Some things some functions may be,
some others can't be.

by

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Klaus Leeb allows himself to speak and wirte for all of us: It is well-known that each permutation is a product of two involutions $f = ab$ (to mean first a than b) with $a^2 = 1$, $b^2 = 1$. As in finite fields, the singular elements cannot satisfy $x^{q-1} = 1$, hence one multiplies by x to also accommodate the singular ones.

Thus Leeb conjectured: Any function $f : X \to X$ is of the form $f = ab$ with $a^3 = a$ and $b^3 = b$. His interest was basically in finite combinatorics. H. Sperber though made it quite clear that "finite" was essential in this statement and sharpened: $f = ab$, $a^2 = 1$, $b^3 = b$ iff x is finite. Leeb proved - angry of this lack of symmetry - also $f = ab$, $a^3 = a$, $b^2 = 1$.

But then he looked also for a way out of the unpleasant features of such conditions: They do not allow **proper** injections. It turned out that in the impossibility proof sometimes some tricky non-commutative algebra is involved, worthy of an exercise book. Leeb-Pirillo checked all the possibilities of adding up to 2 symbols, Leeb in the Brenner Express between 20^{40} and 31^{00} checked all possibilites of adding up to 3 symbols, of course to the original $a^2 = 1$, $1 = b^2$, and such that the resulting system is self-dual.

We present just two examples for the line of argument which shows that an injection satisfying the condition would have to be a permutation:

1) $aba^2b = abb \qquad aab = ab^2ab$
$= aba^2b = a \qquad a = ab^2$
$= abaabbaab = abaaaab = abaabaabaab$
$abaaba = 1$ a bijective $b^2 = 1 \quad ba^2b = 1 \quad a^2 = 1$

2) $a^2ba^2 = a^2b \qquad ab^2 = b^2ab^2$
$a^2\,ba^2b = ab^2ab^2 \quad ab^2$ injective $b^2 = 1$
$= a^2ba^2b = a^2$
$= a^2ba^2bba^2b = a^2baaaab = a^2ba^2ba^2ba^2b$
$a^2ba^2b = 1 \quad a^2b$ surjective $a^2 = 1 \quad b^2 = 1$

Eventually we hope to find a set of conditions satisfied by **all** functions, which for permutations reduces to $a^2 = 1$, $b^2 = 1$.

Reference: The work of Silberger