I}$ or $\mathbf{O}$, respectively, while $\mathcal{L}(\mathcal{H}_{\mathbf{I}/\mathbf{O}})$ is the set of density operators over $\mathcal{H}_{\mathbf{I}/\mathbf{O}}$.

The quantum switch is an example of a supermap that sends any N quantum operations $\mathcal{A}_1[\cdot], ..., \mathcal{A}_N[\cdot]$ to a new quantum operation $\mathcal{S}(\mathcal{A}_1, ..., \mathcal{A}_N)[\cdot]$. More into details, given an ancillary quantum system $\mathbf{C}$, the quantum switch is constructed as a supermap that uses a state ω of $\mathbf{C}$ to coherently control the order in which $\mathcal{A}_1[\cdot], ..., \mathcal{A}_N[\cdot]$ act on the input system in the state ρ:

$$\mathcal{S}(\mathcal{A}_1, ..., \mathcal{A}_N)[\rho \otimes \omega] = \sum_{i_1...i_N} K_{i_1...i_N}(\rho \otimes \omega)K^\dagger_{i_1...i_N}, \quad (1)$$

with

$$K_{i_1...i_N} = \sum_k \mathcal{P}_k(A^{(1)}_{i_1} ... A^{(N)}_{i_N}) \otimes |k\rangle\langle k| \quad (2)$$

denoting the Kraus operators of the output quantum operation of the quantum switch. In (2), $\{A^{(j)}_{i_j}\}_i$ denotes the set of Kraus operators of $\mathcal{A}_j[\cdot]$, $\mathcal{P}_k$ denotes a k-th permutation, and $|k\rangle$ is the k-th basis state of system $\mathbf{C}$.

By restricting the number of controlled operations to two operations $\mathcal{A}[\cdot]$ and $\mathcal{B}[\cdot]$, the supermap in (1) exhibits Kraus operators given by $K_{ij} = A_i B_j \otimes |0\rangle\langle 0| + B_j A_i \otimes |1\rangle\langle 1|$, with $\{A_i\}_i$ and $\{B_j\}_j$ being Kraus operators of $\mathcal{A}[\cdot]$ and $\mathcal{B}[\cdot]$. This new operation implemented by the quantum switch can be represented explicitly in a simple form as [5], [46], [47]:

$$\begin{aligned} \mathcal{S}(\mathcal{A}, \mathcal{B})[\rho \otimes \omega] =& \frac{1}{4} \sum_{ij} \Big(\{A_i, B_j\} \rho \{A_i, B_j\}^\dagger \otimes \omega \\ &+ \{A_i, B_j\} \rho [A_i, B_j]^\dagger \otimes \omega Z \\ &+ [A_i, B_j] \rho \{A_i, B_j\}^\dagger \otimes Z\omega \\ &+ [A_i, B_j] \rho [A_i, B_j]^\dagger \otimes Z\omega Z \Big), \quad (3) \end{aligned}$$

where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote a commutator and an anti-commutator [48], respectively, and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ is the Pauli Z-operator.

III. Controlled Gates via Superposed Orders

In this section, we first exploit the quantum switch to realize superposed orders of arbitrary unitary gates. Then, we exploit these preliminary results to realize controlled gates by resorting to single-qubit gates only.

A. Superposed orders of arbitrary gates

Given two arbitrary unitary gates A and B, the execution of each of them on the input system $\mathbf{I}$ in state ρ can be seen as the action of quantum operations $\mathcal{A}[\rho] = A\rho A^\dagger$ and $\mathcal{B}[\rho] = B\rho B^\dagger$. Therefore, we can realize a superposition of casual orders between the two gates A and B by simplifying (3) as:

$$\begin{aligned}\mathcal{S}(\mathcal{A},\mathcal{B})\,[\rho\otimes\omega] = &\frac{1}{4}\Big(\{A,B\}\rho\{A,B\}^\dagger\otimes\omega \\ &+\{A,B\}\rho[A,B]^\dagger\otimes\omega Z \\ &+[A,B]\rho\{A,B\}^\dagger\otimes Z\omega \\ &+[A,B]\rho[A,B]^\dagger\otimes Z\omega Z\Big). \end{aligned} \tag{4}$$

(4) can be equivalently interpreted as the execution of a new unitary gate $S(A,B)$ – acting on the overall system composed by the input system $\mathbf{I}$ and the ancilla $\mathbf{C}$ – given by:

$$S(A,B) = \frac{1}{2}\Big[\{A,B\}\otimes I_{\mathbf{C}} + [A,B]\otimes Z_{\mathbf{C}}\Big], \tag{5}$$

where $I_{\mathbf{C}}$ and $Z_{\mathbf{C}}$ denote identity and Z-Pauli operators, respectively, that act on $\mathbf{C}$. For taking full advantage of the indefinite causal order among the unitaries, we set system $\mathbf{C}$ in the pure state $\omega = |+\rangle\langle+|$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, so that the input system evolves into a even superposition of the two causal orders among the unitaries.

By assuming for the sake of simplicity that the input system $\mathbf{I}$ is in a pure[1] initial state $\rho = |\psi\rangle\langle\psi|$, the action of the new unitary gate $S(A,B)$ is given by:

$$\begin{aligned}S(A,B)(|\psi\rangle\otimes|+\rangle) = \frac{1}{2}\Big[&(\{A,B\}|\psi\rangle)\otimes|+\rangle + \\ &([A,B]|\psi\rangle)\otimes|-\rangle\Big].\end{aligned} \tag{6}$$

It is straightforward to see that tracing out the ancillary qubit $\mathbf{C}$ results in a probabilistic gate that chooses between well-ordered sequences AB or BA randomly with probability $1/2$. On the other hand, a measurement of the state of $\mathbf{C}$ in the basis spanned by:

$$|\mu(\theta)\rangle = \cos\Big(\frac{\theta}{2}\Big)|0\rangle + i\sin\Big(\frac{\theta}{2}\Big)|1\rangle, \tag{7}$$

$$|\mu^\perp(\theta)\rangle = i\sin\Big(\frac{\theta}{2}\Big)|0\rangle + \cos\Big(\frac{\theta}{2}\Big)|1\rangle, \tag{8}$$

leaves the system in the following states with probability $1/2$

$$|\psi_+(\theta)\rangle = \Big[\cos\Big(\frac{\theta}{2}\Big)AB + i\sin\Big(\frac{\theta}{2}\Big)BA\Big]|\psi\rangle, \tag{9}$$

$$|\psi_-(\theta)\rangle = \Big[i\sin\Big(\frac{\theta}{2}\Big)AB + \cos\Big(\frac{\theta}{2}\Big)BA\Big]|\psi\rangle. \tag{10}$$

Accordingly, once the ancillary qubit is measured, with probability $1/2$, one of two different gates $S_+^{A,B}(\theta)$ and $S_-^{A,B}(\theta)$ is realized:

$$S_+^{A,B}(\theta) = \cos\Big(\frac{\theta}{2}\Big)AB + i\sin\Big(\frac{\theta}{2}\Big)BA, \tag{11}$$

$$S_-^{A,B}(\theta) = i\sin\Big(\frac{\theta}{2}\Big)AB + \cos\Big(\frac{\theta}{2}\Big)BA. \tag{12}$$

[1]The results obtained in what follows can be straightforwardly extended to arbitrary (mixed) state ρ by considering the action of $S(A,B)$ on it as $S(A,B)\rho S^\dagger(A,B)$.

B. Superposed Orders of single-qubit gates

A pre-requisite for the realization of any universal set of quantum gates is the ability of implementing a multi-qubit gate that cannot be reduced to a single tensor product of single-qubit gates only. With the following lemma and corollary, we prove that the overall multi-qubit gate – implemented by combining single-qubit gates via the quantum switch – satisfies the aforementioned property for non-trivial choices of the single-qubit gates and the ancillary measurement bases.

Lemma 1. *Combining N single-qubit gates $A = \bigotimes_{i=1}^N A_i$ and N single-qubit gates $B = \bigotimes_{i=1}^N B_i$ via quantum switch implements one of the following two new N-qubit unitaries:*

$$S_+^{A,B}(\theta) = \cos\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^N A_iB_i + i\sin\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^N B_iA_i, \tag{13}$$

$$S_-^{A,B}(\theta) = i\sin\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^N A_iB_i + \cos\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^N B_iA_i, \tag{14}$$

with the actual implemented gate depending on whether the ancillary qubit is measured as (7) *or* (8).

Proof: See Appendix A. ■

We are now ready to provide the main result of this section with the following corollary.

Corollary 1. *$S_+^{A,B}(\theta)$ and $S_-^{A,B}(\theta)$ in* (13) *and* (14) *cannot be reduced to a single tensor product of single-qubit gates only, unless either: i) $\theta = \pi k$ with $k \in \mathbb{Z}$, or ii) $[A_i, B_i] = 0$ for any $N-1$ gates A_i and B_i.*

Proof: The proof follows directly from Lemma 1. ■

C. Realization of controlled gates

The realization of controlled gates is key in the quantum domain. Indeed, not only entanglement is usually generated within the quantum circuit model via controlled gates (typically, using a controlled-not `CNOT`) but – even more relevant from our perspective – there exists a class of two-qubit controlled gates any one of which is universal for quantum computation [29].

For this, in the following we restrict our attention on two-qubit controlled gates. Accordingly, we denote the controlled gate for an arbitrary single-qubit unitary gate `U` as `CU` (controlled-`U`), which is formally defined as:

$$\mathtt{CU} = |0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes \mathtt{U}. \tag{15}$$

Gate `CU` acts on two input qubits, with the qubit acting as control denoted as Q_0 and the qubit acting as target denoted with Q_1, as depicted in Fig. 3.

In the following, we aim at proving that any arbitrary `CU` can be realized with the gates in (13)-(14), namely, by combining single-qubit gates via the quantum switch. Accordingly, given

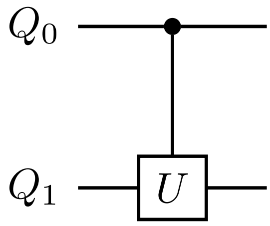


Fig. 3: The CU (controlled-U) logic gate.

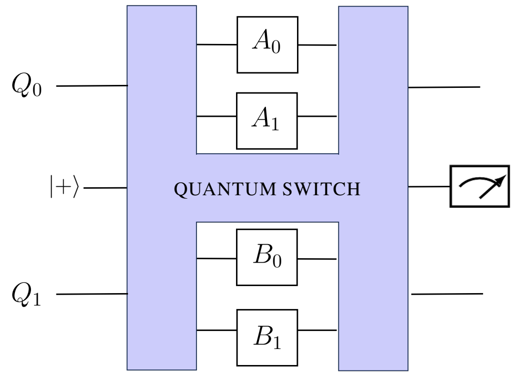


Fig. 4: Representation of two-qubit controlled logic gate via quantum switch, with the switch represented as a H-shape blue box as in [4], [52]–[54]. The quantum switch combines two-qubit gates $A = A_0 \otimes A_1$ and $B = B_0 \otimes B_1$ in superposed orders, with A_0, B_0 denoting the single-qubit unitaries acting on the first qubit Q_0 and A_1, B_1 denoting the single-qubit unitaries acting on the second qubit Q_1.

the two input qubits in an initial state ρ, the quantum switch combines two-qubit gates $A = A_0 \otimes A_1$ and $B = B_0 \otimes B_1$ in superposed orders. As represented in Fig. 4, it is worthwhile to note that A_0, B_0 denote the single-qubit unitaries acting on the first qubit Q_0 of the input state ρ, whereas A_1, B_1 denote the single-qubit unitaries acting on the second qubit Q_1 of the input state ρ.

The following preliminary definitions are needed.

Definition 1. *A single-qubit gate $R_{\mathbf{n}}(\theta)$ that performs a rotation on angle θ around the $\mathbf{n}$-axis defined by the Bloch vector $\mathbf{n} = (n_X, n_Y, n_Z)$ is given by:*

$$R_{\mathbf{n}}(\theta) = \cos\Big(\frac{\theta}{2}\Big) I - i \sin\Big(\frac{\theta}{2}\Big)(\mathbf{n} \cdot \boldsymbol{\sigma}), \tag{16}$$

with I and $\boldsymbol{\sigma} = (X, Y, Z)$ denoting identity matrix and a vector of Pauli matrices, respectively.

Accordingly, gates $R_X(\theta)$, $R_Y(\theta)$, and $R_Z(\theta)$ denote the rotation gate given in (16) with respect to Bloch vectors $\mathbf{n} = (1, 0, 0)$, $\mathbf{n} = (0, 1, 0)$, and $\mathbf{n} = (0, 0, 1)$, respectively.

Definition 2. *A two-qubit rotation gate $R_{\tilde{\mathbf{n}}\mathbf{n}}(\theta)$ with respect to angle θ and Bloch vectors $\tilde{\mathbf{n}}$ and $\mathbf{n}$ is given by:*

$$R_{\tilde{\mathbf{n}}\mathbf{n}}(\theta) = \cos\Big(\frac{\theta}{2}\Big) I \otimes I - i \sin\Big(\frac{\theta}{2}\Big)(\tilde{\mathbf{n}} \cdot \boldsymbol{\sigma}) \otimes (\mathbf{n} \cdot \boldsymbol{\sigma}). \tag{17}$$

Definition 3. *Two-qubit gates A and B are locally equivalent*[2] *if they can be mapped to one another by a tensor product of single-qubit unitary gates $\{V_i\}_{i=1,2}$ and $\{\tilde{V}_i\}_{i=1,2}$:*

$$A = (V_1 \otimes V_2) B (\tilde{V}_1 \otimes \tilde{V}_2). \tag{18}$$

Lemma 2. *The controlled two-qubit gate* CU *given by:*

$$\mathtt{CU} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \mathtt{U}, \tag{19}$$

is locally equivalent to sequences of single-qubit gates put into superposed orders via quantum switch.

Proof: See Appendix B. ■

Lemma 2 shows that any two-qubit controlled gate can be implemented using *single-qubit gates only* by combining them in superposed orders. By exploiting this result, the following Theorem provides a recipe for the implementation of any arbitrary CU gate.

[2]We note that the notion of equivalence given in Def. 3, also referred to in literature as *LU equivalence* [49]–[51], restricts the allowed unitary operators to tensor product of single-qubit unitary gates only. The rationale for this constraint lies in the aim of enabling universal quantum computing via superposed orders of single qubits gates.

Theorem 1. *The arbitrary two-qubit controlled gate* $\mathtt{CU}(\alpha, \theta, \mathbf{n})$

$$\mathtt{CU}(\alpha, \theta, \mathbf{n}) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \mathtt{U}(\alpha, \theta, \mathbf{n}), \tag{20}$$

with $\mathtt{U}(\alpha, \theta, \mathbf{n})$ defined as:

$$\mathtt{U}(\alpha, \theta, \mathbf{n}) = \exp\Big[i\Big(\alpha I + \theta(\mathbf{n} \cdot \boldsymbol{\sigma})\Big)\Big], \tag{21}$$

can be ***deterministically*** *realized through:*

- *superposed orders of single-qubit gates $A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n})$ and $B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n})$ combined via a quantum switch,*
- *preceded by a pre-processing phase $P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n})$, implemented by single-qubit gates $\{P_0, P_1(\mathbf{n})\}$,*
- *followed by a post-processing phase $F_{\pm}(\alpha, \theta, \mathbf{n})$, implemented by single-qubit gates set accordingly to the ancillary-qubit measurement results in the basis spanned by* (7),

as follows:

$$\mathtt{CU}(\alpha, \theta, \mathbf{n}) = F_{\pm}(\alpha, \theta, \mathbf{n}) S_{\pm}^{A(\mathbf{n}), B(\mathbf{n})}(\theta) P(\mathbf{n}). \tag{22}$$

where:

$$F_{\pm}(\alpha, \theta, \mathbf{n}) = e^{i\frac{\alpha}{2}} \Big(R_Z\Big(\alpha \pm \frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(-\theta \pm \frac{\pi}{2}\Big)\Big) \tag{23}$$

$$A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n}) = X \otimes (\mathbf{n}^{\perp} \cdot \boldsymbol{\sigma}), \tag{24}$$

$$B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n}) = R_Z\Big(\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big) \tag{25}$$

$$P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n}) = X \otimes (\mathbf{n}^{\perp} \cdot \boldsymbol{\sigma}) \tag{26}$$

with $\mathbf{n}^{\perp}$ denoting the Bloch vector perpendicular to $\mathbf{n}$.

Proof: The proof follows by exploiting the result of Lemma 2, by plugging (23)-(26) *into the decomposition* (22)*, and by comparing it with the decomposition*

$$\mathtt{CU}(\alpha, \theta, \mathbf{n}) = e^{i\frac{\alpha}{2}} \Big(R_Z(\alpha) \otimes R_{\mathbf{n}}(-\theta)\Big) R_{Z\mathbf{n}}(\theta) \tag{27}$$

of the CU *gate provided in Appendix B.* ■

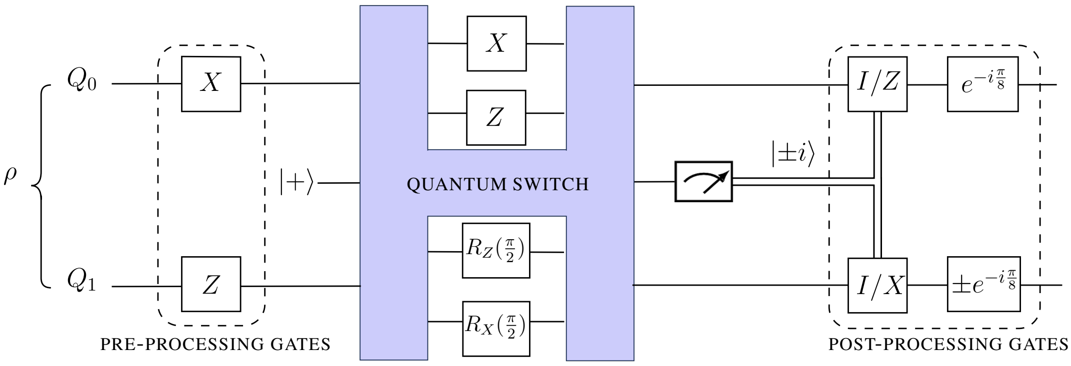


Fig. 5: Abstract representation of the CNOT gate implemented via superposed orders of single-qubit gates, with Q_0 denoting the control qubit and Q_1 denoting the target qubit, respectively. Specifically, the order between the gates is controlled by an ancillary qubit in state $|+\rangle$ that – after a pre-processing phase, implemented by single-qubit gates $P = X \otimes Z$ – implements an even superposition of causal orders between single-qubit gates $A = X \otimes Z$ and $B = R_Z(\frac{\pi}{2}) \otimes R_X(\frac{\pi}{2})$. Once the ancillary qubit is measured in the basis spanned by $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle)$, qubits Q_0 and Q_1 are post-processed by $F_- = -e^{-i\frac{\pi}{4}}\,(Z \otimes X)$ or $F_+ = e^{-i\frac{\pi}{4}}\,(I \otimes I)$, depending on whether the ancilla is measured as $|-i\rangle$ or $|+i\rangle$.

Remark 1. *It is worthwhile to note that the deterministic realization of an arbitrary controlled gate* CU *via superposed orders of single-qubit gates imposes different constraints on the gates acting on the control qubit with respect to the gates acting on the target qubit. Specifically, the single-qubit gates acting on the target – i.e., A_1 and B_1 – depend on the actual definition of gate* U*. Conversely, the single-qubit gates acting on the control qubit – i.e., A_0 and B_0 – do not depend on the gate* U*.*

Theorem 1 is the main instrument, exploited in what follows for the realization of several important examples of two-qubit controlled gates belonging to – or constituting alone as for the Barenco gate – universal sets.

IV. Universal Quantum Computation

In this section, we provide several examples of popular quantum gates that are used to construct universal sets, and we detail how they can be synthesized from single-qubit gates in superposed orders.

A. Controlled NOT

We start by considering the gate CNOT (controlled-NOT), a paramount example of a gate widely used to construct more complex gates and lying at the core of fundamental quantum protocols such as entanglement generation and quantum teleportation.

Formally, CNOT is a controlled-X gate which acts on two qubits as:

$$\texttt{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X. \tag{28}$$

Crucially, CNOT appears in several universal sets as a unique multi-qubit gate together with certain single-qubit gates. Indeed, any other unitary gate can be expressed as a sequence of CNOT gates and some single-qubit gates [48]. Hence, an efficient implementation of the CNOT gate is of paramount importance when it comes to universal quantum computation.

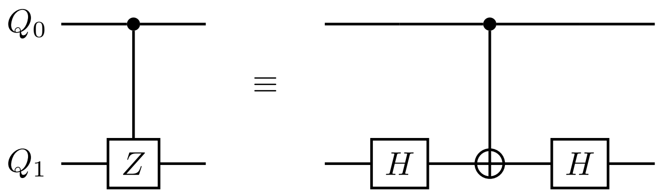


Fig. 6: Example of equivalent quantum circuit for a CZ logic gate exploiting the CNOT logic gate implementation.

To this aim, the following proposition demonstrates that the CNOT gate can be realized using only simple, widely-used single-qubit gates by properly placing some of them into a superposition of orders.

Proposition 1. *The* CNOT *gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:*

$$A = X \otimes Z, \tag{29}$$

$$B = R_Z\left(\frac{\pi}{2}\right) \otimes R_X\left(\frac{\pi}{2}\right) \tag{30}$$

Proof: The proof follows from Theorem 1, by recognizing that the CNOT *gate is given by the decomposition in* (22) *for* $\alpha = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ *and* $\mathbf{n} = (1,0,0)$ *and choosing* $\mathbf{n}^{\perp} = (0,0,1)$. ■

Proposition 1 provides us with the realization of the CNOT gate via superposed orders of single-qubit gates. Specifically, as shown in Figure 5, we first pre-process the input state – namely, control and target qubits Q_0 and Q_1 – with a sequence of single-qubit gates $P = X \otimes Z$ in accordance with (26). Then, the resulting state goes into the quantum switch, which processes it according to simple, widely used single-qubit gates as in (29) and (30). A measurement of the ancillary qubit $\mathbf{C}$ in the basis spanned by states (7) and (8) – which we denote in this case as $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$

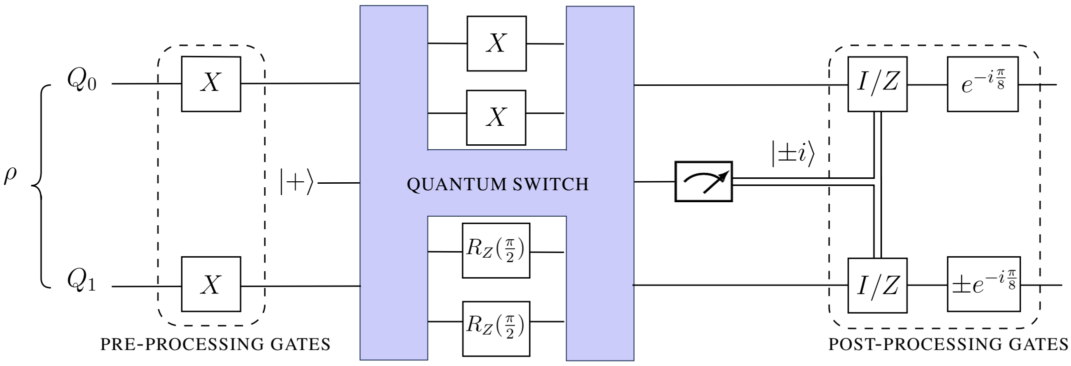


Fig. 7: Abstract representation of the CZ gate realized via superposed orders of single-qubit gates. As in Figure 5, the ancillary qubit is set in $|+\rangle$ to implement an even superposition of causal orders, and the pre-processing phase is implemented by single-qubit gates $P = X \otimes X$. The single-qubit gates in superposed orders via quantum switch are $A = X \otimes X$ and $B = R_Z(\frac{\pi}{2}) \otimes R_Z(\frac{\pi}{2})$, respectively. Once the ancillary qubit is measured in the basis spanned by $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle)$, qubits Q_0 and Q_1 are post-processed by $F_- = -e^{-i\frac{\pi}{4}}\,(Z \otimes Z)$ or $F_+ = e^{-i\frac{\pi}{4}}\,(I \otimes I)$, depending on whether the ancilla is measured as $|-i\rangle$ or $|+i\rangle$.

and $|-i\rangle = \frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle)$ – realizes the gates $S_{\pm}^{A,B}(\frac{\pi}{2})$ with probability $1/2$ in accordance with (13) and (14). Finally, depending on the outcome of the measurement, we perform a post-processing by applying another sequence of single-qubit gates, namely, i) $F_-(-\pi/2, \pi/2, \mathbf{n}) = -e^{-i\frac{\pi}{4}}\,(Z \otimes X)$ whenever the ancillary qubit is found in the state $|-i\rangle$ or ii) the identity $F_+(-\pi/2, \pi/2, \mathbf{n}) = e^{-i\frac{\pi}{4}}\,(I \otimes I)$ otherwise. Accordingly, the overall deterministic implementation of the CNOT via superposed orders can be expressed as:

$$\texttt{CNOT} = \begin{cases} e^{-i\frac{\pi}{4}} S_+^{A,B}(\frac{\pi}{2})(X \otimes Z) & \text{if ancilla in } |+i\rangle\,, \\ -e^{-i\frac{\pi}{4}}(Z \otimes X) S_-^{A,B}(\frac{\pi}{2})(X \otimes Z) & \text{otherwise.} \end{cases} \tag{31}$$

with, again, A, B given in (29) and (30).

B. Controlled Z

While CNOT gate can be directly associated with a classical reversible XOR gate, a logic gate without classical counterpart known as CZ (controlled-Z) gate is also frequently used due to its diagonal form and can constitute a universal set together with the corresponding single-qubit gates [48]. Formally, CZ acts on two qubits as

$$\texttt{CZ} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \texttt{Z}. \tag{32}$$

and is locally equivalent to the CNOT gate via Hadamard gates, as shown in Fig. 6:

$$\texttt{CZ} = (I \otimes H)\texttt{CNOT}(I \otimes H). \tag{33}$$

The following proposition demonstrates that the CZ gate, similarly to the CNOT gate, can be synthesized directly using only single-qubit gates by putting some of them in superposed orders.

Proposition 2. *The CZ gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:*

$$A = X \otimes X, \tag{34}$$

$$B = R_Z\Big(\frac{\pi}{2}\Big) \otimes R_Z\Big(\frac{\pi}{2}\Big). \tag{35}$$

Proof: The proof follows from Theorem 1, by recognizing that the CZ gate is given by the decomposition in (22) for $\alpha = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ and $\mathbf{n} = (0, 0, 1)$, and choosing $\mathbf{n}^{\perp} = (1, 0, 0)$. ■

As represented in Fig. 7, Proposition 2 proves that the CZ gate is obtained via superposed orders of single-qubit gates as follows. We first pre-process the input state with a sequence of single-qubit gates $P = X \otimes X$ in accordance with (26), then the resulted state goes into the quantum switch, which processes it according to the gates in (34) and (35). A measurement of the ancillary qubit $\mathbf{C}$ in basis spanned by the states (7) and (8), which we denote in this case as $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle)$, realizes the gates $S_{\pm}^{A,B}(\frac{\pi}{2})$ with probability $1/2$ in accordance with (13) and (14). Finally, depending on the outcome of the measurement, we perform a post-processing by applying another sequence of single-qubit gates, namely, $F_-(-\pi/2, \pi/2, \mathbf{n}) = -e^{-i\frac{\pi}{4}}\,(Z \otimes Z)$, if the ancillary qubit is found in the state $|-i\rangle$ or $F_+(-\pi/2, \pi/2, \mathbf{n}) = e^{-i\frac{\pi}{4}}\,(I \otimes I)$ otherwise:

$$\texttt{CZ} = \begin{cases} e^{-i\frac{\pi}{4}} S_+^{A,B}(\frac{\pi}{2})(X \otimes X) & \text{if ancilla in } |+i\rangle\,, \\ -e^{-i\frac{\pi}{4}}(Z \otimes Z) S_-^{A,B}(\frac{\pi}{2})(X \otimes X) & \text{otherwise.} \end{cases} \tag{36}$$

C. Barenco gate

Although CNOT and CZ gates are widely used in quantum computing, they still require additional single-qubit gates to realize an arbitrary unitary gate, i.e., to realize universal quantum computation. Differently, there exists a gate named *Barenco gate* – denoted in the following as $\texttt{BAR}(\alpha_{\texttt{B}}, \phi_{\texttt{B}}, \theta_{\texttt{B}})$ – which alone is sufficient for universal quantum computation

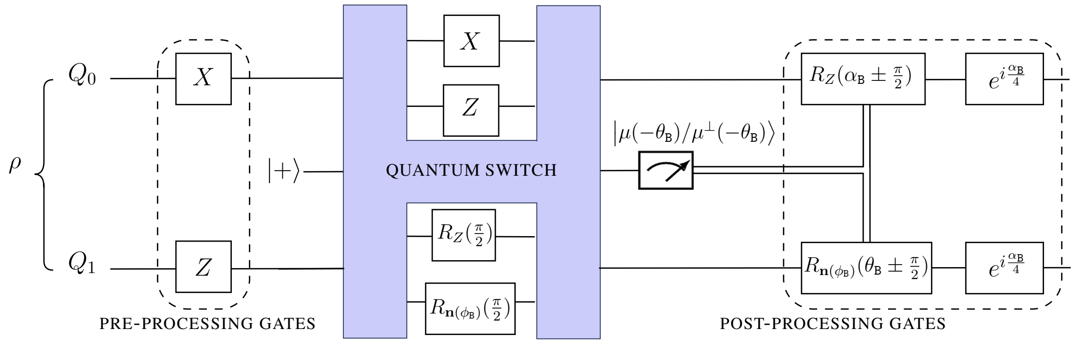


Fig. 8: Abstract representation of the $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ gate realized via superposed orders of single-qubit gates. As in Figure 5, the ancillary qubit is set in $\omega = |+\rangle$ to implement an even superposition of causal orders, and the pre-processing phase is implemented by single-qubit gates $P = X \otimes Z$. The single-qubit gates in superposed orders via quantum switch are $A = X \otimes Z$ and $B = R_Z(\frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\frac{\pi}{2})$, with $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$. Once the ancillary qubit is measured in the basis spanned by the states (7) and (8) with $\theta = -\theta_B$, qubits Q_0 and Q_1 are post-processed by either $F_+ = e^{i\frac{\alpha_B}{2}}\left(R_Z(\alpha_B + \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B + \frac{\pi}{2})\right)$ or $F_- = e^{i\frac{\alpha_B}{2}}\left(R_Z(\alpha_B - \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B - \frac{\pi}{2})\right)$, depending on the ancilla qubit measurement result.

[29]. In other words, Barenco gate forms by itself a universal set of quantum gates. Formally, Barenco gate is a controlled rotation gate, which acts on two qubits as

$$\mathtt{BAR}(\alpha_B, \phi_B, \theta_B) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes e^{i\alpha_B} R_{\mathbf{n}(\phi_B)}(2\theta_B), \quad (37)$$

and is parameterized by the angles $\alpha_B, \phi_B, \theta_B \in [0, 2\pi]$, with $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$. Though its universality, practical implementation of the Barenco gate is known to be highly challenging [55], [56]. The following proposition demonstrates that the $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ gate can be realized for any choice of $\alpha_B, \phi_B, \theta_B$ by using only single-qubit gates in a superposition of orders.

Proposition 3. *The* $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ *gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:*

$$A = X \otimes Z, \quad (38)$$

$$B = R_Z\Big(\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}(\phi_B)}\Big(\frac{\pi}{2}\Big), \quad (39)$$

where $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$.

Proof: The proof follows from Theorem 1, by recognizing that the $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ *gate is given by the decomposition in* (22) *for* $\alpha = \alpha_B, \theta = -\theta_B$, *and* $\mathbf{n} = (\cos(\phi_B), \sin(\phi_B), 0) \equiv \mathbf{n}(\phi_B)$ *and choosing* $\mathbf{n}^\perp = (0, 0, 1)$.

■

As shown in Fig. 8, Proposition 3 proves that the $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ gate is obtained via superposed orders of single-qubit gates as follows. We first pre-process the input state with a sequence of single-qubit gates $P = X \otimes Z$ in accordance with (26). Then, the resulted state goes into the quantum switch, which processes it according to the gates (38) and (39). A measurement of the ancillary qubit $\mathbf{C}$ in basis spanned by the states (7) and (8), with $\theta = -\theta_B$, realizes the gates $S_\pm^{A,B}(-\theta_B)$ with probability $1/2$ in accordance with (13) and (14). Finally, depending on the outcome of the measurement, we perform a post-processing by applying another sequence of single-qubit gates, namely, i) $F_+(\alpha_B, -\theta_B, \mathbf{n}(\phi_B)) = e^{i\frac{\alpha_B}{2}} R_Z(\alpha_B + \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B + \frac{\pi}{2})$, whenever the the ancillary qubit is found in the state $|\mu(-\theta_B)\rangle$ or ii) $F_-(\alpha_B, -\theta_B, \mathbf{n}(\phi_B)) = e^{i\frac{\alpha_B}{2}}\left(R_Z(\alpha_B - \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B - \frac{\pi}{2})\right)$ otherwise. Accordingly, the overall deterministic implementation of the $\mathtt{BAR}(\alpha_B, \phi_B, \theta_B)$ gate via superposed orders can be expressed as:

$$\mathtt{BAR}(\alpha_B, \phi_B, \theta_B) = e^{i\frac{\alpha_B}{2}}\Big(R_Z\Big(\alpha_B \pm \frac{\pi}{2}\Big) \otimes R_{\mathbf{n}(\phi_B)}\Big(\theta_B \pm \frac{\pi}{2}\Big)\Big) \cdot S_\pm^{A,B}(-\theta_B)(X \otimes Z). \quad (40)$$

V. Conclusions

In this paper, we proved that quantum gates in superposed orders via the quantum switch give birth to a novel paradigm for universal quantum computation. Specifically, the quantum switch enables a framework able to implement any two-qubit controlled gate in a deterministic manner, by exploiting only single-qubit gates in superposition of causal orders. The result here proposed paves the way for unleashing the advantages provided by the engineering of the unconventional quantum propagation phenomena towards a computing model based on higher-order quantum operations. And we do hope that this manuscript can fuel further research – both theoretical and experimental – about the powerful setup enabled by superposed orders of quantum gates for photonic quantum computing.

Appendix A
Proof of Lemma 1

For $A = \bigotimes_{i=1}^N A_i$ and $B = \bigotimes_{i=1}^N B_i$, the quantum switch implements the unitary gate given in (5), i.e.:

$$S(A, B) = \frac{1}{2}\Bigg[\Big\{\bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i\Big\} \otimes I_{\mathbf{C}} + \Big[\bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i\Big] \otimes Z_{\mathbf{C}}\Bigg]. \quad (41)$$

Hence, when the input N-qubits are in a pure initial state $\rho = |\psi\rangle\langle\psi|$ and the ancillary system $\mathbf{C}$ is in the pure state $\omega = |+\rangle\langle+|$, the output of the unitary in (41) is equal to:

$$S(A,B)(|\psi\rangle \otimes |+\rangle) = \frac{1}{2}\Big[\Big\{\bigotimes_{i=1}^{N} A_i, \bigotimes_{i=1}^{N} B_i\Big\}\, |\psi\rangle \otimes |+\rangle + \Big[\bigotimes_{i=1}^{N} A_i, \bigotimes_{i=1}^{N} B_i\Big]\, |\psi\rangle \otimes |-\rangle\Big]. \tag{42}$$

Measurement of the ancillary qubit in the basis spanned by the states (7) and (8) leaves the input N qubits in one of the two following states:

$$|\psi_+(\theta)\rangle = \Big[\cos\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^{N} A_iB_i + i\sin\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^{N} B_iA_i\Big]|\psi\rangle, \tag{43}$$

$$|\psi_-(\theta)\rangle = \Big[i\sin\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^{N} A_iB_i + \cos\Big(\frac{\theta}{2}\Big)\bigotimes_{i=1}^{N} B_iA_i\Big]|\psi\rangle, \tag{44}$$

with probability $1/2$. It is straightforward to observe that this is equivalent to the implementation of one of gates in (13) and (14), each with probability $1/2$, and hence the proof follows.

Appendix B
Proof of Lemma 2

An arbitrary unitary gate U can be represented by an Hermitian operator as $\mathtt{U} = e^{iH}$. Accordingly, applying this representation to the CU gate, it results that CU can be expressed as:

$$\mathtt{CU} = e^{\frac{i}{2}(I-Z)\otimes H}. \tag{45}$$

For a single-qubit gate U, the corresponding Hermitian operator H can be expanded into the Pauli basis as:

$$H = \alpha I + \theta(\mathbf{n}\cdot\boldsymbol{\sigma}), \tag{46}$$

where $\alpha, \theta \in \mathbb{R}$, $\mathbf{n}$ denotes a real-valued unit vector known as Bloch vector [48], and $\boldsymbol{\sigma} = (X, Y, Z)$ is a vector of Pauli matrices. Plugging (46) into (45), after some algebraic manipulations, the CU gate can be decomposed into a sequence of gates as:

$$\mathtt{CU} = e^{i\frac{\alpha}{2}}\Big(R_Z(\alpha) \otimes R_{\mathbf{n}}(-\theta)\Big)R_{Z\mathbf{n}}(\theta), \tag{47}$$

where $R_{\mathbf{n}}(\theta) = \cos\big(\frac{\theta}{2}\big)I - i\sin\big(\frac{\theta}{2}\big)(\mathbf{n}\cdot\boldsymbol{\sigma})$ is the rotation operator with respect to the Bloch vector $\mathbf{n}$ defined in (16), $R_Z(\theta)$ is the rotation operator with respect to $\mathbf{n} = (0, 0, 1)$, and $R_{Z\mathbf{n}}(\theta)$ is the two-qubit rotation gate given in (17) when Bloch vector $\tilde{\mathbf{n}}$ is set as $\tilde{\mathbf{n}} = (0, 0, 1)$. Clearly, from (27), it easy to recognize that the CU gate is locally equivalent to the two-qubit rotation gate $R_{Z\mathbf{n}}(\theta)$ given in (17).

From this, let us demonstrate that the two-qubit rotation $R_{Z\mathbf{n}}(\theta)$ is locally equivalent to:

- single-qubit gates $A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n})$ and $B = B_0 \otimes B_1(\mathbf{n})$ combined in superposition of orders via a quantum switch,
- preceded by a sequences $P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n})$ of single-qubit *pre-processing* gates,
- followed by a sequences $F(\mathbf{n}) = F_0 \otimes F_1(\mathbf{n})$ of single-qubit *post-processing* gates

Accordingly, by exploiting Lemma 1, it results that the overall gate implemented by the above-described superposition of orders is either:

$$S_+^{A,B}(\theta)P = \cos\Big(\frac{\theta}{2}\Big)A_0B_0P_0 \otimes A_1B_1P_1 + i\sin\Big(\frac{\theta}{2}\Big)B_0A_0P_0 \otimes B_1A_1P_1, \tag{48}$$

$$S_-^{A,B}(\theta)P = i\sin\Big(\frac{\theta}{2}\Big)A_0B_0P_0 \otimes A_1B_1P_1 + \cos\Big(\frac{\theta}{2}\Big)B_0A_0P_0 \otimes B_1A_1P_1, \tag{49}$$

depending on the measurement of the ancillary qubit $\mathbf{C}$ in the basis spanned by vectors (7) and (8), where the parameter θ matches the corresponding parameter in decomposition (46). By setting the single-qubit gates as:

$$P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n}) = X \otimes (\mathbf{n}^\perp \cdot \boldsymbol{\sigma}), \tag{50}$$

$$A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n}) = X \otimes (\mathbf{n}^\perp \cdot \boldsymbol{\sigma}), \tag{51}$$

$$B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n}) = R_Z\Big(\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big), \tag{52}$$

where $\mathbf{n}^\perp$ denotes the Bloch vector perpendicular to $\mathbf{n}$ (i.e., $\mathbf{n}^\perp \cdot \mathbf{n} = 0$), the operations arising from the causal order AB read as:

$$A_0B_0P_0 = XR_Z\Big(\frac{\pi}{2}\Big)X = R_Z\Big(-\frac{\pi}{2}\Big), \tag{53}$$

$$A_1(\mathbf{n})B_1(\mathbf{n})P_1(\mathbf{n}) = (\mathbf{n}^\perp \cdot \boldsymbol{\sigma})R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big)(\mathbf{n}^\perp \cdot \boldsymbol{\sigma}) = R_{\mathbf{n}}\Big(-\frac{\pi}{2}\Big), \tag{54}$$

while the ones arising from the causal order BA are:

$$B_0A_0P_0 = R_Z\Big(\frac{\pi}{2}\Big)XX = R_Z\Big(\frac{\pi}{2}\Big), \tag{55}$$

$$B_1(\mathbf{n})A_1(\mathbf{n})P_1(\mathbf{n}) = R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big)(\mathbf{n}^\perp \cdot \boldsymbol{\sigma})(\mathbf{n}^\perp \cdot \boldsymbol{\sigma}) = R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big). \tag{56}$$

By substituting the above equations in (48) and (49), it results:

$$S_+^{A,B}(\theta)P(\mathbf{n}) = \cos\Big(\frac{\theta}{2}\Big)R_Z\Big(-\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(-\frac{\pi}{2}\Big) + i\sin\Big(\frac{\theta}{2}\Big)R_Z\Big(\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big), \tag{57}$$

$$S_-^{A,B}(\theta)P(\mathbf{n}) = i\sin\Big(\frac{\theta}{2}\Big)R_Z\Big(-\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(-\frac{\pi}{2}\Big) + \cos\Big(\frac{\theta}{2}\Big)R_Z\Big(\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(\frac{\pi}{2}\Big). \tag{58}$$

Finally, by setting the single-qubit post-processing gates $F(\mathbf{n}) = F_0 \otimes F_1(\mathbf{n})$ equal either to:

$$F_\pm(\mathbf{n}) = F_{\pm,0} \otimes F_{\pm,1}(\mathbf{n}) = R_Z\Big(\pm\frac{\pi}{2}\Big) \otimes R_{\mathbf{n}}\Big(\pm\frac{\pi}{2}\Big), \tag{59}$$

depending on whether the ancillary qubit measurement result is $|\mu(\theta)\rangle$ or $|\mu^\perp(\theta)\rangle$, the following two-qubit rotation is implemented:

$$F_{\pm}(\mathbf{n})S_{\pm}^{A,B}(\theta)P(\mathbf{n}) = R_{Z\mathbf{n}}(\theta). \tag{60}$$

In (60) we exploited the equality $R_Z(\pm\pi)\otimes R_{\mathbf{n}}(\pm\pi) = -Z\otimes(\mathbf{n}\cdot\boldsymbol{\sigma})$. As both $P(\mathbf{n})$ and $F_{\pm}(\mathbf{n})$ are tensor product of single-qubit case, it results that $S_{\pm}^{A,B}(\theta)$ – hence – is locally (single-qubit) equivalent to the two-qubit rotation gate $R_{Z\mathbf{n}}(\theta)$ via (60), which in turn is locally equivalent to the CU gate via (27). Hence, the proof follows.

References

[1] S. Koudia, A. S. Cacciapuoti, K. Simonov, and M. Caleffi, "How deep the theory of quantum communications goes: Superadditivity, superactivation and causal activation," *IEEE Comm. Surv. Tutor.*, vol. 24, pp. 1926–1956, 2022.

[2] G. Chiribella and H. Kristjánsson, "Quantum Shannon theory with superpositions of trajectories," *Proc. R. Soc. A*, vol. 475, no. 2225, p. 20180903, 2019.

[3] A. A. Abbott, J. Wechs, D. Horsman *et al.*, "Communication through coherent control of quantum channels," *Quantum*, vol. 4, p. 333, Sep. 2020.

[4] H. Kristjánsson, G. Chiribella, S. Salek, D. Ebler, and M. Wilson, "Resource theories of communication," *New J. Phys.*, vol. 22, p. 073014, 2020.

[5] G. Chiribella, M. Banik, S. S. Bhattacharya *et al.*, "Indefinite causal order enables perfect quantum communication with zero capacity channels," *New J. Phys.*, vol. 23, p. 033039, 2021.

[6] G. Chiribella, "Perfect discrimination of no-signalling channels via quantum superposition of causal structures," *Phys. Rev. A*, vol. 86, p. 040301(R), 2012.

[7] O. Oreshkov, F. Costa, and Č. Brukner, "Quantum correlations with no causal order," *Nat. Commun.*, vol. 3, no. 1, p. 1092, 2012.

[8] J. Illiano, M. Caleffi, A. Manzalini, and A. S. Cacciapuoti, "Quantum Internet protocol stack: A comprehensive survey," *Computer Networks*, p. 109092, 2022.

[9] A. S. Cacciapuoti, J. Illiano, S. Koudia, K. Simonov, and M. Caleffi, "The quantum internet: Enhancing classical internet services one qubit at a time," *IEEE Network*, vol. 36, no. 5, pp. 6–12, 2022.

[10] A. S. Cacciapuoti, J. Illiano, and M. Caleffi, "Quantum Internet Addressing," *IEEE Network*, 2023.

[11] L. M. Procopio, A. Moqanaki, M. Araújo *et al.*, "Experimental superposition of orders of quantum gates," *Nat. Commun.*, vol. 6, p. 7913, Aug. 2015.

[12] G. Rubino, L. A. Rozema, A. Feix *et al.*, "Experimental verification of an indefinite causal order," *Sci. Adv.*, vol. 3, no. 3, 2017.

[13] K. Goswami, C. Giarmatzi, M. Kewming *et al.*, "Indefinite Causal Order in a Quantum Switch," *Phys. Rev. Lett.*, vol. 121, p. 090503, 2018.

[14] Y. Guo, X.-M. Hu, Z.-B. Hou *et al.*, "Experimental transmission of quantum information using a superposition of causal orders," *Phys. Rev. Lett.*, vol. 124, p. 030502, 2020.

[15] M. Caleffi and A. S. Cacciapuoti, "Quantum switch for the quantum internet: Noiseless communications through noisy channels," *IEEE J. Sel. Areas Commun.*, vol. 38, pp. 575–588, 2020.

[16] D. Chandra, M. Caleffi, and A. S. Cacciapuoti, "The entanglement-assisted communication capacity over quantum trajectories," *IEEE Trans. Wirel. Commun.*, vol. 21, no. 6, pp. 3632–3647, 2021.

[17] M. Caleffi, K. Simonov, and A. S. Cacciapuoti, "Beyond Shannon limits: Quantum communications through quantum paths," *IEEE J. Sel. Areas Commun.*, vol. 41, pp. 2707–2724, 2023.

[18] S. Koudia, A. S. Cacciapuoti, and M. Caleffi, "Deterministic generation of multipartite entanglement via causal activation in the quantum internet," *IEEE Access*, vol. 11, pp. 73 863–73 878, 2023.

[19] Y. Chen and Y. Hasegawa, "EPR pairs from indefinite causal order," arXiv:2112.03233, 2021.

[20] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, "Beyond quantum computers," arXiv:0912.0195, 2009.

[21] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, "Quantum computations without definite causal structure," *Phys. Rev. A*, vol. 88, no. 2, p. 022318, 2013.

[22] J. Wechs, H. Dourdent, A. A. Abbott, and C. Branciard, "Quantum circuits with classical versus quantum control of causal order," *PRX Quantum*, vol. 2, p. 030335, 2021.

[23] T. Colnaghi, G. M. D'Ariano, S. Facchini, and P. Perinotti, "Quantum computation with programmable connections between gates," *Phys. Lett. A*, vol. 376, no. 45, pp. 2940–2943, 2012.

[24] M. Araújo, F. Costa, and Č. Brukner, "Computational advantage from quantum-controlled ordering of gates," *Phys. Rev. Lett.*, vol. 113, p. 250402, 2014.

[25] M. M. Taddei, R. V. Nery, and L. Aolita, "Quantum superpositions of causal orders as an operational resource," *Phys. Rev. Research*, vol. 1, no. 3, p. 033174, 2019.

[26] M. J. Renner and Č. Brukner, "Computational advantage from a quantum superposition of qubit gate orders," *Phys. Rev. Lett.*, vol. 128, no. 23, p. 230503, 2022.

[27] W.-Q. Liu, Z. Meng, B.-W. Song *et al.*, "Experimentally demonstrating indefinite causal order algorithms to solve the generalized Deutsch's problem," *arXiv preprint arXiv:2305.05416*, 2023.

[28] M. M. Taddei, J. Cariñe, D. Martínez *et al.*, "Computational advantage from quantum superposition of multiple temporal orders of photonic gates," *PRX Quantum*, vol. 2, p. 010320, 2021.

[29] A. Barenco, "A universal two-bit gate for quantum computation," *Proc. R. Soc. Lond. A*, vol. 449, p. 679, 1995.

[30] L. M. Procopio, F. Delgado, M. Enríquez, N. Belabas, and J. A. Levenson, "Sending classical information via three noisy channels in superposition of causal orders," *Phys. Rev. A*, vol. 101, p. 012346, 2020.

[31] G. Rubino, L. A. Rozema, F. Massa *et al.*, "Experimental Entanglement of Temporal Orders," *Quantum*, vol. 6, p. 621, 2022.

[32] T. Strömberg, P. Schiansky, R. W. Peterson, M. T. Quintino, and P. Walther, "Demonstration of a quantum SWITCH in a Sagnac configuration," *Phys. Rev. Lett.*, vol. 131, p. 060803, 2023.

[33] K. Goswami, Y. Cao, G. A. Paz-Silva, J. Romero, and A. G. White, "Increasing communication capacity via superposition of order," *Phys. Rev. Research*, vol. 2, p. 033292, 2020.

[34] J. Wang, F. Sciarrino, A. Laing, and M. G. Thompson, "Integrated photonic quantum technologies," *Nat. Photonics*, vol. 14, no. 5, pp. 273–284, 2020.

[35] E. Knill, R. Laflamme, and G. J. Milburn, "A scheme for efficient quantum computation with linear optics," *Nature*, vol. 409, no. 6816, pp. 46–52, 2001.

[36] R. Okamoto, J. L. O'Brien, H. F. Hofmann, and S. Takeuchi, "Realization of a Knill-Laflamme-Milburn controlled-NOT photonic quantum circuit combining effective optical nonlinearities," *Proc. Natl. Acad. Sci. U. S. A.*, vol. 108, no. 25, p. 10067, 2011.

[37] S. Slussarenko and G. J. Pryde, "Photonic quantum information processing: A concise review," *Appl. Phys. Rev.*, vol. 6, p. 041303, 2019.

[38] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum circuit architecture," *Phys. Rev. Lett.*, vol. 101, p. 060401, 2008.

[39] ——, "Theoretical framework for quantum networks," *Phys. Rev. A*, vol. 80, p. 022339, 2009.

[40] A. Bisio and P. Perinotti, "Theoretical framework for higher-order quantum theory," *Proc. R. Soc. A*, vol. 475, p. 20180706, 2019.

[41] L. Apadula, A. Bisio, and P. Perinotti, "No-signalling constrains quantum computation with indefinite causal structure," arXiv:2202.10214, 2022.

[42] K. Goswami and J. Romero, "Experiments on quantum causality," *AVS Quantum Sci.*, vol. 2, p. 037101, 2020.

[43] H. Cao, N. Wang, Z.-A. Jia *et al.*, "Quantum simulation of indefinite causal order induced quantum refrigeration," *Phys. Rev. Research*, vol. 4, p. L032029, 2022.

[44] X. Nie, X. Zhu, C. Xi *et al.*, "Experimental realization of a quantum refrigerator driven by indefinite causal orders," *Phys. Rev. Lett.*, vol. 129, p. 100603, 2022.

[45] M. Antesberger, M. T. Quintino, P. Walther, and L. A. Rozema, "Higher-order process matrix tomography of a passively-stable quantum SWITCH," arXiv:2305.19386, 2023.

[46] K. Simonov, S. Roy, T. Guha, Z. Zimborás, and G. Chiribella, "Activation of thermal states by coherently controlled thermalization processes," arXiv:2208.04034, 2022.

[47] N. Gao, D. Li, A. Mishra *et al.*, "Measuring incompatibility and clustering quantum observables with a quantum switch," *Phys. Rev. Lett.*, vol. 130, p. 170201, 2023.

[48] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. Cambridge University Press, 2011.

[49] J. Zhang, J. Vala, S. Sastry, and K. B. Whaley, "Geometric theory of nonlocal two-qubit operations," *Phys. Rev. A*, vol. 67, p. 042313, 2003.

[50] M. Musz, M. Kuś, and K. Życzkowski, "Unitary quantum gates, perfect entanglers, and unistochastic maps," *Phys. Rev. A*, vol. 87, p. 022111, 2013.

[51] Y. Shen, L. Chen, and L. Yu, "Classification of Schmidt-rank-two multipartite unitary gates by singular number," *J. Phys. A: Math. Theor.*, vol. 55, p. 465302, 2022.

[52] O. Oreshkov, "Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics," *Quantum*, vol. 3, p. 206, 2019.

[53] S. Milz, J. Bavaresco, and G. Chiribella, "Resource theory of causal connection," *Quantum*, vol. 6, p. 788, 2022.

[54] T. van der Lugt, J. Barrett, and G. Chiribella, "Device-independent certification of indefinite causal order in the quantum switch," *Nat. Commun.*, vol. 14, p. 5811, 2023.

[55] C. P. Williams, *Explorations in Quantum Computing*. Springer, 2011.

[56] X.-F. Shi, "Universal Barenco quantum gates via a tunable noncollinear interaction," *Phys. Rev. A*, vol. 97, p. 032310, 2018.