A COMMON GENERALIZATION OF BINOMIAL COEFFICIENTS, STIRLING NUMBERS AND GAUSSIAN COEFFICIENTS

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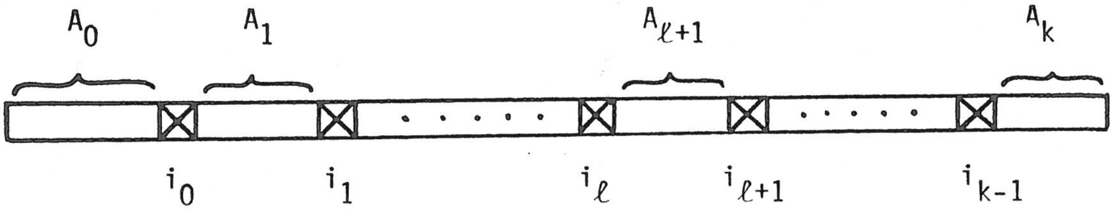
Let $A_0, A_1, A_2, \ldots$ be finite sets. For nonnegative integers n and k denote by $S_k^n(a_0, a_1, a_2, \ldots)$, where $a_i = |A_i|$, the number of words $w = (w_0, \ldots, w_{n-1})$ such that

(1) w contains k labels, say at positions $i_0, \ldots, i_{k-1}$,

(2) all entries in w before position i_0 belong to A_0 , all entries in w between positions i_ℓ and $i_{\ell+1}$, where $\ell = 0, \ldots, k-2$, belong to $A_{\ell+1}$, all entries after position i_{k-1} belong to A_k .



As $$S_k^n(\vec{a}) = \sum_{0 \le i_0 < i_1 < \ldots < i_{k-1} < n} a_0^{i_0} \cdot a_1^{i_1 - i_0 - 1} \cdot \ldots \cdot a_k^{n - i_{k-1} - 1}$$

the numbers S_k^n can obviously be defined for sequences of complex numbers.

Examples:

(1) $S^n_k(1,1,\ldots) = \binom{n}{k}$ (Binomial coefficients)

(2) $S^n_k(0,1,2,\ldots) = S^n_k$ (Stirling numbers of the second kind)

(3) $S^n_k(1,q,q^2,\ldots) = \binom{n}{k}_q$ (Gaussian Binomial coefficients)

(4) $S^n_k(q,q^2,q^3,\ldots) =$ number of affine k-dimensional subspaces in the n-dimensional affine space over $GF(q)$.

(5) $S^n_k(2,3,4,\ldots) =$ number of Boolean sublattices $P(k)$ in $P(n)$ ($P(n) \equiv$ lattice of subsets of an m-element set)

Pascal-identity: $S^{n+1}_k(\vec{a}) = S^n_{k-1} + a_k \cdot S^n_k$

explicitly: $$S^n_k(\vec{a}) = \sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0\\ j\neq i}}^{k} (a_i - a_j)^{-1}$$

provided $a_0,\ldots,a_k$ are mutually distinct.

Let $P^{\vec{a}}_0(x) = 1$ and $P^{\vec{a}}_{k+1}(x) = (x-a_k) \cdot P^{\vec{a}}_k(x)$, i.e.

$P^a_k(x) = (x-a_{k-1}) \cdot \ldots \cdot (x-a_0)$.

Inversion, resp. Interpolation formulae:

$$x^n = \sum_{k\geq 0} S^n_k(\vec{a}) \cdot P^{\vec{a}}_k(x) \quad .$$

Now one can introduce the inverse numbers $s_k^n(\vec{a})$ by

$$\sum_j s_j^n(\vec{a}) \cdot S_k^j(\vec{a}) = \delta_k^n \quad \text{(Kronecker delta)} \quad .$$

<u>Recursion:</u> $s_k^{n+1}(\vec{a}) = s_{k-1}^n - a_n \cdot s_k^n$.

These numbers can be used in order to describe inversion for arbitrary ascending sequences of normalized polynomials:

Let $\mathcal{S}_k^n(\vec{a},\vec{b}) = \sum_j s_j^n(\vec{a}) \cdot S_k^j(\vec{b})$, then

$$P_n^{\vec{a}}(\vec{x}) = \sum_k \mathcal{S}_k^n(\vec{a},\vec{b}) \cdot P_k^{\vec{b}}(x) \quad .$$

<u>Recursion:</u> $\mathcal{S}_k^{n+1}(\vec{a},\vec{b}) = \mathcal{S}_{k-1}^n(\vec{a},\vec{b}) + (b_k - a_n) \cdot \mathcal{S}_k^n(\vec{a},\vec{b})$.

<u>Theorem:</u> $\vec{a}$ as before ℓ a complex number, then

(i) $S_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a} - \ell)$

(ii) $S_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot S_j^n(\vec{a} - \ell)$

(iii) $s_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot s_k^j(\vec{a} + \ell)$

(iv) $s_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot s_j^n(\vec{a} + \ell)$.

<u>Remark:</u> $\vec{a} = (1,q,q^2,\dots)$ and $\ell = 1$ is a theorem of Carlitz [1].

<u>Corollary:</u> number of Boolean sublattices of $\mathcal{P}(n) = B_{n+2}$ (n+2 .nd Bell number)

<u>References</u>

[1] L. Carlitz: On abelian fields, Trans. AMS, 35, 1933, 122 - 136 .