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Lagrange Inversion in infinitely many variables

A *series* $\alpha := \Sigma a_i x^i, i \in \mathbb{Z}$, is the row-vector $(a_i), i \in \mathbb{Z}$.
The symbol $\langle i | \alpha \rangle$ denotes the i-th component of α .
A *Laurent series* is a series α such that $\mathrm{supp}(\alpha)$ has a minimum. A Laurent series is a *power series* if $\mathrm{supp}(\alpha) \subseteq \mathbb{N}$.

The product rule. If α is a Laurent series, the *semicirculant (or Appell) matrix* $S(\alpha)$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix whose i-th row is $x^i \alpha$, $i \in \mathbb{Z}$, that is

row index

$$\langle j | S(\alpha) | i \rangle = \langle j | x^i \alpha \rangle = \langle j-i | \alpha \rangle$$

column-index

$S(\alpha)$ is an upper triangular matrix, and

$$S(\alpha) \times S(\beta) = S(\alpha\beta) .$$

The composition rule. If α is a Laurent series, the *homogeneous (or Jabotinski) matrix* $H(\alpha)$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix whose i-th row is α^i , that is

$$\langle j | H(\alpha) | i \rangle = \langle j | \alpha^i \rangle .$$

If α has positive degree, then $H(\beta)$ is an upper triangular matrix, $\alpha \bullet \beta$ is defined, and moreover

$$H(\alpha) \times H(\beta) = H(\alpha \bullet \beta) .$$

The Schur-Jabotinski inversion rule. If α is a power series which admits compositional inverse $\tilde{\alpha}$ (i.e. if α has degree 1), then for every $m, n \in \mathbb{Z}$, $m \neq 0$:

$$\langle m | \tilde{\alpha}^n \rangle = \frac{n}{m} \langle -n | \alpha^{-m} \rangle .$$

Recursive matrices. If α,β are non-zero Laurent series, the *recursive matrix* $R(\alpha,\beta)$ is the $\mathbb{Z}\times\mathbb{Z}$ matrix whose i-th row is $\alpha^i\beta$, $i\in\mathbb{Z}$, that is:

$$\langle j|R(\alpha,\beta)|i\rangle = \langle j|\alpha^i\beta\rangle \quad .$$

The series α will be called the *recurrence rule*, and β will be the *boundary value*. We have:

$$S(a) = R(x,\alpha)$$

$$H(a) = R(\alpha,1) \quad .$$

If α has positive degree, then $R(\alpha,\beta)$ is an upper triangular matrix and for every Laurent series γ,δ :

Theorem 1: $$R(\gamma,\delta)\times R(\alpha,\beta) = R(\gamma\circ\alpha,(\delta\circ\alpha)\beta) \quad .$$

The pivoting operator. The S-J-inversion rule can be rewritten as follows:

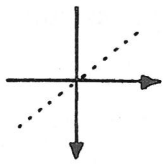
$$\langle m|\tilde{\alpha}^n\rangle = \langle -n|\alpha^{-m}\frac{x}{\alpha}D\alpha\rangle$$

This formula holds even if $m=0$.

If M is a $\mathbb{Z}\times\mathbb{Z}$ matrix, the matrix TM is defined as follows:

$$\langle i|TM|j\rangle = \langle -j|M|-i\rangle \; ,$$

that is, TM is obtained by pivoting M around the "secondary diagonal"



If α is a power series with compositorial inverse $\tilde{\alpha}$, set $P(\alpha) := \frac{x}{\alpha}D\alpha$.

The inversion rule can be restated as follows:

Theorem 2: $$R(\tilde{\alpha},1) = TR(\alpha,P(\alpha)) \quad .$$

A generalization. Let α,β be non-zero Laurent series; then $TR(\alpha,\beta)$ is a recursive matrix if and only if α admits compositorial inverse $\tilde{\alpha}$, and moreover

Theorem 3: $$TR(\alpha,\beta) = R(\tilde{\alpha},P(\alpha)\beta \circ \tilde{\alpha}) .$$

Theorems 1,2,3 can be stated and proved for series with coefficients in a commutative ring with unity of any characteristic (M. Barnabei, A. Brini, G. Nicoletti: *Recursive Matrices and Umbral Calculus*, J. Algebra, 1982).

Moreover, they suggest possible generalizations for series in infinitely many variables over a commutative ring with unity, of any characteristic.

In order to do this, we have to give suitable generalizations of the notions of series, vector series, recursive matrices, and of the operation P.

Monomials and degrees. Let S be a set of any cardinality, $S \neq \emptyset$.
Let $\underline{D}^+$ and $\underline{D}$ denote the free abelian monoid and the free abelian group over S , respectively, that is

$$\underline{D}^+ := \{\underline{d} : S \to \mathbb{N} , \text{supp}(\underline{d}) \text{ is finite}\}$$

$$\underline{D} := \{\underline{d} : S \to \mathbb{Z} , \text{supp}(\underline{d}) \text{ is finite}\}$$

The elements of $\underline{D}$ will be called *degrees*.
$\underline{D}^+$ and $\underline{D}$ are naturally structured as complete, locally finite lattices by setting

$$\underline{f} \leq \underline{g} \quad \text{wherever } \forall s \in S : f_s \leq g_s .$$

The *weigth* of a degree $\underline{f}$ is $w(\underline{f}) := \Sigma f_s$

A *monomial* in the degree $\underline{d} \in \underline{D}$ is the formal writing

$$a x^{\underline{d}} := a \prod_{s \in S} x_s^{d_s}$$

where a belongs to a commutative integral domain $\mathbb{A}$ of any characteristic. $\mathbb{U}$ will be the group of units of $\mathbb{A}$.

Series. A series $\alpha := \Sigma a_{\underline{d}} x^{\underline{d}}$, $\underline{d} \in \underline{D}$ is the row-vector $(a_{\underline{d}})$, $\underline{d} \in \underline{D}$, $a_{\underline{d}} \in \mathbb{A}$. A *Laurent series* is a series $\alpha = (a_{\underline{d}})$ such that the map $\underline{d} \overset{\alpha}{\longmapsto} a_{\underline{d}}$ has support admitting a lower bound. A Laurent series is a *power series* if $\mathrm{supp}(\alpha) \subseteq \underline{D}^t$.

A series whose support admits a minimum element will be called a *principal series* .

The usual sum and product can be defined over power on Laurent series.

A power series α admits multiplicative inverse if and only if

$$\langle \underline{0} | \alpha \rangle \in \mathbb{U}$$

A Laurent series α admits multiplicative inverse if and only if it is principal with leading coefficients in $\mathbb{U}$.

The *weight* of a series α is the integer

$$w(\alpha) := \min\{w(\underline{d}), \underline{d} \in \mathrm{supp}(\alpha)\}$$

The *degree* of a principal series α is the degree

$$\deg(\alpha) := \min \mathrm{supp}(\alpha)$$

The *linear part* of a series $\alpha := \Sigma a_{\underline{d}} x^{\underline{d}}$ is defined as follows:

$$L(\alpha) := \sum_{w(\underline{d})=1} a_{\underline{d}} x^{\underline{d}} .$$

Vector series. A family (α_i) of series will be called *summable* if every degree has non-zero coefficient in at most a finite number of series.

A *power vector series* $\underline{\alpha} := (\alpha_i)$ is a family of series such that:

1) $i \in S$
2) (α_i) is summable
3) $\forall i \in S \quad \alpha_i$ is a power series
4) $\forall i \in S \quad w(\alpha_i) > 0$

A *Laurent vector series* $\underline{\alpha} := (\alpha_i)$ is a family of series such that:

1) $i \in S$
2) (α_i) is summable
3) $\forall i \in S \quad \alpha_i$ is a Laurent series admitting multiplicative inverse
4) $\forall i \in S \quad \deg(\alpha_i) > \underline{0}$

If $\underline{\alpha} := (\alpha_i)$ is a Laurent or a power vector series and $\underline{d} \in \underline{D}$ or $\underline{d} \in \underline{D}^t$, we set

$$\underline{\alpha}^{\underline{d}} := \prod_{i \in S} \alpha_i^{d_i} \qquad \text{Then:}$$

$$\text{The } \textit{composition } \beta \circ \underline{\alpha} := \sum_{\underline{d} \in \underline{D}} b_{\underline{d}} \underline{a}^{\underline{d}} \qquad (\beta := \Sigma b_{\underline{d}} \underline{x}^{\underline{d}})$$

can be defined whenever β is a power series and $\underline{\alpha}$ a power vector series, or β is a Laurent series, and $\underline{\alpha}$ a Laurent vector series.

The composition can be defined ever between power or Laurent vector series:

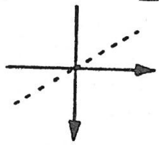
$$\underline{\alpha} \circ \underline{\beta} := (\alpha_i \circ \underline{\beta})$$

The monoids of power and Laurent vector series (under composition) are right regular, that is

$$\underline{\beta} \circ \underline{\alpha} = \underline{\alpha} \Rightarrow \underline{\beta} = \underline{x}$$

<u>The pivoting operator</u>. If M is a $\underline{D}\times\underline{D}$ matrix, the matrix TM is defined as follows: for every $\underline{i},\underline{j}\in\underline{D}$

$$\langle\underline{i}|TM|\underline{j}\rangle := \langle-\underline{j}|M|-\underline{i}\rangle \quad .$$



<u>The normalization</u>. If $\underline{\alpha} := (\alpha_i)$ is an invertible Laurent vector series, with

$$\alpha_i := x_{\sigma(i)}\,\hat{\alpha}_i \ ,$$

its *normalization* will be the vector series $N(\underline{\alpha}) := (N\alpha_i)$ where

$$N\alpha_i := x_i\,\hat{\alpha}_{\sigma^{-1}(i)} \quad .$$

<u>The operator P</u>. If $\underline{\alpha} := (\alpha_i)$ is a normalized invertible Laurent vector series, it is possible to compute

$$P(\underline{\alpha}) := \det\left(\frac{x_i}{\alpha_j}\frac{\partial\alpha_j}{\partial x_i}\right) = \det\left(I + \frac{x_i}{\alpha_j}\frac{\partial\hat{\alpha}_j}{\partial x_i}\right)$$

because every degree $\underline{d}\in\underline{D}$ can be obtained only by rows and columns with indexes $i,j\in\operatorname{supp}(\underline{d})$.

If $\underline{\alpha}$ is an invertible Laurent vector series, we set

$$P(\underline{\alpha}) := P(N(\underline{\alpha})) \quad .$$

We have:

$$P(\underline{\alpha}\circ\underline{\beta}) = (P(\underline{\alpha})\circ\underline{\beta})\cdot P(\underline{\beta})$$

$$P(\tilde{\underline{\alpha}}) = (P(\underline{\alpha})^{-1})\circ\tilde{\underline{\alpha}}$$

and

$$\langle\underline{0}|\underline{\alpha}^{\underline{d}}P(\underline{\alpha})\rangle = \begin{cases} 1 & \text{if } \underline{d}=\underline{0} \\ 0 & \text{otherwise} \end{cases}$$

Compositional inverse. The *linear part* of a vector series $\underline{\alpha} := (\alpha_i)$ is the vector series of the linear parts of the components of $\underline{\alpha}$: $L(\underline{\alpha}) := (L(\alpha_i))$.

A power or Laurent vector series $\underline{\alpha}$ admits two sided compositional inverse if and only if its linear part $L(\underline{\alpha})$ is invertible.

Thus:

Characterizing invertible *power* vector series is equivalent to characterizing invertible infinite matrices.

A *Laurent* vector series $\underline{\alpha} := (\alpha_i)$ admits two sided compositional inverse if and only if

$$\alpha_i = x_{\sigma(i)} \hat{\alpha}_i$$

where σ is a bijection on S and $\hat{\alpha}_i$ are power series admitting multiplicative inverse.

An invertible Laurent vector series will be called *normalized* if σ is the identity on S .

Recursive matrices. Matrices are maps $M : \underline{D} \times \underline{D} \to \mathbb{A}$.
If $\underline{\alpha}$ is a vector series and β a series , the $(\underline{\alpha},\beta)$ recursive matrix $R(\underline{\alpha},\beta)$ is the matrix whose $\underline{i}$-th row is $\underline{\alpha}^{\underline{i}}\beta$, $\underline{i} \in \underline{D}$, that is:

$$\langle \underline{j} | R(\underline{\alpha},\beta) | \underline{i} \rangle = \langle \underline{j} | \underline{\alpha}^{\underline{i}}\beta \rangle \quad .$$

If $\underline{\alpha},\underline{\gamma}$ are Laurent vector series, β and δ are Laurent series, then the product $R(\underline{\alpha},\beta) \times R(\underline{\alpha},\delta)$ is defined, and

Theorem 1: $$R(\underline{\alpha},\beta) \times R(\underline{\gamma},\delta) = R(\underline{\alpha} \circ \underline{\gamma}, (\beta \circ \underline{\gamma})\delta) \quad .$$

If $\underline{\alpha}$ is an invertible Laurent vector series, then:

Theorem 2: $R(\tilde{\underline{\alpha}},1) = TR(\underline{\alpha},P(\underline{\alpha}))$

Theorem 2: $TR(\underline{\alpha},\beta) = R(\tilde{\underline{\alpha}},P(\tilde{\underline{\alpha}})\beta \circ \tilde{\underline{\alpha}})$

Corollary: $\langle \underline{d} \mid \tilde{\alpha}_i \rangle = \langle -\underline{e}_i \mid \underline{\alpha}^{-\underline{d}} P(\underline{\alpha}) \rangle$ (Lagrange inversion),

where the degree $\underline{e}_i$ is defined as follows:

$\underline{e}_i(j) = \delta_{ij}$

Literatur:

Barnabei, M./Brini, A./Nicoletti, G.: "Recursive Matrices and Umbral Calculus", J. Algebra (1982)

---: "A general Umbral Calculus", to appear

---: "A general Umbral Calculus II: Infinitely many variables", to appear