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Recent Results in Partition Theory

Ramsey's theorem: (infinite version)

$$\forall \delta, k \in \mathbb{N}$$
$$\forall \Delta: \binom{\mathbb{N}}{k} \to \{1,\dots,\delta\}$$
$$\exists M \in \binom{\mathbb{N}}{\aleph_0} : \Delta\restriction_{\binom{M}{k}} = \text{const}$$

Ramsey's theorem: (finite version)

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Graham-Leeb-Rothschild: Let $F = GF(q)$, $F\binom{n}{k}$ = the set of all k-dimensional subspaces of F^n.

$$\forall \delta, k, m \in \mathbb{N}$$
$$\exists n \in \mathbb{N}$$
$$\forall \Delta: F\binom{n}{k} \to \{1,\dots,\delta\}$$
$$\exists M \in F\binom{n}{m} : \Delta\restriction_{F\binom{M}{k}} = \text{const}$$

What is the situation for the category FAB of finite abelian groups?

A similar theorem is not true in this generality!

Thus the following question arises:

For which finite abelian groups K is the following valid?

$$\forall M \in \text{FAB} \ \forall \delta \in \mathbb{N}$$
$$\exists N \in \text{FAB}$$
$$\forall \Delta: \binom{N}{K} \to \{1,\dots,\delta\}$$
$$\exists \tilde{M} \in \binom{N}{M} : \Delta\restriction_{\binom{\tilde{M}}{K}} = \text{const}$$

If this holds, then one says, that FAB has the *partition property* with respect to K.

Definition: If G is a group, and f is an automorphism of some subgroup of G , then f is called a *local automorphism*. G has *full symmetry* iff every local automorphism of G has a total extension.

Examples: $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has full symmetry

$\mathbb{Z}_2 \oplus \mathbb{Z}_4$ has not.

Theorem (Voigt): FAB has the partition property with respect to K iff K has full symmetry.

[note: "=>" is an easy exercise]

Note that this theorem is also valid (with the obvious definition of full symmetry) for the categories of

sets (Ramsey),

graphs (note that only the complete graphs and their complements have full symmetry!),

unary algebras,

trees,

.
.
.

but unfortunately not in general:

Fact: The category DIST of finite distributive lattices does not have the partition property with respect to

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II Colorings with arbitrary many colors, equivalence relations, collection of attributes

If we consider - in contrast to the situations in the above theorems - colorings Δ with arbitrarily many colors, i.e. equivalence relations or collections of "attributes", then we find in the simplest case

> If $\sim$ is an equivalence relation on $\mathbb{N}$, then there exists an infinite subset $M \subseteq N$ such that
> - $\sim$ is constant on M, or
> - $\sim$ is injective on M.
>
> (trivial observation!)

Now look at arbitrary equivalence relations on $\binom{\mathbb{N}}{k}$ for $k \geq 1$.
There is one natural way to define attributes for k-sets:

Example: Let $k \geq 7$ and let for $X = \{x_1,\dots,x_k\}_<$, $Y = \{y_1,\dots,y_k\}_<$:

$X \sim_{\{4,7\}} Y$ iff $x_4 = y_4$ and $x_7 = y_7$
(i.e. X and Y coincide on the positions $I = \{4,7\}$).

Any subset I of $\{1,\dots,k\}$ can serve for the definition of an "attribute" $\sim_I$ this way (by the coincidence of positions in I).

Theorem (Erdös-Rado): For any $k \geq 1$, for any equivalence relation $\sim$ on $\binom{\mathbb{N}}{k}$, there exists some $M \in \binom{\mathbb{N}}{\aleph_0}$ and some $I \subseteq \{1,2,\dots,k\}$ such that for all $X,Y \in \binom{M}{k}$ it holds that

$$X \sim Y \quad \text{iff} \quad X \sim_I Y$$

Thus for $k \in \mathbb{N}$ there are 2^k canonical cases which may be described combinatorially.

An analogous result holds for vectorspaces: the finitely many canonical can be given explicitely.