A combinatorial proof of Louck's conjecture

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The following conjecture was recently proposed by J.D.Louck [2] (Oberwolfach, July 1988):

> For (complex) parameters $\mathbf{y} = (y_1, \ldots, y_n)$, integers $n-$tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, and variable u let
>
> $$f_{\boldsymbol{\alpha}}(u, \mathbf{y}) := u(1 + u + \boldsymbol{\alpha} \cdot \mathbf{y})_{|\boldsymbol{\alpha}|-1} \quad ,$$
>
> where $\boldsymbol{\alpha} \cdot \mathbf{y} = \alpha_1 y_1 + \ldots + \alpha_n y_n$ and $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$. [As usual, $(x)_m$ denotes the rising factorial, i.e. $(x)_m = x(x+1) \cdots (x+m-1)$ for $m > 0$ and $(x)_0 = 1$. In the case $\boldsymbol{\alpha} = (0, \ldots, 0) = \mathbf{0}$ put $f_{\mathbf{0}}(u, \mathbf{y}) = 1$]. Then
>
> $$f_{\boldsymbol{\alpha}}(u + v, \mathbf{y}) = \sum_{\boldsymbol{\beta}} \prod_{1 \le i \le n} \binom{\alpha_i}{\beta_i} f_{\boldsymbol{\beta}}(u, \mathbf{y}) f_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(v, \mathbf{y}) \quad ,$$
>
> where v is another (independent) variable, and where the summation runs over all $\boldsymbol{\beta} \in \mathbf{N}^n$ such that $0 \le \boldsymbol{\beta} \le \boldsymbol{\alpha}$ in componentwise order, $\boldsymbol{\alpha} - \boldsymbol{\beta} = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$.

Louck observed in [3] that (an independent proof of) this identity would lead to a fairly elementary approach to Mellin's formula [4] for the series expansion of the principal solution of an algebraic equation with variable coefficients. In this note a proof of this convolution identity will be given by combinatorial means, i.e. by interpreting the polynomials involved as generating polynomials for a class of combinatorial structures, so that the convolution identity expresses a factorization property of these structures. The proof of this factorization property can then be given without any reference to generating functions.

Let $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ be an infinite sequence of variables. For any finite set $A \subset \mathbf{N}$ let $\sigma_A(\mathbf{x}) := \sum\{x_i \,;\, i \in A\}$ and

$$E_A(u,\mathbf{x}) := (u + \sigma_A(\mathbf{x}))_{\sharp A} \quad , \quad F_A(u,\mathbf{x}) := u(1 + u + \sigma_A(\mathbf{x}))_{\sharp A - 1} \quad ,$$

where u is a variable and where $\sharp A$ denotes the cardinality of A (we put $F_\emptyset(u,\mathbf{x}) = 1$). Thus $E_A(u,\mathbf{x})$ and $F_A(u,\mathbf{x})$ are polynomials of degree $\sharp A$ in u with coefficients in $\mathbf{Z}[\mathbf{x}]$.

The following result will be proved by using a combinatorial model for these polynomials:

Theorem 1 $\quad F_A(u+v,\mathbf{x}) = \sum_{B \cup C = A} F_B(u,\mathbf{x}) F_C(v,\mathbf{x})$

Here v is another variable. The summation on the r.h.s. runs over all ordered bipartitions (B,C) of A. Before introducing the combinatorial model I will show that this theorem implies Louck's conjecture.

For this purpose, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ be given and let $\mathbf{y} = (y_1, \ldots, y_n)$ be an n-tuple of variables. For each set $A \subset \mathbf{N}$ of cardinality $\sharp A = |\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$ let $\pi_A^{\boldsymbol{\alpha}} = (A_1, \ldots, A_n)$ denote the unique ordered partition of A which satisfies $\sharp A_i = \alpha_i$ $(1 \le i \le n)$ and where all elements of A_i are less than all elements of A_{i+1} $(1 \le i < n)$.

If (B,C) is an ordered bipartition of A, then there are (unique) $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbf{N}^n$ such that $|\boldsymbol{\beta}| = \sharp B$, $|\boldsymbol{\gamma}| = \sharp C$, $\boldsymbol{\beta} + \boldsymbol{\gamma} = \boldsymbol{\alpha}$ and

$$\pi_B^{\boldsymbol{\beta}} = (B \cap A_1, \ldots, B \cap A_n) \ , \ \pi_C^{\boldsymbol{\gamma}} = (C \cap A_1, \ldots, C \cap A_n).$$

Conversely, any pair $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in (\mathbf{N}^n)^2$ such that $\boldsymbol{\beta} + \boldsymbol{\gamma} = \boldsymbol{\alpha}$ results from precisely $\prod_{1 \le i \le n} \binom{\alpha_i}{\beta_i}$ ordered bipartitions (B,C) of A in this way. Now note that

$$f_{\boldsymbol{\alpha}}(u,\mathbf{y}) = F_A(u,\mathbf{x})\big|_{\mathbf{x} \xleftarrow{\boldsymbol{\alpha}} \mathbf{y}} \quad ,$$

where the substitution $\mathbf{x} \xleftarrow{\boldsymbol{\alpha}} \mathbf{y}$ means that variable x_i has to be replaced by y_j precisely if $i \in A_j$, where A_j is the $j-$th block of $\pi_A^{\boldsymbol{\alpha}}$. Thus Louck's conjecture follows from the identity given in the theorem by applying $\mathbf{x} \xleftarrow{\boldsymbol{\alpha}} \mathbf{y}$ to both sides and by noting that

$$F_B(u,\mathbf{x})\big|_{\mathbf{x} \xleftarrow{\boldsymbol{\alpha}} \mathbf{y}} = F_B(u,\mathbf{x})\big|_{\mathbf{x} \xleftarrow{\boldsymbol{\beta}} \mathbf{y}} = f_{\boldsymbol{\beta}}(u,\mathbf{y})$$

$$F_C(v,\mathbf{x})\big|_{\mathbf{x} \xleftarrow{\boldsymbol{\alpha}} \mathbf{y}} = F_C(v,\mathbf{x})\big|_{\mathbf{x} \xleftarrow{\boldsymbol{\gamma}} \mathbf{y}} = f_{\boldsymbol{\gamma}}(v,\mathbf{y})$$

for any bipartition (B,C) of A. $\diamond$

For any set A let $\mathcal{E}_A$ denote the set of all pairs $\phi = (f,g)$, where $f : A_f \to A$ and $g : A_g \to A$ are functions defined on the parts of some bipartition (A_f, A_g) of A, and where f is (weakly) *decreasing*, i.e. $f(a) \le a$ for all $a \in A_f$.

Looking at the pair $\phi = (f, g)$ as *one* endofunction of A, we see that there are two types of connected components induced by these mapping(s):

- components of the first kind, where the set of recurrent elements consists of just one fixed point of f;

- components of the second kind, where the set of recurrent elements contains at least one element from A_g.

[Note that the monotonicity of f disables the existence of cycles within A_f alone other than fixed points]. Thus each $\phi \in \mathcal{E}_A$ can also be written as a pair $\phi =< \phi', \phi'' >$, where ϕ' comprises all the ϕ-components of the first kind and ϕ'' comprises all the ϕ-components of the second kind. This distiction leads us to introduce

$$\mathcal{E}'_A := \{\phi \in \mathcal{E}_A \,;\, \phi'' = \emptyset\} \ , \ \mathcal{E}''_A := \{\phi \in \mathcal{E}_A \,;\, \phi' = \emptyset\} \ ,$$

and in a suggestive way one may write $\mathcal{E}_A = \bigcup_{B \cup C = A} \mathcal{E}'_B \times \mathcal{E}''_C$. We will now associate a valuation with each $\phi = (f, g) \in \mathcal{E}_A$: let u and $\mathbf{x} = (x_0, x_1, \ldots)$ be variables; we then put

$$w_{u,\mathbf{x}}(\phi) := u^{fix(f)} \mathbf{x}_g \ \ ,$$

where $fix(f)$ is the number of fixed points of f, and where

$$\mathbf{x}_g := \prod_{a \in A_g} x_{g(a)} \ (= \prod_{a \in A} x_a^{\sharp g^{-1}(a)}).$$

Note that $fix(f)$ equals $comp(\phi')$, the number of connected components of the first kind of ϕ. We may also choose to write $\mathbf{x}_\phi$ in place of $\mathbf{x}_g$ (note that *both* kinds of components contribute to this quantity), so that $w_{u,\mathbf{x}}(\phi)$ will also be written as $u^{comp(\phi')}\mathbf{x}_\phi$.

This valuation is multiplicative w.r.t. connected components, i.e. if $\phi \in \mathcal{E}_A, \psi \in \mathcal{E}_B$ (where $A \cap B = \emptyset$), then the union $\phi \cup \psi$ of the connected components from both constituents defines an element of $\mathcal{E}_{A \cup B}$ in the obvious way, and $w_{u,\mathbf{x}}(\phi \cup \psi) = w_{u,\mathbf{x}}(\phi) \cdot w_{u,\mathbf{x}}(\psi)$. In particular: $w_{u,\mathbf{x}}(\phi) = w_{u,\mathbf{x}}(\phi') \cdot w_{u,\mathbf{x}}(\phi'')$ for each $\phi \in \mathcal{E}_A$.

Proposition 1 $\quad E_A(u, \mathbf{x}) = \sum \{w_{u,\mathbf{x}}(\phi) \,;\, \phi \in \mathcal{E}_A\}$

Proof: Let us assume, for simplicity of notation, that $A = \{1, 2, \ldots, n\}$. The elements $\phi \in \mathcal{E}_A$ are precisely the objects constructed by the following nondeterministical procedure, consisting of n rounds to be played:

for $i = 1, 2, \ldots, n$ do:

> select one of the alternatives "$i \in A_f$" or "$i \in A_g$";
>
> if "$i \in A_f$" has been selected: choose $j \in \{1, 2, \ldots, i\}$ and put $f(i) := j$;
>
> if "$i \in A_g$" has been selected: choose $j \in \{1, 2, \ldots, n\}$ and put $g(i) := j$;

During the i-th round, if alternative "$i \in A_f$" has been selected, there are i possibilities for the choice of $f(i)$, one of them, $f(i) = i$, contributing weight u, the $i-1$ other ones each contributing weight 1 to $w_{u,\mathbf{x}}(\phi)$. If alternative "$i \in A_g$" has been selected, each element j of A can be chosen as $g(i)$, contributing $x_{g(i)}$ to the weight $w_{u,\mathbf{x}}(\phi)$, which gives a total of $\sigma_A(\mathbf{x})$ for these possibilities. Thus the total contribution to the weight of all $\phi \in \mathcal{E}_A$ during the i-th round is $u + i - 1 + \sigma_A(\mathbf{x})$, and because the selections made during different rounds are independent, $\sum\{w_{u,\mathbf{x}}(\phi) \,; \phi \in \mathcal{E}_A\}$ is the product, taken over $1 \leq i \leq n$, of these contributions. $\diamond$

A slight modification of the argument just given leads to a combinatorial interpretation of the polynomials $F_A(u, \mathbf{x})$. For this purpose let

$$\mathcal{F}_A := \{\phi \in \mathcal{E}_A \,; \min(A) \in A_f\} \ ,$$

i.e. we consider the construction given above, but we do not allow any choice in the first round: the minimum element of A *must* be a fixed point of f. Then the contribution from the first round is just u instead of $u + \sigma_A(\mathbf{x})$; all the other contributions remain the same, hence:

Proposition 2 $\quad F_A(u, \mathbf{x}) = \sum\{w_{u,\mathbf{x}}(\phi) \,; \phi \in \mathcal{F}_A\}$

The next proposition contains the crucial combinatorial argument needed for the proof of the theorem:

Proposition 3 *There is a bijection* $\mathcal{E}'_A \to \mathcal{F}_A : \phi \mapsto \psi$ *which respects the valuation, i.e.*

$$w_{u,\mathbf{x}}(\phi) = w_{u,\mathbf{x}}(\psi) \ .$$

Proof: Let $\phi = (f, g) = < \{\phi_1, \ldots, \phi_s\}, \emptyset > \in \mathcal{E}'_A$, where all the connected components $\phi_1, \ldots, \phi_s$ of ϕ have a rooted tree structure, with the fixed points of f as their roots. Then ϕ belongs to $\mathcal{F}_A$ precisely if $\min(A)$ is one of these roots. In this case ϕ will be mapped onto itself, i.e. $\psi = \phi$.

Assume now that $a_0 := \min(A)$ is *not* a root of one of these trees, then a_0 appears as a non-root node in one of the components, ϕ_1 say, and certainly belongs to A_g. [Remember that f is a decreasing map, thus $a_0 \in A_f$ would imply $f(a_0) = a_0$, which is not the case]. Thus there is a unique sequence in ϕ_1:

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \ldots \longrightarrow a_{k-1} \longrightarrow a_k$$

where a_k is the root of ϕ_1 (i.e. $a_k \in A_f$ and $f(a_k) = a_k$) and where each arrow $a_i \longrightarrow a_{i+1}$ is either an f-arrow (i.e. $a_i \in A_f$ and $f(a_i) = a_{i+1}$) or a g-arrow (i.e. $a_i \in A_g$ and $g(a_i) = a_{i+1}$), $0 \leq i < k$. Note that there is *at least one* g-arrow, since $a_0 \in A_g$.

The obvious goal is now to make $a_0 = \min(A)$ an $\bar{f}$-fixed point of the structure $\psi = (\bar{f}, \bar{g})$. This is easily achieved by specifying

$$A_{\bar{f}} = (A_f \setminus \{a_k\}) \cup \{a_0\} \ , \ A_{\bar{g}} = (A_g \setminus \{a_0\}) \cup \{a_k\} \ ,$$

and defining $\bar{f}$ by

$$\bar{f}(a) := f(a) \text{ for } a \in A_f \setminus \{a_k\} \ , \ \bar{f}(a_0) := a_0 \ .$$

Thus $\bar{f}$ certainly is a decreasing map.

The definition of $\bar{g}$ employs a technique well-known in constructive combinatorics: transforming linear arrangements of elements of some totally ordered set bijectively into permutations of that set, where the number of left-to-right-maxima of each linear arrangement equals the number of cycles of the corresponding permutation. (This is essentially Foata's *fundamental transformation* [1]; an illustration is given at the end of this note). This transformation will be applied to the sequence $a_0, a_1, \ldots, a_k$ above, but some care has to be taken because we want to be sure to create only cycles containing at least one element of $A_{\bar{g}}$ each. Thus let $0 = i_0 < i_1 < i_2 < \ldots < i_m < k$ be the subsequence of all those indices $i \in \{0, 1, \ldots, k\}$ s.th. $a_i \in A_g$. In addition put $i_{m+1} = k$ and let $b_\mu := a_{i_\mu}$ $(1 \leq \mu \leq m+1)$. Let $0 = r_0 < 1 = r_1 < r_2 < \ldots < r_t \leq m+1$ be the index-subsequence for the left-to-right-maxima of the sequence $b_0, b_1, \ldots, b_{m+1}$, i.e. the sequence of indices r s.th. $b_r = \max\{b_\rho \ ; \rho \leq r\}$. $\bar{g}$ will agree with g on $A_g \setminus \{b_{r_1 - 1}, b_{r_2 - 1}, \ldots, b_{r_t - 1}\}$, and furthermore we define

$$\bar{g}(b_{r_j - 1}) := g(b_{r_{j-1} - 1}) \ (1 < j \leq t) \ , \ \bar{g}(a_k) := g(b_{r_t - 1}) \ .$$

This construction certainly satisfies $\mathbf{x}_g = \mathbf{x}_{\bar{g}}$; furthermore the elements of the sequence $a_1, a_2, \ldots, a_k$ are now arranged in t cycles of $\bar{f}$- and $\bar{g}$-arrows, where each cycle contains at least one element from $A_{\bar{g}}$ (in fact: each one of the left-to-right-maxima $b_{r_1} < b_{r_2} < \ldots < b_{r_t}$ belongs to a different one of the t cycles).

Looking now at $\phi = (f, g)$ and $\psi = (\bar{f}, \bar{g})$ globally, we can state the following result: components $\phi_2, \ldots, \phi_s$ are constituents of both ϕ and ψ. Component ϕ_1 of ϕ decomposes into one component $\bar{\phi}_1$ of the first kind (the one with $a_0 = \bar{f}(a_0) = \min(A)$ as its root), and $t (\geq 1)$ components $\phi_1^{(1)}, \ldots, \phi_1^{(t)}$ of the second kind (where the recurrent elements of $\phi_1^{(i)}$ are the elements of the $(\bar{f}, \bar{g})$-cycle containing $b_{r_i}, 1 \leq i \leq t$). We thus have:

$$\begin{aligned}
\psi &= (\bar{f}, \bar{g}) = < \psi', \psi'' > \text{ with} \\
\psi' &= \{\bar{\phi}_1, \phi_2, \ldots, \phi_s\} \ , \ \psi'' = \{\phi_1^{(1)}, \ldots \phi_1^{(t)}\} \text{ and} \\
w_{u,\mathbf{x}}(\phi) &= u^{comp(\phi')} \mathbf{x}_g = u^{comp(\psi')} \mathbf{x}_{\bar{g}} = w_{u,\mathbf{x}}(\psi) \ .
\end{aligned}$$

ψ belongs to $\mathcal{F}_A$ because $a_0 = \min(A)$ is the root of the component $\bar{\phi}_1$. Finally it should be remarked that the mapping $\phi \to \psi$ can be reversed for precisely the same reason as for the so-called fundamental transformation. $\diamond$

Proof of the theorem: By proposition 2, $F_A(u+v,\mathbf{x})$ is the generating polynomial for structures $\phi \in \mathcal{F}_A$ under the valuation

$$w_{u+v,\mathbf{x}}(\phi) = (u+v)^{comp(\phi')}\mathbf{x}_\phi .$$

Putting a weight $u+v$ on each of the ϕ'-components means that each ϕ'-component is given weight u or weight v in all $(= 2^{comp(\phi')})$ possible ways, and then summing over all these possibilities. Collecting under the name ϕ'_u (ϕ'_v resp.) all the ϕ'-components which receive weight u (v resp.) we may write

$$F_A(u+v,\mathbf{x}) = \sum \{ u^{comp(\phi'_u)} v^{comp(\phi'_v)} \mathbf{x}_\phi \} ,$$

where the sum on the r.h.s. is taken over all $< \phi'_u, \phi'_v, \phi'' >$ such that $\phi'_u \in \mathcal{E}'_{B_u}$, $\phi'_v \in \mathcal{E}'_{B_v}$, $\phi'' \in \mathcal{E}''_C$ for some tripartition (B_u, B_v, C) of A, and such that $\phi =< (\phi'_u \cup \phi'_v), \phi'' > \in \mathcal{F}_A$. The last condition says that $\min(A)$ is a recurrent element (fixed point) of either ϕ'_u or ϕ'_v:

> - in the first case, $\psi := < \phi'_u, \phi'' >$ belongs to $\mathcal{F}_{B_u \cup C}$, and via bijection (prop.3) ϕ'_v is equivalent to some χ in $\mathcal{F}_{B_v}$ s.th. $w_{v,\mathbf{x}}(\phi'_v) = w_{v,\mathbf{x}}(\chi)$;
>
> - in the second case, $\chi := < \phi'_v, \phi'' >$ belongs to $\mathcal{F}_{B_v \cup C}$, and via bijection (prop.3) ϕ'_u is equivalent to some ψ in $\mathcal{F}_{B_u}$ s.th. $w_{u,\mathbf{x}}(\phi'_u) = w_{u,\mathbf{x}}(\psi)$.

Note that in both cases we have

$$u^{comp(\phi'_u)} v^{comp(\phi'_v)} \mathbf{x}_\phi = w_{u,\mathbf{x}}(\phi'_u) \cdot w_{v,\mathbf{x}}(\phi'_v) \cdot w_{\bullet,\mathbf{x}}(\phi'') = w_{u,\mathbf{x}}(\psi) \cdot w_{v,\mathbf{x}}(\chi) \ .$$

ψ belongs to some $\mathcal{F}_B$ and χ belongs to some $\mathcal{F}_D$ where (B, D) is a bipartition of A, with $\min(B) < \min(D)$ in the first case and $\min(D) < \min(B)$ in the second. This mapping

$$< \phi'_u, \phi'_v, \phi'' > \mapsto (\psi, \chi)$$

is reversible, as can be routinely checked. This proves the theorem. $\diamond$

Concluding remarks: Louck's conjecture has been proved independently by P. Paule [5] , who used a suitable version of multvariable Lagrange-inversion, and by J. Zeng [8], who use induction. The combinatorial proof given above resembles the one I had given earlier [6] in order to combinatorially explain a result on rooted trees that G. Kreweras had obtained by inductive methods. In fact, a common setting for both of these proofs can be established, with the extra benefit that many of the classical convolution identities similar to the one proved here can be derived via specialization within a common combinatorial setting; the details are given in [7].

References

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