Shellability of Exponential Structures

by

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SIMPLICIAL COMPLEXES

Definition: An n-simplex σ is an n-dimensional tetrahedron i.e. the convex hull of n+1 points: vert $\sigma = \{a_0, a_1, \ldots, a_n\}$ in general position. A face of σ is any simplex f s.t. vert $f \subseteq$ vert σ. A (geometric) simplicial complex Δ is formed by identifying (glueing together) simplices along faces. Let $M(\Delta) = \{\sigma \in \Delta \mid \sigma$ is a maximal simplex$\}$ then Δ is pure of dimension d, written $\dim \Delta = d$, if every $\sigma \in M(\Delta)$ has dimension d. If $\dim \Delta = d$ then Δ is shellable if there is a permutation $\sigma_1\sigma_2\ldots\sigma_s$ of $M(\Delta)$ s.t. for all j : σ_j intersects

$$\bigcup_{i<j} \sigma_j$$

in a union of d-1 faces i.e. $\forall_j$ and $\forall_{i<j}$ there is a $k<j$ such that

$$\sigma_j \cap \sigma_i \subseteq \sigma_j \cap \sigma_k \text{ and } \dim(\sigma_j \cap \sigma_k) = d-1 .$$

An (abstract) simplicial complex Δ ist $\Delta = \{\sigma \mid \sigma$ a set$\}$
s.t. (1) if $\sigma \in \Delta$ and $f \subseteq \sigma \Rightarrow f \in \Delta$
(2) if $\sigma, \tau \in \Delta \Rightarrow \sigma \cap \tau \in \Delta$

If $\sigma \in \Delta$ then define $\dim \sigma = |\sigma| - 1$. $M(\Delta)$, purity and shellability are defined as before.

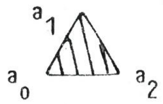
Examples:

1. Simplices & Faces

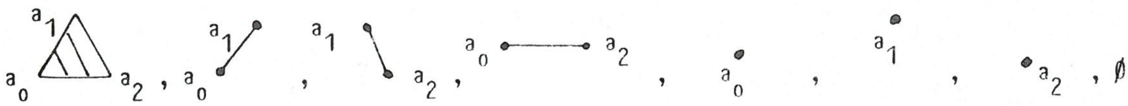
-1-simplex = $\emptyset$

0-simplex = point = .

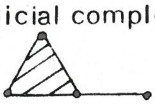
1-simplex = 

2-simplex = 

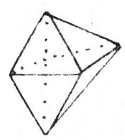
with faces



2. Simplicial complexes and purity

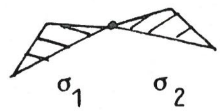
Δ =  ,

not pure

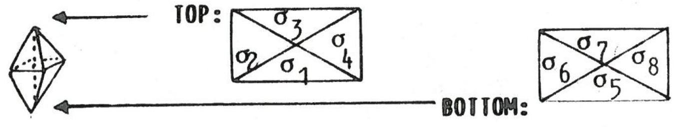
Δ = octahedron = 

pure & $\dim \Delta = 2$

3. Shellability

$\Delta =$ 

is not shellable because $\dim(\sigma_1 \cap \sigma_2) = 0 < 1 = \dim \Delta - 1$.

$\Delta =$ 

The octahedron is shellable with the shelling $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8$.
Note: $\sigma_1\sigma_7\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_8$ is not a shelling as $\sigma_1 \cap \sigma_7 = \emptyset$.

POSETS AND ORDER COMPLEXES

Definition: Let $P = (P, \leq)$ be a poset, i.e.,

(1) $x \leq x \qquad \forall x \in P$

(2) $x \leq y \;\&\; y \leq x \Rightarrow x = y \qquad \forall x,y \in P$

(3) $x \leq y \;\&\; y \leq z \Rightarrow x \leq z \qquad \forall x,y,z \in P$

Let $\to$ be the covering relation in P, i.e., $x \to y$ if $x < y$ and $\nexists z \in P$ s.t. $x < z < y$.

The Hasse Diagram of P is the directed graph with vertices = P and arcs = $x \to y$ written $\overset{y}{\underset{x}{|}}$.

P has a $\hat{0}$ (resp. $\hat{1}$) if P has a unique minimal (resp. maximal) element

$$\therefore \hat{0} \leq x \leq \hat{1} \quad \forall x \in P .$$

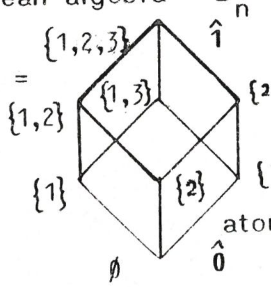
The atom set of P is $A(P) = \{a \in P \mid 0 \to a\}$. A chain of length n in P is a totally ordered subset $c = \{x_0 < x_1 < \ldots < x_n\} \subseteq P$, notation: $l(c) = n$.
The order complex of P is the abstract simplicial complex $\Delta(P) = \{c \subseteq P \mid c \text{ is a chain}\}$.
Note: the dimension of a chain is its length and if $\dim \Delta(P) = d$ we write $l(P) = d$.
Let $M(P) = M(\Delta(P)) = \{m \subseteq P \mid m \text{ a maximal chain}\}$ and call P shellable if $\Delta(P)$ is shellable.

Examples:

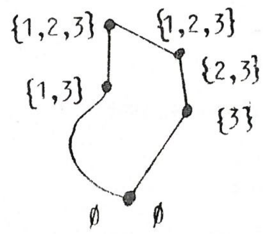
1. The n-chain $C_n = (\{0,1,\ldots,n\}, \leq)$

e.g. $C_4 =$ 

2. The Boolean algebra B_n = (subsets of $\{1,2,\ldots,n\}, \subseteq$)

e.g. $B_3 =$ 

chain c $l(c) = 2$



maximal chain m
$l(m) = l(B_3) = 3$

RECURSIVE ATOM ORDERINGS

Definition: A poset P is graded if

(1) P is finite

(2) P has a $\hat{0}$ and $\hat{1}$

(3) ΔP is pure: $m_1, m_2 \in M(P) \Rightarrow l(m_1) = l(m_2)$.

For now, all our posets will be graded. Any $x,y \in P$ define an interval $[x,y] = \{z \mid x \leq z \leq y\}$. P admits a recursive atom ordering (RAO) if there is a permutation $a_1 a_2 \ldots a_p$ of A(P) s.t.

(R1) $\forall_j$: $[a_j,\hat{1}]$ admits an RAO where the atoms of $[a_j,\hat{1}]$ covering some $a_i, i < j$, come first

(R2) $\forall i < j$: if $a_i, a_j < y \Rightarrow$ there is a $k < j$ and a $z \in P$ such that $a_k, a_j \rightarrow z \leq y$.

Example: In B_n , an RAO of the atoms of $[S,\hat{1}]$ is $S_1 S_2 \ldots S_p$ where $S_i = S \cup \{t_i\}$, $t_1 < t_2 < \ldots < t_p$.

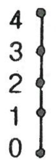
Definition: With each $m = \{\hat{0} = x_o \rightarrow x_1 \rightarrow x_2 \ldots \rightarrow x_d = 1\} \in M(P)$ we associate the sequence $x_o x_1 x_2 \ldots x_d = (x_i)$.

If $m_1 = (x_i)$, $m_2 = (y_i)$ we have the lexicogrpahic order:
Let s be the least index s.t. $x_s = y_s$ & $x_{s+1} \neq y_{s+1}$
$\therefore m_1 <_L m_2$ iff x_{s+1} comes before y_{s+1} in the RAO of $[x_s,\hat{1}]$.

Examples:

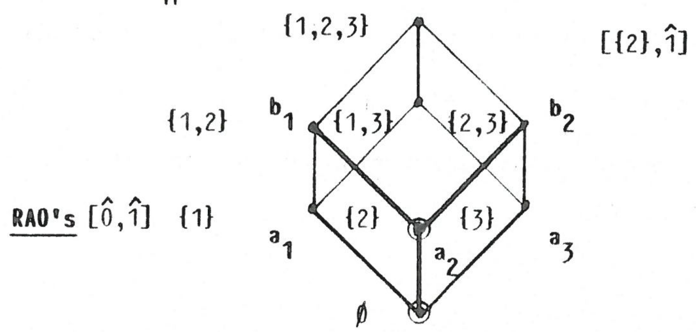
1. The n-chain $C_n = (\{0,1,\ldots,n\}, \leq)$

e.g. C_4 =

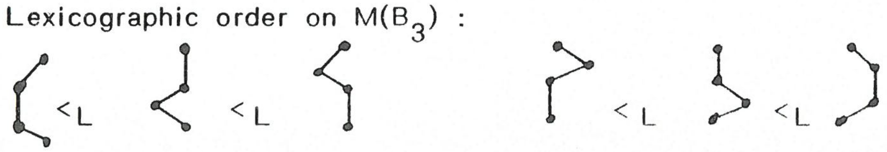


2. The Boolean algebra B_n = (subsets of $\{1,2,\ldots,n\}$,)

e.g. B_3 =



Lexicographic order on $M(B_3)$:



Theorem [B-W]:

P admits an RAO => P shellable .

Proof: We show that the order $<_L$ is a shelling. Given m_1, m_2 and s as before we must find $m \in M(P)$ s.t. $m <_L m_2$ and $m_2 \cap m_1 \subseteq m_2 \cap m = m_2 - \{y_i\}$.

Let t be the least index s.t. $x_t = y_t$, $t > s$.

and let $x = x_s = y_s$, $y = x_t = y_t$. We induct on t-s , noting that $t - s \geq 2$.

If $t - s = 2$, then let $m = m_2 - \{y_{s+1}\} + \{x_{s+1}\}$

$\therefore m \in M(P)$ also $m_1 <_L m_2 \Rightarrow m <_L m_2$ and $m_2 \cap m_1 \subseteq m_2 - \{y_{s+1}\} = m_2 \cap m$.

If $t - s > 2$, then let the RAO of $[x, \hat{1}]$ be $a_1 a_2 \ldots a_p$ with $x_{s+1} = a_i$ & $y_{s+1} = a_j, i < j$.

Case 1: $y_{s+1} \leftarrow a_k$ for some $k < j$. Same as in the case t-s = 2 with a_k in the rôle of x_{s+1} .

Case 2: $y_{s+1} \nleftarrow a_k$ for all $k < j$. By (R2) pick k and z with $a_k, a_j \to z \leq y$.

Consider any maximal chain of the form $\tilde{m} = y_0 y_1 \ldots y_s y_{s+1} z^{***} y_t y_{t+1} \ldots y_d$, ***arbitrary.

Now $\tilde{m} <_L m_2$ by (R1) in $[y_{s+1}, \hat{1}]$, & by induction

$$\exists m \text{ with } m <_L m_2 \ \& \ m_2 \cap m \subseteq m_2 \cap \tilde{m} \subseteq m_2 \cap m = m_2 - \{y_i\} \ .$$

PARTITION LATTICES AND RAO's

Let $[n] = \{1,2,\ldots,n\}$.

A partition π of $[n]$, $\pi \vdash [n]$, is a collection of sets $B_1, B_2, \ldots, B_k$ with $\dot{\bigcup} B_i = [n]$.

We write $\pi = B_1/B_2/\ldots/B_k$ & call the B_i blocks. Let Π_n = all $\pi \vdash [n]$ ordered by refinement i.e., $\pi = B_1/\ldots/B_k \leq \lambda = C_1/\ldots/C_l$ if each C_i is a union of B_j's.

Note that Π_n is in fact a lattice, i.e. if $\pi, \lambda \in \Pi_n$ then they have a least upper bound or join, $\pi \vee \lambda$, and a greatest lower bound or meet, $\pi \wedge \lambda$.

Specifically if $\pi = B_1/\ldots/B_k$, $\lambda = C_1/\ldots/C_l$.

Then $\pi \wedge \lambda$ has blocks $B_i \cap C_j \quad \forall_{i,j}$

And $\pi \vee \lambda$ has blocks $B \cup (\bigcup_i C_i) \cup (\bigcup_j B_j) \cup \ldots$ where $\bigcup_i C_i$ is over all C_i s.t. $C_i \cap B \neq \emptyset$.

$\bigcup_j B_j$ is over all B_j s.t. $B_j \cap C_i \neq \emptyset$, etc

A poset P is strongly recursive if, given any permutation of σ of A(P) there is an RAO of P agreeing with σ on A(P) .

Lemma:

Π_n is strongly recursive.

Proof: Induct on n, the lemma being trivial if $n \leq 3$.

Given A permutation of $A(\Pi_n)$:

$$\sigma = a_1 a_2 \ldots a_p \quad \text{where} \quad p = \binom{n}{2},$$

we first give an arbitrary RAO to $[a_1, \hat{1}]$. Trivially this is an RAO of $\hat{0} \cup [a_1, \hat{1}] \subseteq \Pi_n$.
Assuming that for $i < j$ we have given an RAO to $[a_i, \hat{1}]$ which extends to an RAO of $\hat{0} \cup (\bigcup_{i<j} [a_i, \hat{1}])$ agreeing with σ on $a_1 a_2 \ldots a_{j-1}$ we extend the RAO to $[a_j, \hat{1}]$ as follows.

Let $b_i = a_i \vee a_j$ for $1 \leq i < j$ and let $c_1, \ldots, c_{a_o}$ be the rest of the atoms of $[a_j, \hat{1}]$

since $[a_j, \hat{1}] \cong \Pi_{n-1}$ is strongly recursive by induction, it admits an RAO agreeing with

$$b_1 b_2 \ldots b_{j-1} c_1 c_2 \ldots c_8$$

We claim that $\hat{0} \cup (\bigcup_{i \leq j} [a_i, \hat{1}])$ still satisfies R1 and R2, agreeing with σ on $a_1 a_2 \ldots a_j$.

R1 is true by construction. To verify R2 note:

$$a_i, a_j \leq y \Rightarrow a_i \vee a_j \leq y \Rightarrow a_i, a_j \rightarrow b_i \leq y$$

$\therefore$ R2 holds with $k = i, z = b_i$.

EXPONENTIAL STRUCTURES

Let $\mathbb{P} = \{1,2,3,\ldots\}$ & given $f : \mathbb{P} \rightarrow \mathbb{P}$ define $f_\pi(x) = \sum_{n>1} \frac{f(n)x^n}{n!}$.

Theorem:

If $f,g : \mathbb{P} \rightarrow \mathbb{P}$ satisfy $g(n) = \sum_{\pi \in \Pi_n} f(|B_1|) \ldots f(|B_k|)$ then $1 + g_\pi(x) = e^{f_\pi(x)}$.

If $(P_1, \leq_1)$, $(P_2, \leq_2)$ are posets, their product $(P, \leq)$ is $P = P_1 \times P_2$ and $(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq_1 y_1$ & $x_2 \leq_2 y_2$.

If $\pi = B_1/\ldots/B_k \in \Pi_n$ then the type of π is $|\pi| = (m_1, \ldots, m_n)$ where $m_i \neq$ # of blocks of size i.

Proposition: If $|\pi| = (m_1, \ldots, m_n)$, $\sum_i m_i = k \Rightarrow [\hat{0}, \pi] \cong \Pi_1^{m_1} \times \ldots \times \Pi_n^{m_n}$ and $[\pi, \hat{1}] \cong \Pi_k$.

An exponential structure is a sequence of posets with $\hat{1}$: $Q = (Q_1, Q_2, \ldots)$ s.t.

(1) If $p \in Q_n$ is minimal => $[p, \hat{1}] \cong \Pi_n$

(2) If $\pi \in Q_n$ => $|\pi| = (m_1, \ldots, m_2)$ is the same in all copies of Π_n in which π lies & $\{q \mid q \leq \pi\} \cong Q_1^{m_1} \times \ldots \times Q_n^{m_n}$.

Let Q be exponential, $M(n)$ = # of minimal $p \in Q_n$.

Theorem[S]

$f, g : \mathbb{P} \to \mathbb{P}$ satisfy $g(n) = \sum_{\pi \in Q_n} f(1)^{m_1} \ldots f(n)^{m_n}$

$$\Rightarrow 1 + g_Q(x) = e^{f_Q(x)} \quad \text{where } f_Q(x) = \sum_{n \leq 1} f(n) \frac{x^n}{n!M(n)}$$

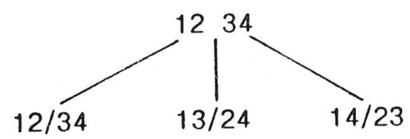
Examples:

1. d-divisible Partitions

$\Pi_n^{(d)} = \{\pi = B_1/\ldots/B_k \mid \pi \vdash [nd], d \text{ divides } |B_i| \ \forall_i\}$

ordered by refinement

$\Pi_2^{(2)}$ =

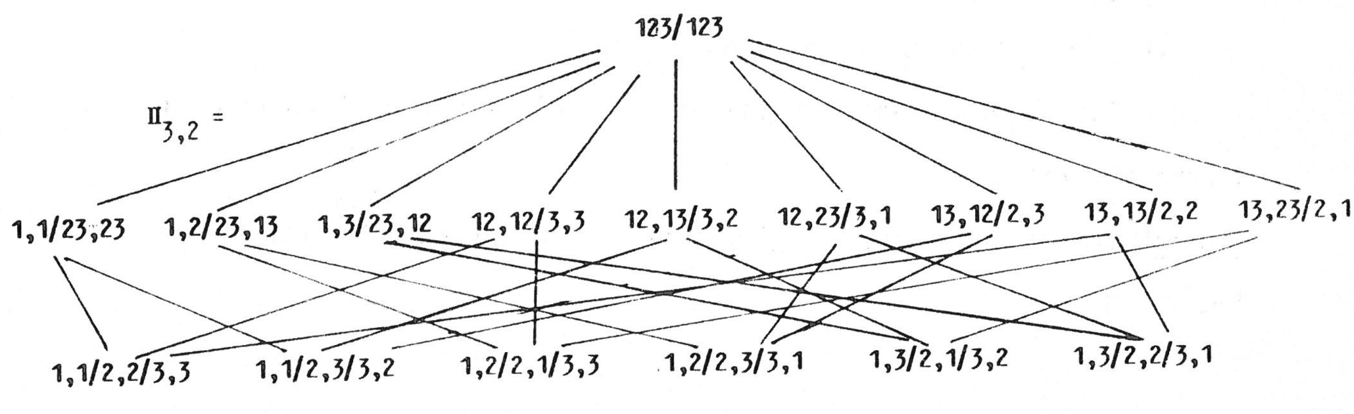


2. Vector partitions of $[n]^r = ([n], [n], \ldots, [n])$

$\Pi_{n,r} = \{\pi = B_1/\ldots/B_k \vdash [n]^r \mid B_i = (B_{i1}, \ldots, B_{ir}), |B_{i1}| = \ldots = |B_{ir}|\}$

We write $\pi = B_{11}, \ldots, B_{1r}/\ldots/B_{k1}, \ldots, B_{kr}$

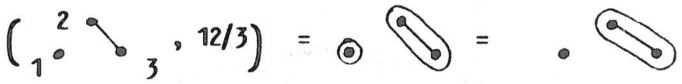
$\Pi_{3,2}$ =

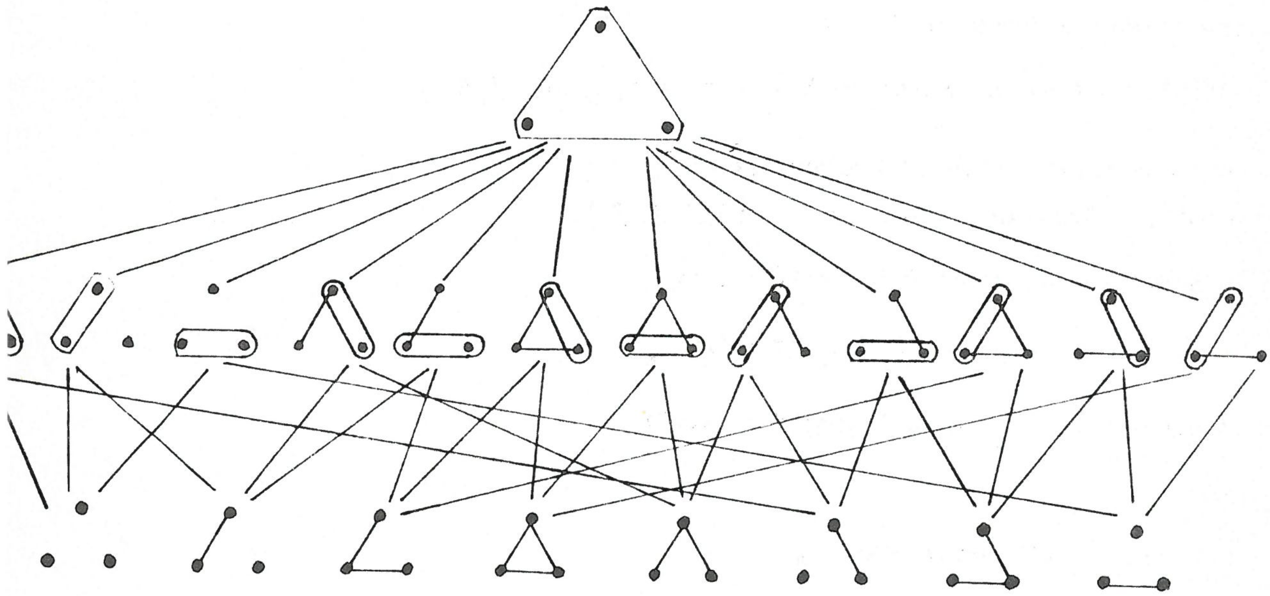


3. Colored Graphs

$\chi_n = \{(G,\pi) \mid G$ a graph on $[n]$, $\pi = B_1/\ldots/B_k \vdash [n]$ s.t. B_i is independent in $G\ \forall_i\}$

$(G,\pi) \le (H,\rho)$ iff $\pi \le \rho = B_1/\ldots/B_k$ and whenever $u \in B_i$ & $v \in B_j$, $i \neq j$ then $uv \in E(G) \Leftrightarrow uv \in E(H)$, where E(G), E(H) are the edge sets of G,H.

We write 



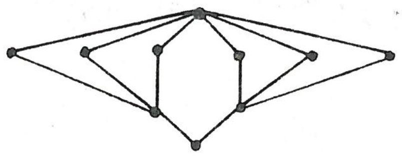
χ_3

RECURSIVE EXPONENTIAL STRUCTURES

If $Q = (Q_1, Q_2, \ldots)$ is exponential, let $\hat{Q} = (\hat{Q}_1, \hat{Q}_2 \ldots)$

Where $\hat{Q}_n = Q_n$ with a $\hat{0}$ adjoined ∴ Q_n is graded. It is not true that every exponential structure admits an RAO

Example:

$\hat{Q}_3 =$ 

However the 3 previous "natural" examples do admit RAO's constructed as follows: Given any totally orderes set S, let

$$W(S) = \text{set of all words on } S = \{\sigma \mid \sigma = t_1 t_2 \ldots t_k,\ t_i \in S\}$$

Hence W(S) is totally ordered lexicographically by $\leq_L$.
Also let B(S) = Boolean algebra on S = $\{T \mid T \subseteq S\}$.

There is a natural injection $B(S) \hookrightarrow W(S)$, by

$$T = \{t_1 < t_2 < \ldots < t_k\} \mapsto t_1 t_2 \ldots t_k$$

∴B(S) is ordered by $\leq_L$ as a "subset" of W(S) .

Theorem:

$\hat{\Pi}_n^{(k)}$, $\hat{\Pi}_{n,r}$ and $\hat{\chi}_n$ all admit RAO's .

Proof: It follows from the Lemma (Π_n strongly recursive) that we need only show that in each case $A([\hat{0},\hat{1}])$ can be ordered to satisfy R2. In each example $\leq_L$ will be used.

1. $\hat{\Pi}_n^{(d)}$: since $\Pi_n^{(d)} \subseteq B(B[n])$, the atoms of $\hat{\Pi}_n^{(d)}$ can be ordered by $\leq_L$, say the order is $\pi_1 \pi_2 \ldots \pi_p$. Given $i < j$ and $\pi_i = (B_1, \ldots, B_n)$, $\pi_j = (C_1, \ldots, C_n)$ there are 2 cases

   (i) $B_1 \nmid C_1$: Hence $B_1 <_L C_1$ so consider $l = \min(B_1 - C_1)$ and $m = \max C_1$.

   Note that $l < m$. Find the block C_t in π_j s.t. $l \in C_t$ and construct the k-sets

   $$C_1' = C_1 - \{m\} + \{l\},\ C_t' = C_t - \{l\} + \{m\}$$

   and the atom $\pi_k = \pi_j - \{C_1, C_t\} + \{C_1', C_t'\}$.
   Now $l < m \Rightarrow C_1' <_L C_1 \Rightarrow \pi_k <_L \pi_j \Rightarrow k < j$

   Furthermore

   $$\pi_j \vee \pi_k = C_1 \cup C_t / C_2 / C_3 / \ldots / \hat{C}_t / \ldots \leftarrow \pi_j, \pi_k$$

Finally if $y > \pi_i,\pi_j \Rightarrow l,m$ are in the same block of $y \Rightarrow C_1 \cup C_t$ is contained in a block of y

$$\therefore y \geq \pi_j V \pi_k .$$

(ii) $B_1 = C_1$: find the first S s.t. $B_s \not\Vdash C_s$ and apply the same construction as in (i) to B_s, C_s . The only additional verification required is that $l \in C_t >_L C_s$ which is true since $B_q = C_q$ for $q < S$.

2. $\hat{\Pi}_{n,r}$: since $\Pi_{n,r} \subseteq B(W(B([n])))$ we can order $A(\hat{\Pi}_{n,r})$ by $\leq_L$ to obtain $\pi_1 \pi_2 \ldots \pi_p$ given $i < j$ and $\pi_i = 1, b_{12}, \ldots, b_{1r}/2, b_{22}, \ldots, b_{2r}/\ldots$ and $\pi_j = 1, c_{12}, \ldots, c_{1r}/2, c_{22}, \ldots, c_{2r}/\ldots$ There are 2 cases

(i) $(1, \ldots, b_{1r}) \nmid (1, \ldots, c_{1r})$: Hence $(1, \ldots, b_{1r}) <_L (1, \ldots, c_{1r})$ so consider the smallest index m s.t. $b_{1m} \nmid c_{1m} \Rightarrow b_{1m} < c_{1m}$. Now find the unique element $c_{lm}, 2 \leq l \leq r$, s.t. $c_{lm} = b_{1m}$ and construct

$$\pi_l = \pi_j \text{ with } c_{1m} \text{ and } c_{lm} \text{ interchanged.}$$

Hence $c_{lm} = b_{1m} < c_{1m} \Rightarrow \pi_k <_L \pi_j \Rightarrow k < j$. Also

$$\pi_j V \pi_k = 1l, \ldots, c_{1m} c_{lm}, \ldots, c_{1r} c_{lr}/2, c_{22}, \ldots, c_{2r}/\ldots \leftarrow \pi_j, \pi_k$$

Finally if $y > \pi_i, \pi_j \Rightarrow c_{1m}$ and $c_{lm} = b_{1m}$ are in the same block of $y \Rightarrow (1l, \ldots, c_{1m}, c_{lm}, \ldots c_{1r} c_{lr})$ is contained in some block of $y \Rightarrow y > \pi_j V \pi_k = z$.

(ii) $(1, \ldots, b_{1r}) = (1, \ldots, c_{1r})$: find the smallest index s s.t. $(s, \ldots, b_{sr}) \nmid (s, \ldots, c_{sr})$ and apply the same construction to these 2 sequences. The only additional verification required is that c_{lm} has index $l > s$ but this follows from $(t, \ldots, b_{tr}) = (t, \ldots, c_{tr})$ for $t < s$.

3. $\hat{X}_n$: the atoms of $\hat{X}_n$ are $(G_i, \hat{0})$, $\hat{0} = 1/2/\ldots/n$. Each G_i can be viewed as a set of edges $\{u,v\} = uv$

$\therefore G_i \in B(B([n]))$ and so $\leq_L$ orders the $(G_i, \hat{0})$.

if $(G_i, \hat{0}), (G_j, \hat{0}) \in A(\hat{X}_n)$ with $i < j$, there are 2 cases.

(i) $G_i \subset G_j$: consider the edge $uv = \max G_j$ and the graph $G_k = G_j - uv$.

(ii) $G_i \not\subset G_j$: consider the edge $uv = \min (G_i - G_j)$ and the graph $G_k = G_j + uv$.

By construction, in both cases $(G_k, \hat{0}) <_L (G_j, \hat{0})$ and $(G_k, \hat{0}) V (G_j, \hat{0}) = (G, 1/\ldots/uv/\ldots/n) \leftarrow (G_k, \hat{0}), (G_j, \hat{0})$

where $G = G_k$ (in case (i)) or $G = G_j$ (in case (ii)).

Now if $y = (H, \pi) \geq (G_i, \hat{0}), (G_j, \hat{0}) \Rightarrow H \subseteq G_i \cap G_j \subseteq G$

and u,v are both in the same block of π since uv is an edge in exactly one of G_i, G_j

$$\therefore y \geq x = (G_k,\hat{0}) \vee (G_j,\hat{0})$$

BIBLIOGRAPHY

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