Luigi Cerlienco - Francesco Piras (Cagliari)

A few properties and applications of the linearly recursive sequences.

§1. Basic properties.

We shall identify $\mathbb{C}[x]$ with $\mathbb{C}^{(\mathbb{N})}$ and $\mathbb{C}[[x]]$ with $\mathbb{C}^{\mathbb{N}}$.

Let $g(x) = x^k - a_{k-1}x^{k-1} - \ldots - a_o = \prod_{i=1}^{h} (x-\rho_i)^{m_i} \in \mathbb{C}[x]$; a sequence $u=(u_n)_{n\in\mathbb{N}}$ of complex numbers(*) is a <u>linearly recursive sequence</u> (<u>l.r.s.</u>) with <u>scale of recurrence</u> g if, for every $n\in\mathbb{N}$,

(1) $$u_{n+k} = a_{k-1}u_{n+k-1} + \ldots + a_o u_n$$

or equivalently, if

(1') $$g(E)u = 0$$

where E is the shift operator over $\mathbb{C}^{\mathbb{N}}$.

The set of the scales for a given l.r.s. u is an <u>ideal</u> of $\mathbb{C}[x]$, whose monic generator is called the <u>minimal scale</u> of u.

The set S_g of all l.r.s. with scale g is a vector space of dimension $k=\deg(g)$. S_g is the subspace of $\mathbb{C}^{\mathbb{N}}$ orthogonal to the subspace (g) (= the ideal generated by g) of $\mathbb{C}^{(\mathbb{N})}$:

(2) $$S_g = (g)^{\perp} .$$

§2. Some examples of l.r.s.

a) The Fibonacci sequence: $u_o=0$, $u_1=1$, $u_{n+2}=u_{n+1} + u_n$.

b) The coefficients of the expansion

(*) The substitutions of $\mathbb{C}$ with an arbitrary field K and that of the set of indices $\mathbb{N}$ with $\mathbb{Z}$ present no particular difficulty.

$$\frac{\beta(x)}{\gamma(x)} = \Sigma\, u_n\, x^n$$

where $\beta,\gamma\in C[x]$, $\deg(\beta)<\deg(\gamma)=k$; the scale is $g(x)=x^k\cdot\gamma(x^{-1})$.

c) Elementary homogeneous functions. Let $x_i\in C$, $i\in N$. Let $A=(a_k^n)$ and $H=(h_k^n)$ be the $\omega\times\omega$ lower triangular matrices defined by

$$a_k^n=a_k^n(x_o,\dots,x_{n-1}) = \begin{cases} (-1)^{n-k} \sum\limits_{0\leq i_1< \dots <i_{n-k}<n} x_{i_1} \dots x_{i_{n-k}} & \text{if } n\geq k \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_k^n=h_k^n(x_o,\dots,x_k) = \begin{cases} \sum\limits_{i_o+\dots+i_k=n-k} x_o^{i_o} \dots x_k^{i_k} & \text{if } n\geq k \\ 0 & \text{otherwise} \end{cases}$$

(a_k^n elementary symmetric functions; h_k^n elementary homogeneous functions).

Obviously, $H = A^{-1}$.

For every $k\in N$ the k-th column in H is a l.r.s. with minimal scale $P_{k+1}(x) = (x-x_o)\dots(x-x_k) = a_o^{k+1} + a_1^{k+1}x + \dots + a_{k+1}^{k+1}x^{k+1}$.

If we set:

- $x_i=1$ then $H=T$ is the Pascal triangle;
- $x_i=n$ then $H=T^n$;
- $x_i=q^i$ then H is the triangle of q-binomial coefficients;
- $x_i=i$ then H is the triangle of the Stirling numbers of the second kind.

(Cfr. [6])

§3. The bialgebra of l.r.s.

Theorem (B.Peterson-E.J.Taft, 1980, [8]). The space S of all l.r.s. is a bialgebra with multiplication $m(u\otimes v) = w$ defined by

$$w_n = \sum_{k=0}^{n} \binom{n}{k} u_k\, v_{n-k}$$

and diagonalization Δ defined by

$$\Delta u = \begin{pmatrix} u_o & u_1 & u_2 & \dots & u_n & \dots \\ u_1 & u_2 & u_3 & \dots & u_{n+1} & \dots \\ u_2 & u_3 & u_4 & \dots & u_{n+2} & \dots \\ \dots & & & & & \end{pmatrix} \quad \text{(Hankel matrix of u)}$$

Such a bialgebra is the dual of the usual bialgebra of polynomials.

§4. Some algorithmic uses of l.r.s.

4.1. The <u>fundamental l.r.s.</u> $\underline{r}^i = (r^i_n)$, $i=1,\dots,k$ with scale $g(x)$ are defined by the boundery values $r^i_j = \delta^i_{j+1}$, $1 \leq i, j+1 \leq k$. They form a basis for S_g. The matrices

$$R_g = \begin{bmatrix} 1 & 0 & \dots & 0 & r^1_k & r^1_{k+1} & \dots \\ 0 & 1 & \dots & 0 & r^2_k & r^2_{k+1} & \dots \\ \dots & & & & & & \\ 0 & 0 & \dots & 1 & r^k_k & r^k_{k+1} & \dots \end{bmatrix}$$

and

$$Q_g = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & r^k_k & r^k_{k+1} & r^k_{k+2} & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & r^k_k & r^k_{k+1} & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & r^k_k & \dots \\ \dots & & & & & & & & \end{bmatrix}$$

represent (with reference to the basis x^n) the linear maps "remainder" and "quotient"

$$r: \mathbb{C}[x] \longrightarrow \mathbb{C}[x]/(g) \qquad \text{and} \qquad q: \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$
$$p \longmapsto r(p) \qquad\qquad\qquad\qquad p \longmapsto q(p)$$

in the division by g: $p = g \cdot q(p) + r(p)$.

Moreover:

M=companion matrix of g ↘ $\qquad$ M^n ↘

$$R_g = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & r_k^1 & \dots & r_n^1 & \dots\dots & r_{n+k-1}^1 & \dots \\ 0 & 1 & 0 & \dots & 0 & r_k^2 & \dots & r_n^2 & \dots\dots & r_{n+k-1}^2 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & r_k^k & \dots & r_n^k & \dots\dots & r_{n+k-1}^k & \dots \end{bmatrix}$$

$$s_n = \sum_{i=1}^{h} m_i \rho_i^n = r_n^1 + \dots + r_{n+k-1}^k$$

(Cfr. [4]).

4.2. Let us now assume $u_n \in \mathbb{R}$, $g \in \mathbb{R}[x]$ and denote by σ_j, $1 \leq j \leq k$, the zeros of g; furthermore, suppose $|\sigma_1| \geq |\sigma_2| \geq \dots \geq |\sigma_k|$. Consider a l.r.s. $u=(u_n)$ with minimal scale $g(x)$ and put

$$\Delta_n^{(j)} = \begin{vmatrix} u_n & u_{n+1} & \dots & u_{n+j-1} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n+j-1} & u_{n+j} & \dots & u_{n+2j-2} \end{vmatrix} , \quad j=1,2,\dots,k;$$

if the sequence $\Theta_n^{(j)} = \Delta_{n+1}^{(j)}/\Delta_n^{(j)}$ converges, we have

$$\lim_{n\to\infty} \Theta_n^{(j)} = \sigma_1\sigma_2\dots\sigma_j \quad ;$$

otherwise, if $\Theta_n^{(j)}$ does not converge, then $|\sigma_j| = |\sigma_{j+1}|$.

For j=1, this gives the well known "Bernoulli method".

(Cfr. [3]).

§5.

What follows is part of a work in progress made in collaboration with Prof.G.Nicoletti.

Let $s_o=0$, $s_1=1$, s_2, s_3, ... $(i\neq 0 \Rightarrow s_i \neq 0)$ be any given sequence of complex numbers. We put:

$$s_o! = 1 \ , \quad s_{n+1}! = s_n!\, s_{n+1}$$

and

$$\left\{ \begin{matrix} m_1 + m_2 + \ldots + m_\ell \\ m_1, m_2, \ldots, m_\ell \end{matrix} \right\} = \frac{s_{m_1+\ldots+m_\ell}!}{s_{m_1}! \ \ldots \ s_{m_\ell}!} \qquad (m_j \in \mathbb{N});$$

we write $\left\{ \begin{matrix} h+k \\ h \end{matrix} \right\}$ instead of $\left\{ \begin{matrix} h+k \\ h,k \end{matrix} \right\}$.

Obviously:

a) if $s_n = n$ then $\left\{ \begin{matrix} n \\ h \end{matrix} \right\} = \binom{n}{h}$;

b) if $s_n = 1+q+q^2+\ldots+q^{n-1}$ then $\left\{ \begin{matrix} n \\ h \end{matrix} \right\} = \binom{n}{h}_q$;

c) if $s_n = 1$ then $\left\{ \begin{matrix} n \\ h \end{matrix} \right\} = 1$.

Furthermore, using a theorem of E. Lucas [7] about the l.r.s. of second order, we have:

if (s_n) is a l.r.s. with scale $g(x) = x^2+ax+b$, with $a,b \in \mathbb{Z}$, then all the coefficients $\left\{ \begin{matrix} m_1+\ldots+m_\ell \\ m_1, \ldots, m_\ell \end{matrix} \right\}$ are in Z.

The vector space V spanned by $(b_i)_{i \in \mathbb{N}}$ can be structured as a (coassociative, cocommutative) coalgebra $V = (V, \Delta, \varepsilon)$ setting

$$\Delta(b_h) = \sum_{i=o}^{h} \left\{ \begin{matrix} h \\ i \end{matrix} \right\} b_i \otimes b_{h-i}$$

$$\varepsilon(b_h) = \delta_h^0 .$$

Denote by V^* the dual algebra of V, and by (b^i) the pseudobasis dual of (b_i).

If $\alpha = \sum_{i \geq o} \alpha_i b^i \in V^*$ and if we set $\langle i|\alpha\rangle = \alpha_i$, we have:

(Mendelpasssatz) if $\alpha \in V^*$, $\langle o|\alpha\rangle \neq 0$, then for every $h,n \in \mathbb{N}$

$$\langle h|\alpha^n\rangle = \sum_{|\underline{m}|=h} \left\{ \begin{matrix} h \\ m_1, \ldots, m_i \end{matrix} \right\} \alpha_{\underline{m}} \binom{n}{\# \underline{m}} \alpha_o^{n-\#\underline{m}}$$

where $\underline{m} = (m_1, \ldots, m_i)$ is an ordered multiset of positive integers, $|\underline{m}| = \Sigma_j m_j$, $\#\underline{m} = i$, $\alpha_{\underline{m}} = \Pi_j \alpha_{m_j}$.

As a corollary, recalling that α_o^n, $\binom{n}{1}\alpha_o^{n-1}$, ..., $\binom{n}{\ell}\alpha_o^{n-\ell}$ are indipendent l.r.s. with scale $(x-\alpha_o)^{\ell+1}$, we get that, for any given $h\in\mathbb{N}$, the sequence $\langle h|\alpha^n\rangle$, $n\in\mathbb{N}$, is a l.r.s. with scale $g_{h+1}=(x-\alpha_o)^{h+1}$. Moreover, if $\alpha_1\neq 0$, g_{h+1} is its minimal scale.

As a final remark, note that the assumed condition $\alpha_o\neq 0$ ensures that α admits multiplicative inverse and that the above properties hold for $n\in\mathbb{Z}$.

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