AND ANALOGS OF THE UMBRAL CALCULUS

by

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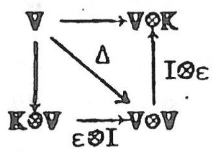
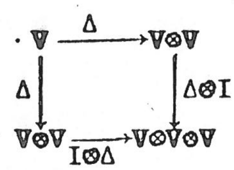
Summary: Automorphisms of properly graded coalgebras are characterized, and applications to the Umbral Calculus, and its analogs, are exhibited.

§1 Definitions and examples

A $\mathbb{K}$-*coalgebra* is a triple $C:=(\mathbb{V},\varepsilon,\Delta)$, where $\mathbb{V}$ is a $\mathbb{K}$-vector space, $\varepsilon:\mathbb{V}\to\mathbb{K}$ and $\Delta:\mathbb{V}\to\mathbb{V}\otimes\mathbb{V}$ are linear maps, called the *counit* and the *comultiplication*, respectively, such that the following diagrams commute:



counitary property



coassociativity.

The *dual algebra* of the coalgebra $C:=(\mathbb{V},\varepsilon,\Delta)$ is the triple $C^*:=(\mathbb{V}^*,i,\mu)$, where $i:=\varepsilon^*$ and μ is the restriction of Δ^* to the canonical imbedding of $\mathbb{V}^*\otimes\mathbb{V}^*$ into $(\mathbb{V}\otimes\mathbb{V})^*$. i and μ are called the *identity map* and the *multiplication*. Now suppose $\mathbb{V}$ has a countable basis $(\mathbf{b}_i)$, $i\in\mathbb{N}$, and define $\varepsilon:\mathbb{V}\to\mathbb{K}$, $\Delta:\mathbb{V}\to\mathbb{V}\otimes\mathbb{V}$ such that

$$\varepsilon(\mathbf{b}_i):=\delta_{i,0}$$

and

$$\Delta(\mathbf{b}_i):=\sum_{h+k=i} a_{h,k}\mathbf{b}_h\otimes\mathbf{b}_k$$

where the *structure coefficients* $a_{h,k}\in\mathbb{K}$ satisfy

(1) $$a_{h,0}=a_{0,k}=1$$

and

(2) $$a_{h,j}a_{h+j,k}=a_{h,j+k}a_{j,k}$$

then, $C:=(\mathbb{V},\varepsilon,\Delta)$ is a coalgebra which will be named a (*co*)*graded coalgebra*.

In the dual algebra C^* identity and multiplication are defined as follows:

$$i(c):=c\mathbf{b}_0^*$$

$$\mu(\mathbf{b}_i^*\otimes\mathbf{b}_j^*):=a_{i,j}\mathbf{b}_{i+j}^*$$

(recall that $(\mathbf{b}_i^*)$ is a pseudobasis for the vector space $\mathbb{V}^*$ endowed of a suitable topology).

In the following we shall identify $\mathbb{V}$ and $\mathbb{V}^*$ with $\mathbb{K}^{(\mathbb{N})}$ and $\mathbb{K}^{\mathbb{N}}$, respectively, i.e. the elements of $\mathbb{V}$ will be regarded as sequences of scalars, with finite support, while the elements of $\mathbb{V}^*$ will be regarded as sequences (c_i), $i\in\mathbb{N}$ of scalars or "series" $\sum_{i\in\mathbb{N}} c_i\mathbf{b}_i^*$. The *canonical bases* $(\mathbf{b}_i)$ and $(\mathbf{b}_i^*)$ are

$$\mathbf{b}_i=\mathbf{b}_i^*:=(\delta_{n,i}).$$

Example 1

Setting $a_{h,k}=1$ for every $h,k\in\mathbb{N}$ we get a graded coalgebra C called the *standard (co)commutative coalgebra* or the *coälgebra of divided powers*. The dual algebra C is the *algebra of formal power series*.

Example 2

Let $\mathrm{char}(\mathbb{K})=0$ and set $a_{h,k}:=\binom{h+k}{k}$: we get a graded coalgebra C called the *binomial* or *polynomial coalgebra*.

Example 3

Let $\mathrm{char}(\mathbb{K})=p$ and set $a_{h,k}:=\binom{h+k}{k}_{\mathrm{mod}\ p}$: this coalgebra is called the *mod p-binomial* or *mod p-polynomial coalgebra*.

Example 4

Let $\mathrm{char}(\mathbb{K})=0$ and set $0!_q:=1$ and, recursively, $(n+1)!_q:=n!_q(1+q+\cdots+q^{n-1})$; now set $a_{h,k}:=\begin{bmatrix}h+k\\k\end{bmatrix}_q:=(h+k)!_q/h!_q k!_q$. This coalgebra is called the *q-eulerian* or *q-nomial coalgebra*.

Example 5

Let $\gamma:=(c_i)\in\mathbb{V}^*$ be a sequence such that $c_0=c_1=1$ and $c_i\neq0$ for every $i\in\mathbb{N}$; for every $h,k\in\mathbb{N}$ let us define $a_{h,k}:=c_{h+k}/c_h c_k$. These structure coefficients define a coalgebra which will be called the *γ-nomial coalgebra*. In the following, the matrix whose entries are the $a_{h,k}$'s now defined will be denoted by A_γ.

If $\mathrm{char}(\mathbb{K})=0$ and $c_i=i!$ or $c_i=i!_q$, we get Example 2 and Example 4, respectively.

In the following, we will denote by C_γ the γ-nomial coalgebra, for a given sequence γ, and by C_I the standard cocommutative coalgebra.

A graded coalgebra will be called *proper* or *properly graded* if its structure coefficients satisfy

(3) $$a_{h,k}\neq 0 \quad .$$

1,2,4 and 5 are examples of properly graded coalgebras, while 3 is not.

Moreover, 5 is a prototypical example, because of the following result:

Theorem 1

Let $A:=(a_{h,k})$ be the matrix whose entries are the structure coefficients of a properly graded $\mathbb{K}$-coalgebra: then $A=A_\gamma$, where $\gamma:=(c_i)$ and

$$c_0:=1 \quad , \quad c_i:=\prod_{n=0}^{i-1} a_{1,n} \ .$$

Proof By conditions (1), (2) and (3) we get, for every $h,k\in\mathbb{N}$:

$$a_{h,k}=\frac{\prod_{i=0}^{h+k-1} a_{1,i}}{\prod_{i=0}^{h-1} a_{1,i}\ \prod_{i=0}^{k-1} a_{1,i}}$$

and this gives the assertion. ∎

Theorem 1 allows us to show that any two properly graded $\mathbb{K}$-coalgebras are isomorphic. In fact, we have:

Theorem 2

Let C_γ and $C_{\gamma'}$ be properly graded coalgebras over $\mathbb{V}$, related to the sequences $\gamma:=(c_i)$ and $\gamma':=(c_i')$, and let C_γ^* and $C_{\gamma'}^*$ be their dual algebras: then the map $\phi:\mathbb{V}\to\mathbb{V}$ and its dual $\phi^*:\mathbb{V}^*\to\mathbb{V}^*$ defined by

$$\phi(b_i):=\frac{c_i}{c_i'}\, b_i$$

and

$$\phi^*(b_i^*):=\frac{c_i}{c_i'}\, b_i^*$$

are a coalgebra isomorphism of C_γ and $C_{\gamma'}$ and an algebra isomorphism of $C_{\gamma'}^*$ and C_γ^*, respectively. ∎

We recall that the *Hadamard product* ⊠ and the *Hadamard division* ⊡ of two sequences (or series) $\rho:=(r_i)$ and $\sigma:=(s_i)$ are defined as

$$\rho\boxtimes\sigma:=(r_i s_i)$$
$$\rho\boxdot\sigma:=(r_i/s_i) \quad \text{if } s_i\neq 0.$$

Let us denote by $\rho\times_\gamma\sigma$ and $\rho^{n\gamma}$ the product and the power performed in the algebra C_γ dual of the properly graded coalgebra C_γ, and by $\rho\times\sigma$ and ρ^n the product and the power performed in the algebra of formal power series; then, by Theorem 2, we get:

Corollary 3

For every ρ and $\sigma\in\mathbb{V}^*$ we have:

(4) $$\rho \underset{\gamma}{\times} \sigma = ((\rho ⊡ \gamma)\times(\sigma ⊡ \gamma)) ⊠ \gamma$$

(5) $$\rho^{n\gamma} = (\rho ⊡ \gamma)^n ⊠ \gamma .$$ ■

Corollary 4

In the algebra C_γ^*, dual of the properly graded coalgebra C_γ^* related to the sequence $\gamma:=(c_i)$, the following identity holds:

(6) $$(\mathbb{b}_1)^{n\gamma} = c_n \mathbb{b}_n ,$$

which implies that $\mathbb{b}_1$ is a pseudogenerator of C_γ^*. ■

§2 Automorphisms

In order to characterize and represent the group of all automorphisms of a properly graded coalgebra, we first recall the following result (see [1]):

Theorem 5

The linear map $\phi:\mathbb{V}\rightarrow\mathbb{V}$ is an automorphism of the properly graded coalgebra C_γ if and only if its dual map $\phi^*:\mathbb{V}^*\rightarrow\mathbb{V}^*$ is a continuous automorphism of the dual algebra C_γ^*. ■

For any given two series $\rho:=(r_i)$ and $\sigma:=(s_i)$, with $s_0=0$, we define the *formal composition* as usual by

(7) $$\rho\circ\sigma := \sum_{i\in\mathbb{N}} a_i \sigma^i .$$

If $\gamma:=(c_i)$, $c_i\neq 0$ is given, the γ-*composition* is defined by

(8) $$\rho \underset{\gamma}{\circ} \sigma := \sum_{i\in\mathbb{N}} (a_i/c_i)\sigma^{n\gamma} = \rho\circ(\sigma ⊡ \gamma) .$$

Theorem 6

Let C_γ^* be the dual algebra of the properly graded coalgebra C_γ related to the sequence $\gamma:=(c_i)$, $c_i\neq 0$; then the continuous automorphisms of C_γ^* are the maps

$$\mathbb{V}^* \ni \rho \longrightarrow \rho \underset{\gamma}{\circ} \sigma$$

where $\sigma:=(s_i)$, $s_0=0$, $s_1\neq 0$, and conversely.

Proof Let ϕ be a continuous automorphism of C_γ^* and set $\sigma:=\phi(\mathbb{b}_1^*)=(s_i)$: it is easy to see that $s_0=0$, $s_1\neq 0$ and for every $\rho\in\mathbb{V}^*$: $\phi(\rho)=\rho \underset{\gamma}{\circ} \sigma$.

Conversely, if $\sigma=(s_i)$, $s_0=0$, $s_1\neq 0$, the map ϕ sending $\rho\circ\sigma$ to each $\rho\in\mathbb{V}^*$ is clearly a continuous endomorphism of C_γ^* which can be proved to be invertible by recursively computing - because $s_1\neq 0$ - the "γ-compositional inverse" $\tilde{\sigma}^\gamma$ such that $\tilde{\sigma}^\gamma \underset{\gamma}{\circ} \sigma = \sigma \underset{\gamma}{\circ} \tilde{\sigma}^\gamma = \mathbb{b}_1^*$. ■

An *invertible series* will be a series $\sigma:=(s_i)$ such that $s_0=0$ and $s_1\neq 0$; the preceding result can be rephrased as follows: the continuous automorphisms of the algebra C_γ are precisely the γ-compositions with invertible series.
Given an invertible series σ and a series $\gamma:=(c_i)$, $c_i\neq 0$, the *γ-recursive matrix with recurence rule* σ will be the $\mathbb{N}\times\mathbb{N}$-matrix $R_\gamma(\sigma)$ whose rows are the sequences of the coefficients of $\sigma^{n\gamma}$. If $c_i=1$ for every $i\in\mathbb{N}$, the matrix will be denoted by $R(\sigma)$ and it will be called a *recursive matrix*.
The symbol D_γ will denote the $\mathbb{N}\times\mathbb{N}$-matrix whose entries $(m_{h,k})$ are

$$m_{h,k}:=c_h\delta_{h,k}$$

where $(c_i)=\gamma$.

Let us denote by U_γ and U_γ^* the automorphism group of C_γ and the group of all continuous automorphisms of C_γ^*, with $\gamma:=(c_i)$, $c_i\neq 0$. If $c_i=1$ we will set $U:=U_\gamma$. The preceding results give us the following representation theorem:

Theorem 7

Let $\phi\in U_\gamma$; the matrix M which represent ϕ with respect to the canonical basis (b_i) is

(9) $$M=(D_\gamma)^{-1}\times R_\gamma(\sigma)=(D_\gamma)^{-1}\times R(\sigma\boxdot\gamma)\times D_\gamma$$

where $\sigma:=\phi^*(b_1^*)$.

Proof. Let us regard the elements of $\mathbb{V}$ as column-sequences with finite support, and the elements of $\mathbb{V}^*$ as row-sequences: then M represents both ϕ and ϕ^*; by results 4,5 and 6 we get that the rows of M are

$$\phi^*(b_i^*)=\phi^*(1/c_i)(b_1^*)^i \qquad \blacksquare$$

Denote by R the set of all recursive matrices: the preceding result proves that R is a group, under matrix multiplication, and that R and $(D_\gamma)^{-1}\times R\times D_\gamma$ represent U and U_γ, with respect to the canonical basis (b_i).

§3 Convolutorial bases

A *graded basis* is a basis (v_i) of $\mathbb{V}$ such that

(10) $$v_i=\sum_{i,j\in\mathbb{N}} d_{i,j}b_j$$ with

(11) $j>i$ implies $d_{i,j}=0$ and

(12) $d_{i,i}\neq 0$.

A graded basis will be said to be *homogeneous* when, for every $i\in\mathbb{N}$

(13) $d_{i,0}=0$.

A graded basis $(\mathbf{v}_i)$ will be called a *convolutorial basis of type* (γ,η), with $\gamma:=(c_i)$, $\eta:=(e_i)$, $c_i\neq 0\neq d_i$, or a (γ,η)-*basis* for short, if

(14) $$\Delta_\gamma \mathbf{v}_i = \sum_{h+k=i} a'_{h,k}\mathbf{v}_h\otimes\mathbf{v}_k$$

where Δ_γ is the comultiplication in the properly graded coalgebra C_γ, and the $a'_{h,k}$ are the structure coefficients of the properly graded coalgebra C_η. Characterizing all (γ,γ)-bases, for a given γ, is equivalent to find all the linear maps $T\in \mathrm{Aut}(\mathbb{V})$ mapping the canonical basis into a graded basis and preserving the structure coefficients of the comultiplication Δ_γ: such maps are nothing but the automorphisms of the coalgebra C_γ. This proves the following result:

Theorem 8

A graded basis $(\mathbf{v}_i):=(T(\mathbf{b}_i))$, with $T\in \mathrm{Aut}(\mathbb{V})$, is a $(\gamma,\check{\gamma})$-basis, for a given γ, if and only if $T\in (D_\gamma)^{-1}\times \mathcal{R}\times D_\gamma$. ■

Similarly, we can easily prove that:

Theorem 9

A graded basis $(\mathbf{v}_i):=(T(\mathbf{b}_i))$, with $T\in \mathrm{Aut}(\mathbb{V})$, is a (γ,η)-basis, for given γ,η, if and only if $T\in (D_\gamma)^{-1}\times \mathcal{R}\times D_\eta$. ■

Corollary 10

Every (γ,η)-basis is a homogeneous basis. ■

§4 The Umbral Calculus and its analogs

In the Umbral Calculus, $\mathbb{K}$ is a field of characteristic zero, $\mathbb{V}$ is the vector space $\mathbb{K}_{[x]}$, and $\mathbb{V}\otimes\mathbb{V}$ is $\mathbb{K}_{[y,z]}$. The canonical basis of $\mathbb{V}$ is (x^i).
A very natural comultiplication in $\mathbb{K}_{[x]}$ is the map Δ_π defined by

(15) $$p(x)\xrightarrow{\Delta_\pi} p(y+z),$$

whose structure coefficients, with respect to the canonical basis, are the usual binomial coefficients (see Example 2), and hence $\pi:=(p_i)$ with $p_i:=i!$.
A graded basis (p_i) is said to be of *binomial type* if

(16) $$\Delta_\pi(p_i) = \sum_{h+k=i}\binom{h+h}{k}p_h(y)p_k(z).$$

In the vein of the previous section, binomial bases are precisely the (π,π)-bases.

In fact, the Umbral Calculus is - roughly speaking - the study of the coalgebra C_π defined above, that is the study of $\mathbb{K}_{[x]}$ under the action of the so-called Umbral Group, which, in our terminology is $\mathcal{U}_\pi$, or - equivalently - the study of the dual algebra C_π of the exponential series.
After the papers by G.C.Rota and co-workers, many different "analogs" of the Umbral Calculus have been proposed. Among them, only two classes look have the features to be properly traited as effective analogs of the Umbral Calculus. In the light of section 3, we can describe them as follows:
a) for a given $\gamma:=(c_i)$, $c_i\neq 0$, let Δ_γ be the comultiplication whose structure coefficients with respect to (x^i) are the entries of A_γ (see Example 5).
A graded basis is said to be *of γ-GJ-type* (following [7]) if the structure coefficients of Δ_γ with respect to it are still the entries of A_γ.
These bases are precisely the (γ,γ)-bases.
b) for a given $\gamma:=(c_i)$, $c_i\neq 0$, a graded basis will be called *of γ-BBN-type* if, with respect to it, the structure coefficients of the "natural" comultiplication Δ_π are the entries of the matrix A_γ(cfr.[5]).

In these two cases the typical concepts of the Umbral Calculus can be developed to build up an effective analog of the Umbral Calculus, while the same is not possible for other suggested analogs.
The analogs we described above are easily generalized as follows: a graded basis of $\mathbb{K}_{[x]}$ is called a basis *of (γ,η) -type* if it is a (γ,η)-basis.
In a forthcoming paper we shall develop an "analog" for these types, which generalizes both GJ- and BBN- analogs.

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