

What makes a causal loop consistent?

Hippolyte Dourdent,^{1,2,*} Andreas Leitherer,¹ Emanuel-Cristian Boghiu,^{3,4} Kyrlo Simonov,⁴ Ravi Kunjwal,⁵ and Antonio Acín^{1,6}

¹*ICFO-Institut de Ciències Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels, Barcelona, Spain*

²*Department for Quantum Technology, Universität Münster, Heisenbergstraße 11, 48149 Münster, Germany*

³*Barcelona Supercomputing Center, Plaça d'Eusebi Güell, 1-3 08034, Barcelona, Spain*

⁴*Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*

⁵*Aix-Marseille University, CNRS, LIS, Marseille, France*

⁶*ICREA, Passeig Lluís Companys 23, 08010 Barcelona, Spain*

(Dated: October 1, 2026)

Causal loops escape paradoxes if they generate valid probabilities under arbitrary independent local interventions. We show that a deterministic classical process is logically consistent if and only if it contains no signaling loops: cyclic signaling is forbidden but cyclic causation is not, allowing indefinite causal order. Logical consistency is then found to hold exactly when the list of events of a process is pairwise exclusive and complete, providing the first event-based, intervention-free, non-recursive characterization of consistent causal loops. Our results repair a previously introduced intervention-free criterion that misses global loops and reveal that what makes deterministic classical communications without causal order consistent is a multipartite form of Specker's principle: if any two of several questions can be answered jointly, so can all of them.

Introduction. The assumption that physical events unfold within a well-defined causal order underlies both classical and quantum physics. Yet, general relativity admits spacetime geometries containing closed timelike curves (CTCs) [1, 2]. The latter famously give rise to the *grandfather paradox*, a (hypothetically) physical contradiction induced by the suppression of a cause by its own effect. What principle, then, prevents causal loops from generating such contradictions? Hawking's *chronology protection* conjecture [3] answers this question by forbidding CTCs altogether, whereas Novikov's *self-consistency* principle permits them, provided that events along the loop are self-consistent, i.e. never lead to a contradiction [4, 5].

Self-consistency, however, introduces a second puzzle. A causal loop may contain an effect that is also its own cause, producing an apparently self-originating piece of information, the *bootstrap paradox*. Although such loops might be dismissed as a benign underdetermination, information-theoretic considerations show that, under the principle of *freedom of choice*—the ability of local agents to choose their interventions independently—bootstrap and grandfather paradoxes become operationally equivalent [6]. A satisfactory notion of consistency must therefore exclude not only contradictions, but also tautologies.

This criterion is captured by *logical consistency*, the information-theoretic requirement that a communication process should always generate valid joint probabilities under arbitrary, independent local operations. Surprisingly, logical consistency does not require a fixed global causal order. It permits indefinite causal structures, including correlations that violate causal inequalities [7]. Such *non-causal* behaviors do not solely arise from quan-

tum [8] or probabilistic [9] dynamics. They also manifest in fully classical, deterministic single-round communication channels known as *process functions* [10, 11]. Process functions without global past [10–17], such as the canonical tripartite Lugano process [11, 12], dynamically route information through a mesh of mutual controls of causal structures, such that no party's input remains independent of the others' outputs, while still preserving freedom of choice.

By definition, logical consistency is the mechanism that shields these causal loops from paradoxes. But can it be characterized by the communication events alone, without invoking the agents or checking every group of them? Its relation to paradoxes is explicitly formalized through *fixed-point uniqueness* (Theorem 1) [13, 18]: logically consistent causal loops must yield a unique fixed point for any local interventions, while grandfather and bootstrap antinomies corresponding precisely to dynamics with zero or multiple fixed points, respectively. However, evaluating fixed points requires checking the induced dynamics for every choice of local interventions. Instead, *unipartite reducibility* (Theorem 2) [13] captures the intuitive idea that reducibility under the free local operations of a single agent provides another core representation of logical consistency. It remains however recursive and intervention-dependent. Inspired by the idea that non-causal process functions rely on a non-trivial mesh of classical controls of causal orders, earlier works sought an intervention-free alternative in *output-reducibility* (Proposition 1) [13, 14]—a criterion that, as we show below, turns out to be incomplete.

Results. First, we reformulate fixed point uniqueness as an explicit *prohibition of any signaling loop*, whether it is *local*, *internal*, or *global* (Corollary 1). It reveals

that output-reducibility is incomplete, as it fails to rule out global loops. We then show how output-reducibility can be rectified by *non-erasing reducibility* (Theorem 3), based on reductions that preserve dependencies, detecting all signaling loops. Alternatively, it can be completed with a linear *process-matrix* constraint (Theorem 4), also formulated as function-level *phase-erasure* conditions (Theorem 5). In fact, as process functions are classical deterministic process matrices, all linear constraints characterizing the latter can be transformed into a *phase erasure characterization* (Theorem 6). This generalizes to arbitrary local dimension the constraints of the Boolean “parity-erasure” principle, recently proposed in Ref. [19] as a fundamental principle underlying indefinite causal order. Finally, we introduce the first purely event-based characterization of process functions (Theorem 7): a process function is valid if and only if it forms a *complete list of pairwise exclusive events*, where two global events are defined as exclusive whenever some party assigns them distinct outcomes for the same local setting. Thus, consistency of a causal loop only requires the communication structure to be well-defined for every tuple of local outcomes. This provides a complete, conceptually parsimonious, non-recursive and intervention-free characterization of logical consistency. It identifies the latter as a multipartite form of Specker’s principle [20, 21], the motivational basis [22–24] of the local orthogonality [25] and exclusivity [26] principles, which play a central role in explaining why quantum correlations are not more nonlocal than they are.

Process functions. Consider n parties, labeled $k \in \{1, \dots, n\} = \mathcal{N}$. Each party receives classical inputs $x_k \in \mathcal{X}_k$ and $i_k \in \mathcal{I}_k$ and produces classical outputs $a_k \in \mathcal{A}_k$ and $o_k \in \mathcal{O}_k$. For compactness, we will use the notations $\mathcal{X} := \times_{k=1}^n \mathcal{X}_k$, $\mathcal{A} := \times_{k=1}^n \mathcal{A}_k$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{a}_{\setminus i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ etc. Here¹ $\mathcal{X}_k, \mathcal{A}_k$ are exchanged with a shared communication resource—analogous to hidden variables mediating the correlations in a Bell scenario—while $\mathcal{I}_k, \mathcal{O}_k$ are the local setting and outcome registers. The parties are operationally isolated, interacting only through the shared resource; their settings i_k are freely chosen, and each party applies a local stochastic channel $P(a_k, o_k | x_k, i_k)$ exactly once. The observed correlations are

$$P(\mathbf{o} | \mathbf{i}) = \sum_{\mathbf{a}, \mathbf{x}} \left(\prod_{k=1}^n P(a_k, o_k | x_k, i_k) \right) P(\mathbf{x} | \mathbf{a}), \quad (1)$$

where $P(\mathbf{x} | \mathbf{a})$ is the classical process implemented by the resource. The process is *logically consistent* if Eq. (1) yields a valid distribution, i.e. $P(\mathbf{o} | \mathbf{i}) \geq 0$, $\sum_{\mathbf{o}} P(\mathbf{o} | \mathbf{i}) = 1$, for every choice of local operations.

Definition 1 (Process function). A deterministic logically consistent process defines a map $\mathbf{w}$ called the **process function**, composed of a set of local functions $\mathbf{w} := (w_k)_k$ where $w_k : \mathcal{A} \rightarrow \mathcal{X}_k$, which specifies the relation between outputs and inputs in Eq. (1) via $P(\mathbf{x} | \mathbf{a}) = \delta_{\mathbf{x}, \mathbf{w}(\mathbf{a})}$, where $\delta_{i,j}$ is the Kronecker delta. Similarly, deterministic local channels, also called **local interventions**, define functions $\mathbf{f} := (f_k)_k$ where $f_k : \mathcal{X}_k \times \mathcal{I}_k \rightarrow \mathcal{A}_k \times \mathcal{O}_k$ such that $P(a_k, o_k | x_k, i_k) = \delta_{(a_k, o_k), f_k(x_k, i_k)}$.

Fixed point uniqueness. Let us consider the simplified scenario where $|\mathcal{I}_k| = |\mathcal{O}_k| = 1$ for all parties k . Logical consistency then reduces to $\sum_{\mathbf{x}} \prod_k \delta_{x_k, w_k(\mathbf{f}(\mathbf{x}))} = 1$, equivalently $\sum_{\mathbf{a}} \prod_k \delta_{a_k, f_k(w_k(\mathbf{a}))} = 1$ for any local interventions $\mathbf{f}$. This yields the following characterization:

Theorem 1 (Fixed point uniqueness). A quasi-process function $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$, that is, a set of local functions $\mathbf{w} := (w_k)_k$ where $w_k : \mathcal{A} \rightarrow \mathcal{X}_k$, is a process function if and only if for every $\mathbf{f} = (f_1, \dots, f_n)$, $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$, the composed map $\mathbf{w} \circ \mathbf{f}$ has a unique fixed point

$$\forall \mathbf{f} : \mathcal{X} \rightarrow \mathcal{A}, \exists! \mathbf{x}^{\mathbf{f}} \text{ s.t. } \forall k, x_k^{\mathbf{f}} = w_k(\mathbf{f}(\mathbf{x}^{\mathbf{f}})), \quad (2)$$

or equivalently, $\mathbf{f} \circ \mathbf{w}$ has a unique fixed point.

Proof. Eq. (2) is proven in Refs. [13, 18]. $\square$

Fixed-point uniqueness excludes both the grandfather antinomy (a collection of interventions $\mathbf{f}$ is such that $\mathbf{f} \circ \mathbf{w}$ has no fixed point, a self-contradiction, e.g. $a_k = a_k \oplus 1$) and the bootstrap antinomy (a collection of interventions $\mathbf{f}$ is such that $\mathbf{f} \circ \mathbf{w}$ has several fixed points, a tautology, e.g. $a_k = a_k$), which are anyway equivalent under local operations [6]. It thus acts as a “time-policing” principle compatible with freedom of choice. It relies, however, on every possible local intervention.

Unipartite reducibility. Another complete characterization [13, Lemma 3] replaces the global composition $\mathbf{w} \circ \mathbf{f}$ of Theorem 1 by a composition with a single local operation f_k , yielding a reduced process function.

Definition 2. (Reduced process function) Let $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ be non-self-signaling, i.e. $w_k(a) = w_k(\mathbf{a}_{\setminus k})$ for all k . For a local operation $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$, the reduced process function $w^{f_k} : \mathcal{A}_{\setminus k} \rightarrow \mathcal{X}_{\setminus k}$ is obtained by feeding $x_k = w_k(\mathbf{a}_{\setminus k})$ into f_k and closing the loop (Fig. 1): $w^{f_k}(\mathbf{a}_{\setminus k}) := \mathbf{w}_{\setminus k}(\mathbf{a}_{\setminus k}, f_k(w_k(\mathbf{a}_{\setminus k})))$.

Theorem 2 (Unipartite reducibility). Let $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ be a non-self-signaling function. Then $\mathbf{w}$ is an n -partite process function if and only if there exists a party k such that w^{f_k} is a valid $(n-1)$ -partite process function for every f_k . (Necessity in fact holds for every party k .)

¹ Our notation is distinct from that adopted in Refs. [9, 11, 13, 14, 27].

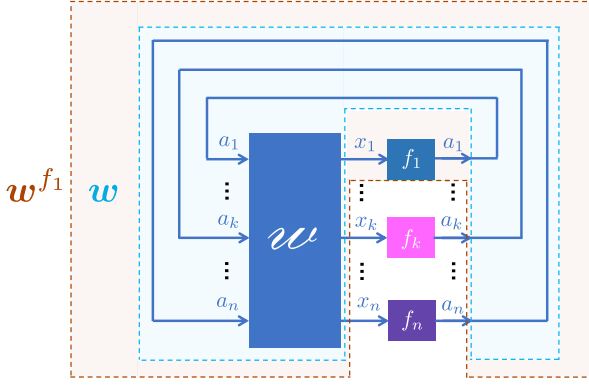


FIG. 1. Visualization of the reduced process function as defined in Def. (2) for the case of fixing f_1 , a local operation of the first party.

Proof. Proven in Ref. [13, Lemma 3]. $\square$

While losing the explicit relation with the prohibition of paradoxes, unipartite reducibility is structurally informative. It expresses the idea that logical consistency is conserved when a party is absorbed into the process. It is however hard to apply, as it is recursive and quantifies over *all* local operations f_k .

Output reducibility. Refs. [13, 14] proposed a simplification using only constant interventions $f_k(x_k) = a_k$ for all x_k , where party k feeds to the shared communication resource the same value a_k irrespective of the input x_k received from the resource. The resulting *output-reduced function* is

$$\mathbf{w}^{a_k}(\mathbf{a}_{\setminus k}) := (w_j(\mathbf{a}_{\setminus k}, a_k))_{j \neq k}. \quad (3)$$

Proposition 1 (Output-reducibility). *A quasi-process function $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ is a process function only if $\mathbf{w}^{a_k}$ is a valid $(n-1)$ -partite process function for every party k and every $a_k \in \mathcal{A}_k$.*

Proof. Necessity follows directly from Theorem 2. $\square$

Crucially, this condition is operationally intervention-free: one can fix outputs without invoking local operations. Nevertheless, output-reducibility is *not* sufficient: it may incorrectly classify certain logically inconsistent quasi-processes [9] as valid. The simplest counterexamples are the Guess-Your-Neighbor's-Input (GYNI) [28] processes, $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ where each component w_k depends on a single output $a_{j \neq k}$, and every party's output is fed into exactly one other party's input, forming a signaling loop through all parties. A standard tripartite example is (cf. Fig. 2)

$$\mathbf{w}_{3\text{-GYNI}} : \quad x_1 := a_3, \quad x_2 := a_1, \quad x_3 := a_2, \quad (4)$$

with straightforward n -partite generalizations. Such loops satisfy output-reducibility: fixing any output, say

a_1 , yields the bipartite reduction $\mathbf{w}_{3\text{-GYNI}}^{a_1}(a_2, a_3) = (x_2 = a_1, x_3 = a_2)$, which is one-way signaling, hence valid, and the same holds for the other reductions. However, GYNI processes are manifestly inconsistent. For instance, if all parties apply an identity channel $a_k := x_k$, Eq. (4) gives $x_1 = a_3 = x_3 = a_2 = x_2 = a_1 = x_1$, i.e. the bootstrap paradox arises. Thus, this inconsistency evades detection here.

In order to understand more generally what makes output-reducibility incomplete, let us define a *signaling loop* on a nonempty subset $\mathcal{K} \subseteq \mathcal{N}$ as follows. Fix the outputs $\bar{\mathbf{a}} := \mathbf{a}_{\mathcal{N} \setminus \mathcal{K}}$ of all parties outside $\mathcal{K}$, and choose local interventions $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$ for $k \in \mathcal{K}$. These induce the closed-loop map $L_{\mathcal{K}}^{\bar{\mathbf{a}}, \mathbf{f}} : \mathcal{A}_{\mathcal{K}} \rightarrow \mathcal{A}_{\mathcal{K}}$,

$$(L_{\mathcal{K}}^{\bar{\mathbf{a}}, \mathbf{f}}(\mathbf{a}_{\mathcal{K}}))_k = (f_k(w_k(\mathbf{a}_{\mathcal{K}}, \bar{\mathbf{a}})))_k. \quad (5)$$

We say that $\mathcal{K}$ forms a signaling loop whenever this map fails to have exactly one fixed point. This establishes a scale of loops, see Fig. 2, which can be local ($|\mathcal{K}| = 1$, a party signaling to itself), internal ($\mathcal{K} \subsetneq \mathcal{N}$, manifested in a reduced function), and global ($\mathcal{K} = \mathcal{N}$, as in Eq. (4)). Hence, the following corollary can be derived:

Corollary 1 (No- $\mathcal{K}$ -signaling loop). *A quasi-process function, i.e. any tuple $\mathbf{w} = (w_k)$, $w_k : \mathcal{A} \rightarrow \mathcal{X}_k$, is a process function if and only if, no subset of $\mathcal{K} \subseteq \mathcal{N}$ parties form a signaling loop, for any fixed remaining outputs.*

Proof. By definition, a signaling loop yields paradoxes. The claim therefore follows directly from Theorem 1, and conversely. $\square$

Forbidding signaling loops do not forbid all kinds of cyclic dependencies. In the Lugano process [11, 12],

$$x_1 := a_3(a_2 \oplus 1), \quad x_2 := a_1(a_3 \oplus 1), \quad x_3 := a_2(a_1 \oplus 1), \quad (6)$$

each bipartite signaling relation is controlled by the third party, such that no loop survives. Yet, this process function admits no global past, i.e. no x_k is constant. Non-causality of process functions therefore rests not on signaling loops, but on a mesh of classically controlled causal orders.

Corollary 1 sheds light on the incompleteness of output-reducibility. The latter does rule out local and internal loops. A global loop, however, is cut by the very act of fixing an output, and thereby escapes every reduction. This diagnosis comes nevertheless with its cure.

Non-erasing reducibility. On the one hand, one can correct output-reducibility at its source, by requiring reducibility under operations that preserve dependencies rather than erase them.

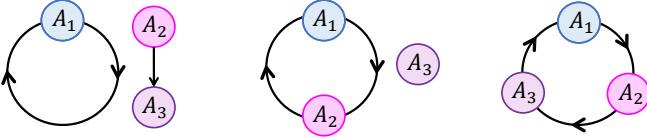


FIG. 2. Examples of three non-valid $n = 3$ quasi-processes structures containing (left) a local loop on A_1 , e.g., $x_1 := a_1$, $x_2 := 0$, $x_3 := a_2$; (middle) an internal loop between A_1 and A_2 , e.g., $x_1 := a_2$, $x_2 := a_1$, $x_3 := 0$; (right) a global loop, e.g., the GYNI quasi-process $x_1 := a_3$, $x_2 := a_1$, $x_3 := a_2$. For each of these forbidden structures, there exists a set of local interventions that lead to a self-contradiction $a_1 = a_1 \oplus 1$ (e.g. for $f_1 : x_1 \rightarrow a_1 := x_1 \oplus 1$, $f_2 : x_2 \rightarrow a_2 := x_2$ and $f_3 : x_3 \rightarrow a_3 := x_3$) or a tautology $a_1 = a_1$ (e.g. for $f_k : x_k \rightarrow a_k := x_k$ for all k).

Definition 3 (Non-erasing operation). A local operation $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$ is non-erasing for $\mathbf{w}$ if it preserves at least one dependence on a_k : there exist a party j , an assignment $\mathbf{a}_{\setminus k}$, and inputs $x_k, x'_k \in \mathcal{X}_k$ such that

$$w_j(f_k(x_k), \mathbf{a}_{\setminus k}) \neq w_j(f_k(x'_k), \mathbf{a}_{\setminus k}). \quad (7)$$

Two compositions must be distinguished. Testing whether f_k is non-erasing probes the substitution $a_k \mapsto f_k(x_k)$ with x_k kept free, isolating which dependencies survive. The reduction w^{f_k} instead closes the loop, feeding $x_k = w_k(\mathbf{a}_{\setminus k})$, i.e. $a_k \mapsto f_k(w_k(\mathbf{a}_{\setminus k}))$. This yields:

Theorem 3 (Non-erasing reducibility). A quasi-process function $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ is a valid n -partite process function if and only if

- (i) for $n = 1$, w is constant;
- (ii) for $n > 1$, for every party k and every non-erasing operation $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$, the reduced function $\mathbf{w}^{f_k} : \mathcal{A}_{\setminus k} \rightarrow \mathcal{X}_{\setminus k}$ is well-defined and is a valid $(n-1)$ -partite process function.

Proof. See Appendix. $\square$

Completing output-reducibility. On the other hand, output-reducibility can be made sufficient by adding a constraint ruling out genuine global signaling loops, i.e. non-valid quasi-process that do not contain local nor internal loops. Since process functions are equivalent to deterministic classical process matrices, it is natural to consider the latter framework to formalize the missing constraint. By Theorem 4 of Ref. [15], $\mathbf{w}$ can be encoded as a process matrix $W_{\mathbf{w}}$ (Supplemented Material (SM), Eq. (19)). In the trace-and-replace notation of Eq. (18), with A_k^O the output space of party k , the single linear constraint

$$\Pi_k [1 - A_k^O] W_{\mathbf{w}} = 0, \quad (8)$$

forbids $W_{\mathbf{w}}$ from having component depending jointly on all n outputs. The GYNI process function (Eq. (4)) violates Eq. (8), while all its output-reductions are valid. Eq. (8) completes precisely what output-reducibility, which excludes every loop among proper subsets of parties, leaves out: it is violated by any quasi-process whose sole defect is a global loop. It is not, however, a stand-alone diagnostic for global loops. See Appendix.

Theorem 4 (Output-reducibility and no global loop). A quasi-process function $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ is a valid n -partite process function if and only if

- (i) Proposition 1 holds and

$$(ii) \Pi_k [1 - A_k^O] W_{\mathbf{w}} = 0.$$

Proof. As $W_{\mathbf{w}}$ is automatically positive and normalized, validity reduces to the linear constraints $\Pi_{k \in \mathcal{K}} [1 - A_k^O] A_{\mathcal{N} \setminus \mathcal{K}}^{IO} W_{\mathbf{w}} = 0$ for all $\emptyset \neq \mathcal{K} \subseteq \mathcal{N}$ [29–31]. Averaging a reduced process $\mathbf{w}^{a_k}$ over a_k amounts to tracing out the input and output spaces A_k^{IO} of party k . Hence (i) implies the process-matrix constraints for every $\mathcal{K} \subsetneq \mathcal{N}$, choosing any $k \notin \mathcal{K}$, while (ii) is the remaining case $\mathcal{K} = \mathcal{N}$. Conversely, the full family of constraints implies both (i) and (ii). More details can be found in Appendix. $\square$

Because Eq. (8) is defined on the process matrix $W_{\mathbf{w}}$, while $\mathbf{w}$ is purely a classical function, one may ask whether this constraint can be expressed without invoking Hilbert spaces. Indeed, it admits an equivalent, purely classical representation in an explicit phase-erasure form:

Theorem 5 (Output-reducibility and phase erasure). A quasi-process function $\mathbf{w} : \mathcal{A} \rightarrow \mathcal{X}$ is a valid n -partite process function if and only if

- (i) Proposition 1 holds and

$$(ii') \text{ for every } \mathbf{x} \in \mathcal{X} \text{ and every } \boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \text{ with } \nu_k \in \{1, \dots, d_k - 1\}, d_k := |\mathcal{A}_k|,$$

$$\sum_{\mathbf{a}} \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}} \prod_{k=1}^n e^{2\pi i \nu_k a_k / d_k} = 0. \quad (9)$$

Proof. See SM. $\square$

Eq. (9) is the Fourier transform of Eq. (8). For each global setting $\mathbf{x}$, the set of outputs producing it carries no Fourier component in which every party has a nonzero frequency. The inputs generated by $\mathbf{w}$ are thus never controlled by all n outputs jointly, and genuine global loops are ruled out.

A complete phase erasure characterization. Noting that $W_{\mathbf{w}}$ satisfies all process-matrix linear constraints, the same translation applies to each subset $\mathcal{K}$, yielding a complete family of phase-erasure constraints, which completely characterize process functions:

Theorem 6 (Phase erasure). *A quasi-process function $w : \mathcal{A} \rightarrow \mathcal{X}$ is a valid n -partite process function if and only if, for every nonempty $\mathcal{K} \subseteq \mathcal{N}$, every $\mathbf{x}_{\mathcal{K}} \in \mathcal{X}_{\mathcal{K}}$ and every $(\nu_k)_{k \in \mathcal{K}}$ with $\nu_k \in \{1, \dots, d_k - 1\}$,*

$$\sum_{\mathbf{a} \in \mathcal{A}} \prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k} e^{2\pi i \nu_k a_k / d_k} = 0. \quad (10)$$

Proof. See SM. $\square$

Theorems (1–6) are multiple faces of the same condition: the deterministic classical special case of the process-matrix constraints that detects signal loops at every scale. Remarkably, in the Boolean case, $\nu_k = 1$ and Eq. (10) reduces to the parity condition

$$\sum_{\mathbf{a} \in \mathcal{A}} \prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k} (-1)^{\sum_{k \in \mathcal{K}} a_k} = 0, \quad (11)$$

recovering the constraints on which the *parity-erasure* principle for higher-order processes [19] is based on.

All the above criteria ultimately chase signaling loops (and thus paradoxes), in the signaling structure (Corollary 1), through local reductions (Theorems 3–5) or in the Fourier coefficients (Theorem 6) of w . All inherit either a recursion over parties or a quantification over their subsets. Is logical consistency expressible without these features?

Pairwise exclusive completeness. Here, we find such a criterion, phrased on w alone, based on a single check over all pairs of events, with no intervention, recursion, nor subset involved.

An *event* is a pair of tuples $(\mathbf{a} | \mathbf{x}) \in \mathcal{A} \times \mathcal{X}$, read as “the parties produce the outputs $\mathbf{a}$ upon receiving the inputs $\mathbf{x}$ ”. Two events are *exclusive* if some party k satisfies $x_k = x'_k$ and $a_k \neq a'_k$: receiving the same input, it would have to produce two different outputs, which no local function can do. Exclusive events therefore never occur under a common intervention. A list $E \subseteq \mathcal{A} \times \mathcal{X}$ is *complete* if $|E| = |\mathcal{A}|$.

Theorem 7 (Pairwise-exclusive completeness). *A list of events $E \subseteq \mathcal{A} \times \mathcal{X}$ is the event list $\{(\mathbf{a} | w(\mathbf{a}))\}_{\mathbf{a} \in \mathcal{A}}$ of an n -partite process function if and only if it is complete and pairwise exclusive.*

Proof. Two events sharing the same $\mathbf{a}$ are never exclusive, so a pairwise exclusive list has pairwise distinct outputs, hence at most $|\mathcal{A}|$ events. If the list is moreover complete, its outputs exhaust $\mathcal{A}$, and E is associated with a total function $w : \mathcal{A} \rightarrow \mathcal{X}$. Conversely the event list of a total function is automatically complete. It remains to show that a total quasi-process function w is a process function if and only if E is pairwise exclusive.

($\Leftarrow$) Sufficiency follows from Ref. [17, Theorem 3]. Let $\mathbf{f} = (f_k)_k$, $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$, be an arbitrary intervention,

and suppose $\mathbf{a} \neq \mathbf{a}'$ are both fixed points of $\mathbf{f} \circ w$. Exclusivity of the corresponding events gives a party k with $a_k \neq a'_k$ and $w_k(\mathbf{a}) = w_k(\mathbf{a}')$, hence

$$a_k = f_k(w_k(\mathbf{a})) = f_k(w_k(\mathbf{a}')) = a'_k, \quad (12)$$

a contradiction. Every intervention thus admits *at most one* fixed point. Hence the quasi-process function does not generate bootstrap paradoxes. By Ref. [6, Theorem 1], any total quasi-process function w that does not suffer from the bootstrap paradox has a unique fixed point. Hence, w is a process function by Theorem 1.

($\Rightarrow$) Let $w : \mathcal{A} \rightarrow \mathcal{X}$ be a process function with event list $E = \{(\mathbf{a} | \mathbf{x}) \mid \mathbf{x} = w(\mathbf{a})\}$. Assume now *ad absurdum* that there exist two events $(\mathbf{a} | \mathbf{x}) \neq (\mathbf{a}' | \mathbf{x}')$ in E violating exclusivity, i.e. for every party k ,

$$x_k = x'_k \Rightarrow a_k = a'_k \text{ and } a_k \neq a'_k \Rightarrow x_k \neq x'_k. \quad (13)$$

Let $D = \{k : a_k \neq a'_k\} \neq \emptyset$ because $(\mathbf{a} | \mathbf{x}) \neq (\mathbf{a}' | \mathbf{x}')$. Define the following intervention for every k ,

$$f_k(x_k) := \begin{cases} a'_k & x_k = x'_k, \\ a_k & \text{otherwise.} \end{cases} \quad (14)$$

For $k \in D$ the inputs $x_k \neq x'_k$ are distinct, hence $f_k(x'_k) = a'_k$ and $f_k(x_k) = a_k$; for $k \notin D$, f_k is the constant map $a_k = a'_k$ and both equalities hold as well. Since $w_k(\mathbf{a}) = x_k$ and $w_k(\mathbf{a}') = x'_k$, both $\mathbf{a}$ and $\mathbf{a}'$ are distinct fixed points of $\mathbf{f} \circ w$ with $\mathbf{f} := (f_k)_k$, contradicting Theorem 1. Hence, the process function w satisfies pairwise exclusivity. $\square$

Discussion. The characterizations of process functions brought together in Table I resolve the central question of this work on three distinct levels.

On one hand, logical consistency requires forbidding closed signaling loops across all scales (Corollary 1). Notably, this restriction against cyclic signaling does not enforce global causality: cyclic dependence may remain viable if it is mediated by classical control, where each party’s causal placement is dynamically selected by prior outputs (Eq. (6)).

On the other hand, traditional criteria establish validity based on interventions or a quantification over subsets of parties. In contrast, pairwise-exclusive completeness (Theorem 7) bypasses this requirement: a causal loop is valid precisely when its allowed events form a partition of the output space into mutually exclusive outcomes. By eliminating any direct reliance on loops, interventions, or recursive calls, this property is the simplest to state, inherently non-sequential and thus straightforward to parallelize. Because pairwise exclusivity always holds for complete Boolean event lists, this drastically simplifies exhaustive enumeration—retrieving the 12 bipartite and 744 tripartite Boolean process functions (64 of which possess no global past) corresponding to the deterministic extremal points of the classical processes polytope [11].

Characterization	Condition on w	Type	Ref.
Fixed-point uniqueness	$f \circ w$ has a unique fixed point, $\forall f$	interventions	[18]
Unipartite reducibility	w^{f_k} valid for some k and all f_k	recursive	[13]
No signaling loop	no $\mathcal{K} \subseteq \mathcal{N}$ forms a signaling loop	signaling structure	Cor. 1
Non-erasing reducibility	w^{f_k} valid $\forall k$ and all non-erasing f_k	recursive	Thm. 3
Output-red. + no global loop	w^{a_k} valid $\forall k, a_k$ [13, 14], and $\prod_k [1 - A_k^O] W w = 0$	recursive + matrix	Thm. 4
Output-red. + phase erasure	w^{a_k} valid $\forall k, a_k$, and Eq. (9)	recursive + functions	Thm. 5
Phase erasure	Eq. (10) for all $\emptyset \neq \mathcal{K} \subseteq \mathcal{N}$	functions	Thm. 6
Pairwise-exclusive completeness	$ E = \mathcal{A} $, all events of E pairwise exclusive	combinatorial	Thm. 7

TABLE I. Complete characterizations of process functions: those previously known (top) and those established here (bottom). Only phase erasure and pairwise-exclusive completeness avoid recursion, and only the latter is a condition on pairs of events, hence testable in parallel.

This criterion recovers the correspondence between process functions and unambiguous complete product bases [17], which specify a list of events that is pairwise exclusive (by unambiguity) and complete. With it comes the relation between process functions and non-signaling inequalities of the local orthogonality type [17, 25]. Pairwise exclusive completeness is in fact a *structural* [22] reading of Specker’s principle [20, 21]—“if you have several questions and you can answer any two of them, then you can also answer all of them”—in a Bell scenario, i.e. a structural version of the local orthogonality principle [25, 32]. Known as *orthocoherence*, it also plays a key role in the discussion of tensor products in quantum logic [33–35]. As we show in forthcoming work, pairwise exclusive completeness anchors in fact process functions within the hypergraph-theoretic language of contextuality scenarios [36].

These verifiable membership criteria enable a more systematic study of logically consistent classical processes with indefinite causal order. They may, for instance, help establish the conjecture of Ref. [37] that a “siblings-on-cycles” property suffices for a causal structure to be admissible. Moreover, the physical realizability of these consistent loops remains fundamentally tied to their spacetime embeddability [38]. Since process matrices—and by extension process functions—form a linear subset of both post-selected closed timelike curves [39–41] and pre-and-post-selected quantum states [42], process functions can be simulated via quantum post-selection [43]. Nevertheless, identifying the precise physical mechanisms that realize the full class of logically consistent classical loops—be it non-trivial spacetime topologies or dynamical feedback architectures—remains a compelling direction for future research.

Acknowledgments. We thank Ognian Oreshkov, Alastair Abbott, Cyril Branciard, Pierre Pocreau, Marco Túlio Quintino, Nasra Daher Ahmed, Alexei Grinbaum, Kuntal Sengupta, Amin Baumeler, and Zixuan Liu for enlightening discussions. We are also grateful to Fabio

Costa, whose input helped us identify a loophole in an earlier version of our complete output-reducibility proposal and develop the non-erasing reducibility characterization (Theorem 3). We acknowledge the use of the AI assistant Claude (Anthropic), interaction with which helped identifying the loophole mentioned above, developing the non-erasing reducibility characterization and proving Theorem 5. The assistant was further used to improve the English of the manuscript. All mathematical statements and proofs were independently derived, checked, and are the sole responsibility of the authors.

H.D., A.L., and A.A. acknowledge financial support from the Government of Spain (Severo Ochoa CEX2019-000910-S, NextGenerationEU PRTR-C17.I1 and FUNQIP), Fundació Cellex, Fundació Mir-Puig, Generalitat de Catalunya (CERCA program). A.A. was also supported by the ERC AdG CERQUITE, the AXA Chair in Quantum Information Science. K.S., A.L., and E.C.B. acknowledge: This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/PAT4559623. For open access purposes, the author has applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission. A.L. acknowledges the European Union (PASQuanS2.1, 101113690). R.K. acknowledges: This work received support from the French government under the France 2030 investment plan, as part of the Initiative d’Excellence d’Aix-Marseille Université-A*MIDEX, AMX-22-CEI-01.

Proof of Theorem 3

For $n = 1$, the statement follows directly from Theorem 1: a unipartite process function is necessarily constant. Let $n > 1$. If w is a process function, then every reduction of w by a local operation is again a process function (Theorem 2). This holds, in particular, for every non-erasing operation, proving the forward implication.

For the converse we show the contrapositive: if the quasi-process $\mathbf{w}$ is not a process function, some non-erasing f_k has a reduction $\mathbf{w}^{f_k}$ that is not well defined or not a valid $(n-1)$ -partite process function, i.e. violates (ii).

Assume first that the non-valid quasi-process $\mathbf{w}$ is self-signaling, i.e. there exists k such that x_k depends on a_k , i.e. $\exists \mathbf{a}_{\setminus k}, a_k \neq a'_k$ such that $x_k := w_k(a_k, \mathbf{a}_{\setminus k}) \neq x'_k := w_k(a'_k, \mathbf{a}_{\setminus k})$. Take a function f_k such that $f_k(x'_k) := a_k$ and $f_k(x_k) := a'_k$ for $x_k \neq x'_k$. Then f_k is non-erasing (its image is $\{a_k, a'_k\}$, on which w_k is non-constant), yet the one-party loop $b = f_k(w_k(b, \mathbf{a}_{\setminus k}))$ has no solution: it takes values in $\{a_k, a'_k\}$, and $a_k \mapsto a'_k, a'_k \mapsto a_k$. Hence $\mathbf{w}^{f_k}$ is not well defined and (ii) fails.

Assume now that the non-valid quasi-process $\mathbf{w}$ is non-self-signaling, i.e. every $\mathbf{w}^{f_k}$ is well-defined. By Theorem 1, there exists a collection of local operations $\mathbf{f} = (f_k)_{k \in \mathcal{N}}, f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$, such that $\mathbf{f} \circ \mathbf{w}$ has no fixed point or several. Let us find a non-erasing operation f_k such that $\mathbf{f}_{\setminus k} \circ \mathbf{w}^{f_k}$ has also no fixed point or several.

Let $\mathcal{B}_k := \text{Im } f_k$ (a proper subset of $\mathcal{A}_k$ in general, as f_k need not be surjective) and $\mathcal{B} := \prod_k \mathcal{B}_k$. Since $\mathbf{f} \circ \mathbf{w}$ takes values in $\mathcal{B}$, all its fixed points lie in $\mathcal{B}$, and $(\mathbf{f} \circ \mathbf{w})|_{\mathcal{B}}$ has the same fixed points; in particular it still fails to have exactly one.

Let us assume that among the considered collection of $\mathbf{f}$, no f_k is non-erasing for $\mathbf{w}$. Intuitively, each f_k then severs party k 's outgoing dependences, so that no signaling loop, and hence no paradox, can be detected by reducing with it. The precise definition follows from the negation of Definition 3: for every j , every $k \neq j$, every $\mathbf{a}_{\setminus k}$ and all $x_k, x'_k, w_j(f_k(x_k), \mathbf{a}_{\setminus k, j}) = w_j(f_k(x'_k), \mathbf{a}_{\setminus k, j})$, where we have used $w_j(\mathbf{a}) = w_j(\mathbf{a}_{\setminus j})$ from non-self-signaling. Since $f_k(x_k)$ ranges over $\mathcal{B}_k$, this means that $a_k \mapsto w_j(a_k, \mathbf{a}_{\setminus k, j})$ is constant on $\mathcal{B}_k$ for every $\mathbf{a}_{\setminus k, j} \in \mathcal{B}_{\setminus k, j}$. Because $\mathcal{B} = \prod_k \mathcal{B}_k$, $\mathbf{w}$ is constant on $\mathcal{B}$. Therefore $(\mathbf{f} \circ \mathbf{w})|_{\mathcal{B}}$ is constant and always has a unique fixed point, contradicting our assumption on $\mathbf{f}$. We conclude that some f_k is non-erasing for $\mathbf{w}$.

Finally, let us show that the map $\mathbf{a} \mapsto \mathbf{a}_{\setminus k}$ is a bijection from the fixed points of $\mathbf{f} \circ \mathbf{w}$ onto those of $\mathbf{f}_{\setminus k} \circ \mathbf{w}^{f_k}$. A tuple $\mathbf{a}$ is a fixed point of $\mathbf{f} \circ \mathbf{w}$ if and only if $a_k = f_k(w_k(\mathbf{a}_{\setminus k}))$ and $\mathbf{a}_{\setminus k} = \mathbf{f}_{\setminus k}(\mathbf{w}_{\setminus k}(\mathbf{a}_{\setminus k}))$. The first equation determines a_k from $\mathbf{a}_{\setminus k}$ alone, so the map is injective; substituting it into the second gives exactly $\mathbf{a}_{\setminus k} = \mathbf{f}_{\setminus k}(\mathbf{w}_{\setminus k}^{f_k}(\mathbf{a}_{\setminus k}))$, so the image is contained in the fixed-point set of $\mathbf{f}_{\setminus k} \circ \mathbf{w}^{f_k}$. Conversely, given such a fixed point $\mathbf{a}_{\setminus k}$, setting $a_k := f_k(w_k(\mathbf{a}_{\setminus k}))$ recovers a fixed point of $\mathbf{f} \circ \mathbf{w}$.

Hence, $\mathbf{f}_{\setminus k} \circ \mathbf{w}^{f_k}$ has the same number of fixed points as $\mathbf{f} \circ \mathbf{w}$, which is not unique by assumption. By Theorem 1, $\mathbf{w}^{f_k}$ is not a valid $(n-1)$ -partite process function, and (ii) fails. $\square$

Process-matrix constraints and output-reducibility (Theorem 4)

The process-matrix constraint indexed by $\mathcal{K} \subsetneq \mathcal{N}$ traces out the complementary systems A_j^{IO} and hence *averages* over the outputs $a_j, j \notin \mathcal{K}$. By contrast, output-reducibility (Proposition 1), requires validity for every *fixed* assignment of these outputs. Indeed, for any $k_0 \notin \mathcal{K}$, the sum over a_{k_0} in Eq. (26) can be split into separate contributions, one for each value of a_{k_0} . Each contribution is precisely the output-reducibility constraint evaluated on the reduced function $\mathbf{w}^{a_{k_0}}$. Consequently, output-reducibility implies every process-matrix constraint indexed by a proper subset $\mathcal{K} \subsetneq \mathcal{N}$.

The converse does not hold constraint by constraint, because defects arising at different fixed outputs may cancel under averaging. For example, the $\mathcal{K} = \mathcal{N}$ process-matrix constraint Eq. (8) may hold despite the presence of a global loop, in quasi-process also containing internal or local loops. For example, $x_1 = a_2 \oplus a_3, x_2 = a_1 \oplus a_3, x_3 = a_1 \oplus a_2$ contains a global signaling loop, although Eq. (8) is satisfied. However, it also contains internal loops: e.g. fixing $a_3 = 0$ yields $x_1 = a_2, x_2 = a_1$. Output-reducibility detects the defect because it tests this reduced function separately.

Thus, the whole set of process-matrix constraints and output-reducibility completed with Eq. (8) encode the same validity requirement when imposed as complete families, but their individual conditions do not correspond term by term. A process-matrix constraint indexed by $\mathcal{K}$ averages over the outputs of the complementary parties $\mathcal{N} \setminus \mathcal{K}$, whereas output-reducibility fixes those outputs and tests each resulting reduced function separately.

* hippolyte.dourdent@uni-muenster.de

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Supplemental Material

Process functions as process matrices

Process functions are the deterministic classical counterpart of process matrices [8], which are the Choi representations [44] of quantum supermaps that yield valid probabilities under arbitrary quantum operations.

Each party k performs a single local quantum operation

$$M_{|i_k} : \mathcal{L}(\mathcal{H}^{A_k^I}) \times \mathcal{I}_k \rightarrow \mathcal{L}(\mathcal{H}^{A_k^O}) \times \mathcal{O}_k, \quad (15)$$

where the classical spaces $\mathcal{X}_k$ and $\mathcal{A}_k$ are replaced by spaces of linear operators $\mathcal{L}(\mathcal{H}^{A_k^I})$ and $\mathcal{L}(\mathcal{H}^{A_k^O})$. Concretely, $M_{|i_k}$ is the Choi operator [44] of a quantum instrument $M_{|i_k} := (M_{o_k|i_k}^{A_k^I A_k^O})_{o_k|i_k}$, where each element $M_{o_k|i_k}^{A_k^I A_k^O} \in \mathcal{L}(\mathcal{H}^{A_k^I A_k^O}) := \mathcal{L}(\mathcal{H}^{A_k^I} \otimes \mathcal{H}^{A_k^O})$ for any $i_k \in \mathcal{I}_k$ satisfies $M_{o_k|i_k}^{A_k^I A_k^O} \geq 0$ and $\text{Tr}_{A_k^O} \sum_{o_k} M_{o_k|i_k}^{A_k^I A_k^O} = \mathbb{1}^{A_k^I}$.

With the notations $\mathbf{A} = A_1 \dots A_n$, $A_k = A_k^I A_k^O$, $\mathbf{o} = (o_1, \dots, o_n)$ and $\mathbf{i} = (i_1, \dots, i_n)$, within the process matrix framework, the correlations established by the parties are given by the probabilities

$$P(\mathbf{o}|\mathbf{i}) = \text{Tr} \left[\left(\bigotimes_{k=1}^n M_{o_k|i_k}^{A_k} \right)^T \cdot W^{\mathbf{A}} \right] = \bigotimes_{k=1}^n M_{o_k|i_k}^{A_k} * W^{\mathbf{A}}, \quad (16)$$

where $W \in \mathcal{L}(\mathcal{H}^{\mathbf{A}})$ is the “process matrix”, and $*$ denotes the “link product” [45, 46]. For two operators $M \in \mathcal{L}(\mathcal{H}^{XY})$ and $N \in \mathcal{L}(\mathcal{H}^{YZ})$, the latter is defined as $M * N = \text{Tr}_Y[(M^{T_Y} \otimes \mathbb{1}^Z)(\mathbb{1}^X \otimes N)]$ where T_Y denotes partial transposition on Y . This product is both commutative (up to reordering of tensor products) and associative (as long as each space involved in the product appears at most twice); a tensor product $M^X \otimes N^Y$ is a special case with trivial link.

For Eq. (16) to define valid probabilities under any arbitrary operations $M_{|i_k}^{A_k}$, the process matrix $W^{\mathbf{A}}$ must be positive semidefinite $W^{\mathbf{A}} \geq 0$ and belong to a nontrivial subspace $\mathcal{L}^{\mathcal{N}}$ of $\mathcal{L}(\mathcal{H}^{\mathbf{A}})$ [29–31]:

$$W \in \mathcal{L}^{\mathcal{N}} \Leftrightarrow \forall \mathcal{K} \subseteq \mathcal{N}, \forall \mathcal{K} \neq \emptyset, \quad \prod_{k \in \mathcal{K}} [1 - A_k^O] A_{\mathcal{N} \setminus \mathcal{K}}^{I O} W = 0, \quad (17)$$

with the trace-and-replace notation

$$\begin{aligned} [1-X]W &= W -_X W, \\ {}_X W &= \frac{\mathbb{1}^X}{d_X} \otimes \text{Tr}_X W. \end{aligned} \quad (18)$$

By [15, Theorem 4], any process function $w : \mathcal{A} \rightarrow \mathcal{X}$ admits a unique encoding as a process matrix:

$$W_w^{\mathbf{A}} = \sum_{\mathbf{a}} \bigotimes_k |a_k\rangle\langle a_k|^{A_k^O} \otimes |w_k(\mathbf{a}_{\setminus k})\rangle\langle w_k(\mathbf{a}_{\setminus k})|^{A_k^I}. \quad (19)$$

Setting $|\mathcal{I}_k| = |\mathcal{O}_k| = 1$, arbitrary local operations $f_k : \mathcal{X}_k \rightarrow \mathcal{A}_k$ and $g_{j \neq k} : \mathcal{X}_j \rightarrow \mathcal{A}_j$ are encoded as quantum channels

$$M_k^{A_k} = \sum_{x_k \in \mathcal{X}_k} |x_k\rangle\langle x_k|^{A_k^I} \otimes |f_k(x_k)\rangle\langle f_k(x_k)|^{A_k^O}, \quad (20)$$

$$M_j^{A_j} = \sum_{x_j \in \mathcal{X}_j} |x_j\rangle\langle x_j|^{A_j^I} \otimes |g_j(x_j)\rangle\langle g_j(x_j)|^{A_j^O}. \quad (21)$$

From the no global loop to phase erasure constraints (Theorem 5)

Let us show that the process matrix constraint Eq. (8) under the matrix encoding Eq. (19) translates into the phase erasure conditions Eq. (9).

Proof. Denote $W_{\mathbf{w}} = \sum_{\mathbf{a} \in \mathcal{A}} P_{\mathbf{a}} \otimes Q_{\mathbf{w}(\mathbf{a})}$, where $P_{\mathbf{a}} := \bigotimes_{k \in \mathcal{N}} |a_k\rangle\langle a_k|^{A_k^O}$, $Q_{\mathbf{x}} := \bigotimes_{k \in \mathcal{N}} |x_k\rangle\langle x_k|^{A_k^I}$. For each party k , define the average of $\mathbf{a} \mapsto Q_{\mathbf{w}(\mathbf{a})}$ over a_k , and the centering C_k by

$$(\mathbb{E}_k Q_{\mathbf{w}})(\mathbf{a}) := \frac{1}{d_k} \sum_{a'_k=0}^{d_k-1} Q_{\mathbf{w}(a'_k, \mathbf{a}_{\setminus k})}, \quad C_k := \text{id} - \mathbb{E}_k. \quad (22)$$

Using the trace-and-replace map (Eq. (18)) on A_k^O and

$$\text{Tr} |a_k\rangle\langle a_k| = 1, \quad \frac{\mathbb{1}^{A_k^O}}{d_k} = \frac{1}{d_k} \sum_{a_k=0}^{d_k-1} |a_k\rangle\langle a_k|^{A_k^O}, \quad (23)$$

we obtain

$$\begin{aligned} A_k^O W_{\mathbf{w}} &= \frac{1}{d_k} \sum_{\mathbf{a}_{\setminus k}} \sum_{a'_k} P_{(a_k, \mathbf{a}_{\setminus k})} \otimes Q_{\mathbf{w}(a'_k, \mathbf{a}_{\setminus k})} \\ &= \sum_{\mathbf{a}} P_{\mathbf{a}} \otimes (\mathbb{E}_k Q_{\mathbf{w}})(\mathbf{a}). \end{aligned} \quad (24)$$

Consequently,

$$[1 - A_k^O] W_{\mathbf{w}} = \sum_{\mathbf{a}} P_{\mathbf{a}} \otimes (C_k Q_{\mathbf{w}})(\mathbf{a}). \quad (25)$$

The averaging operators commute, and hence so do the centering operators. It follows that

$$\prod_k [1 - A_k^O] W_{\mathbf{w}} = \sum_{\mathbf{a}} P_{\mathbf{a}} \otimes \left(\prod_{k \in \mathcal{N}} C_k \right) Q_{\mathbf{w}}(\mathbf{a}). \quad (26)$$

Since the operators $P_{\mathbf{a}}$ are linearly independent, Eq. (8) is equivalent to

$$\left(\prod_{k \in \mathcal{N}} C_k \right) Q_{\mathbf{w}}(\mathbf{a}) = 0 \quad \forall \mathbf{a} \in \mathcal{A}. \quad (27)$$

Moreover, $Q_{\mathbf{w}(\mathbf{a})} = \sum_{\mathbf{x} \in \mathcal{X}} \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}} Q_{\mathbf{x}}$, and the operators $Q_{\mathbf{x}}$ are linearly independent. Thus the matrix constraint separates into the scalar conditions

$$\left(\prod_{k \in \mathcal{N}} C_k \right) F_{\mathbf{x}} = 0 \quad \forall \mathbf{x} \in \mathcal{X}, \quad (28)$$

where $F_{\mathbf{x}}(\mathbf{a}) := \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}}$.

It remains to express Eq. (28) in the complex Fourier basis. Let us remark that the latter has no intrinsic physical significance here; it is merely a convenient simultaneous eigenbasis for the averaging operators $\mathbb{E}_k$. The essential fact is that $C_k = \text{id} - \mathbb{E}_k$ projects onto the subspace of functions that are non-constant in a_k . Thus $\prod_k C_k$ selects the component that depends jointly on all outputs.

Label each output alphabet as $\mathcal{A}_k \simeq \mathbb{Z}_{d_k}$ and define the characters

$$\chi_{\boldsymbol{\nu}}(\mathbf{a}) := \prod_{k=1}^n e^{2\pi i \nu_k a_k / d_k}, \quad \nu_k \in \{0, \dots, d_k - 1\}. \quad (29)$$

The Fourier coefficients of $F_{\mathbf{x}}$ are

$$\hat{F}_{\mathbf{x}}(\boldsymbol{\nu}) := \frac{1}{|\mathcal{A}|} \sum_{\mathbf{a} \in \mathcal{A}} F_{\mathbf{x}}(\mathbf{a}) \chi_{\boldsymbol{\nu}}(\mathbf{a}). \quad (30)$$

Orthogonality, $\frac{1}{d_k} \sum_{a_k=0}^{d_k-1} e^{2\pi i \nu_k a_k / d_k} = \delta_{\nu_k, 0}$, shows that $\mathbb{E}_k$ retains precisely the Fourier modes with $\nu_k = 0$. Consequently, $C_k = \text{id} - \mathbb{E}_k$ retains precisely those modes with $\nu_k \neq 0$. The product $\prod_k C_k$ therefore retains exactly the Fourier modes for which every frequency is nonzero.

Equation (28) is thus equivalent to

$$\widehat{F}_{\mathbf{x}}(\boldsymbol{\nu}) = 0 \quad \forall \mathbf{x} \in \mathcal{X}, \quad \forall \boldsymbol{\nu} \text{ such that } \nu_k \neq 0 \quad \forall k. \quad (31)$$

Substituting $F_{\mathbf{x}}(\mathbf{a}) = \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}}$ and omitting the irrelevant normalization factor $|\mathcal{A}|^{-1}$ gives

$$\sum_{\mathbf{a}} \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}} \prod_{k=1}^n e^{2\pi i \nu_k a_k / d_k} = 0, \quad (32)$$

for every $\mathbf{x}$ and every $\boldsymbol{\nu} \in \prod_k \{1, \dots, d_k - 1\}$.

This is precisely Eq. (9). $\square$

For binary outputs, the result immediately reduces to the parity condition, since the only nonzero frequency is $\nu_k = 1$ and $e^{2\pi i a_k / 2} = (-1)^{a_k}$. Hence Eq. (9) becomes

$$\sum_{\mathbf{a}} \delta_{\mathbf{w}(\mathbf{a}), \mathbf{x}} (-1)^{\sum_k a_k} = 0 \quad \forall \mathbf{x}. \quad (33)$$

Proof of Theorem 6

We show that, for a classical process matrix $W_{\mathbf{w}}$, the complete family of process-matrix constraints

$$\prod_{k \in \mathcal{K}} [1 - A_k^O] A_{\mathcal{N} \setminus \mathcal{K}}^{IO} W_{\mathbf{w}} = 0 \quad \forall \emptyset \neq \mathcal{K} \subseteq \mathcal{N} \quad (34)$$

is equivalent to the phase-erasure conditions

$$\sum_{\mathbf{a} \in \mathcal{A}} \prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k} e^{2\pi i \nu_k a_k / d_k} = 0 \quad (35)$$

for every nonempty $\mathcal{K} \subseteq \mathcal{N}$, every $\mathbf{x}_{\mathcal{K}} \in \mathcal{X}_{\mathcal{K}}$, and every collection of nonzero frequencies $\boldsymbol{\nu}_{\mathcal{K}} \in \prod_{k \in \mathcal{K}} \{1, \dots, d_k - 1\}$.

Proof. The argument is the same as for the full-set constraint $\mathcal{K} = \mathcal{N}$. Fix a nonempty subset $\mathcal{K} \subseteq \mathcal{N}$. Applying $A_{\mathcal{N} \setminus \mathcal{K}}^{IO}(\cdot)$ traces and replaces the systems of the complementary parties. Ignoring nonzero normalization factors and identity operators, the resulting diagonal matrix has coefficients $F_{\mathbf{x}_{\mathcal{K}}}(\mathbf{a}_{\mathcal{K}}) := \sum_{\mathbf{a}_{\mathcal{N} \setminus \mathcal{K}}} \prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k}$. Thus, the process-matrix constraint indexed by $\mathcal{K}$ becomes

$$\left(\prod_{k \in \mathcal{K}} C_k \right) F_{\mathbf{x}_{\mathcal{K}}} = 0 \quad \forall \mathbf{x}_{\mathcal{K}}, \quad (36)$$

where $C_k = \text{id} - \mathbb{E}_k$ removes the part that is constant in a_k .

As shown for $\mathcal{K} = \mathcal{N}$, the product $\prod_{k \in \mathcal{K}} C_k$ selects precisely the Fourier terms whose frequencies are nonzero for every $k \in \mathcal{K}$. The above condition is therefore equivalent to

$$\sum_{\mathbf{a}_{\mathcal{K}}} F_{\mathbf{x}_{\mathcal{K}}}(\mathbf{a}_{\mathcal{K}}) \exp \left(2\pi i \sum_{k \in \mathcal{K}} \frac{\nu_k a_k}{d_k} \right) = 0 \quad (37)$$

for every $\mathbf{x}_{\mathcal{K}}$ and every $\nu_k \in \{1, \dots, d_k - 1\}$, $k \in \mathcal{K}$.

Substituting the definition of $F_{\mathbf{x}_{\mathcal{K}}}$ and combining the sums over $\mathbf{a}_{\mathcal{K}}$ and $\mathbf{a}_{\mathcal{N} \setminus \mathcal{K}}$ gives

$$\begin{aligned} 0 &= \sum_{\mathbf{a} \in \mathcal{A}} \left(\prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k} \right) \exp \left(2\pi i \sum_{k \in \mathcal{K}} \frac{\nu_k a_k}{d_k} \right) \\ &= \prod_{k \in \mathcal{K}} \delta_{w_k(\mathbf{a}), x_k} e^{2\pi i \nu_k a_k / d_k}. \end{aligned} \quad (38)$$

Since this holds for every nonempty $\mathcal{K} \subseteq \mathcal{N}$, the complete family of process-matrix constraints is equivalent to a complete family of phase-erasure conditions. $\square$