Symbolic Methods for exhausting Discrete Functions

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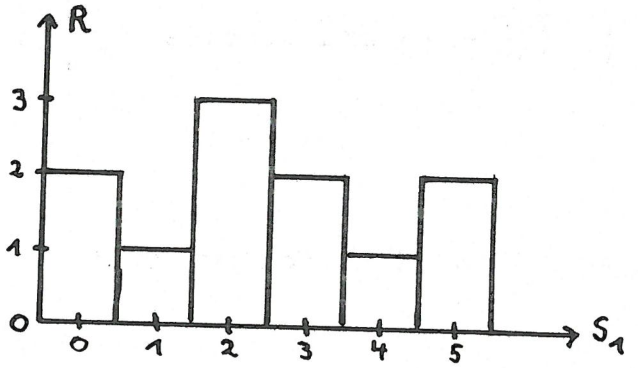
Let $f: S_1 \times \ldots \times S_n \rightarrow R$ be a discrete function, where the domains S_i, R are totally ordered and finite and therefore, e.g., of the form $\{0,1,\ldots,s_i - 1\}$, $\{0,1,\ldots,r-1\}$. Using the natural ring-and lattice-structure, we consider block functions $B = b \wedge \bigwedge_{i=1}^{n} x_i^{[b_i,b_i']}$, where $x_i \in S_i$, $b \in R$ and $x_i^{[b_i,b_i']}$ means the test function, which is $r-1$ for x_i in the "circular" interval $b_i \leq x_i \leq b_i'$ and 0 otherwise. $\wedge$ is the minimum (meet) in R.

If $B \leq f$, B is called implicand of f. We want to find all prime implicands of f. (Having done this, an appropriate choice of prime implicands yields a cost optimal disjunctive representation of f. In the Boolean case this process - which involves a NP-complete set covering problem - is the classical Quine-McCluskey procedure)



There is a well known symbolic algorithm: Define (according to Davio, Deschamps and Thayse ≈ 1973 - 76) a derivative

$$\frac{Mf}{Mx_i} := f(\ldots,x_i \oplus 1,\ldots) \wedge f(\mathfrak{x}),$$ where $\oplus$ is ring addition in S_i.

Analogously to other difference and differential operators we measure local properties of f. It is in the Lotharingian spirit that we study discrete analogues of differential operators. There are, of course, many formal analogies to differential calculus. We consider mixed derivations. The Order of deriving is immaterial. After applying the maximal $(s_i - 1)$th derivation no further change occurs. Let vector $\mathbf{k} = (k_1,\ldots,k_n)$ symbolize, that we have derived k_i times in x_i-direction.

Theorem (DDT): The equation $\left(\frac{M^{\mathbf{k}} f}{M\mathfrak{x}^{\mathbf{k}}}\right)_{\mathfrak{x}=\mathfrak{x}_0} = b$ indicates, that the block function defined above with the extension $\mathbf{k}$ starting in $\mathfrak{x}_0$

"from left to right" is an implicand of f, which is maximal by value b (but possible extendible by size and therefore in general no prime block).

We are able to give a characterisation of *prime* blocks in these terms:

Theorem: Unless there is a nonvanishing maximal derivative in at least one direction (which means independence from this direction and which is the normal situation in the Boolean case), a block of size k is a prime block iff the value of the k-derivative in the upper left corner is a local *proper* maximum.

Proof-idea: Under the tremendous calculation efforts of DDT there is a simple observation: In the non-maximal case derivation means making a block successively smaller. The proper maximum recognizes a prime block as a peak just before it is disappearing. The calculation of derivatives is - as always - nothing but a clever principle of bookkeeping local properties, where higher derivatives enlarge the neighborhood boundaries.

As an application, we can dispense with the somewhat troublesome containment comparison of blocks, which follows the original DDT-procedure in order to get the list of prime blocks. We only have to test for a local proper maximum in a derivation table, which is a local task.