U M B R A L C A L C U L U S

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Lavoro finanziato in parte con fondi ministeriali per la ricerca (40% e 60%) ed in parte dal Progetto Finalizzato Trasporti del C.N.R.

CHAPTER 1. UMBRAL METHOD.

On doit considérer le calcul symbolique comme une méthode rapide pour l'écriture des formules dans une suite de déductions théoriques; mais, lorsqu'il s'agit de déterminer les valeurs des nombres fournis par ce calcul, il est indispensable de remplacer la formule symbolique par le développement ordinaire. On fait de même lorsque la suite des raisonnements laisse dans l'esprit une certaine obscurité; alors on remplace encore la formule par les notations ordinaires. C'est donc, en quelque sorte, pour le développement des nouvelles théories, une sténographie des formules de l'Arithmétique et de l'Algèbre.

Cette méthode est déjà ancienne; on la trouve comme procédé mnémonique dans les écrits de LEIBNIZ, pour les dérivées successives d'un produit de deux ou de plusieurs facteurs; on la retrouve dans la série de TAYLOR étendue au cas de plusieurs variables; [...].

Dévevoppée plus tard par LAPLACE, par VANDERMONDE et par HERSCHEL, elle a été considérablement augmentée par les travaux de CAYLEY et de SYLVESTER, dans la théorie des formes.

E. LUCAS [3]

In its primitive form, umbral notation, or symbolic notation as it was called by invariant theorists in the past century, is an algorithmic device for treating a sequence $a_1, a_2, a_3, \ldots$ as a sequence of powers $a, a^2, a^3, \ldots$ Computationally, the technique turned out to be very effective in the hands of BLISSARD (after whom the device is sometimes named), BELL, and above all SYLVESTER, to name only a few. Several authors attempted to set the "calculus", as it somewhat improperly came to be called, on a rigorous foundation; the last unsuccessful attempt is BELL's paper of 1941.

G.-C. ROTA [6]

[UMBRAL NOTATION FOR SEQUENCES] Given two sequences (a_i), (b_i) of real numbers, for every $n \in N$ we set:

1) $$A^n := a_n$$

2) $$B^n := b_n$$

3) $$(A+B)^n := \Sigma_i \binom{n}{i} A^i B^{n-i};$$

the symbols A and B are called the *umbrae* of the given sequences. Note that, in the identities 1), 2) and 3), the powers are not powers, and the sum is not a sum.

With this notation, the usual *exponential generating function* of the sequence (a_i) can be written as follows:

4) $$\sum_i a_i \frac{x^i}{i!} = \sum_i A^i \frac{x^i}{i!} = \exp(Ax).$$

In [2], A.P.GUINAND states two rules of the umbral method:

RULE 1: INTERPRETATION Expressions involving one or several umbrae are to be interpreted by expanding as power series in the umbrae and replacing exponents by suffixes.

RULE 2: MANIPULATION Additions or linear combinations of equations involving umbrae are permissible, but multiplication is only valid when the factors have no umbra in common. In general, any step in manipulation is valid if and only if it remains valid when interpreted in non-umbral form.

Loosely speaking, the basic idea of the umbral method, in this form, is: *interchange exponents with suffixes: maybe you will get a correct result.*

Some classical examples:

THE BERNOULLI NUMBERS (see [3]) The Bernoulli numbers are defined as the elements of the sequence (b_i) such that

5) $$\exp(Bx) = (\exp(x)-1)^{-1} \cdot x, \quad B^i := b_i;$$

obviously

6) $$b_0 = 1;$$

from 5) we have:

7) $$\exp((B+1)x) - \exp(Bx) = x;$$

changing x with -x in 5) we get :

8) $$\exp(-Bx) = -x(\exp(-x)-1)^{-1} = \exp(x) \cdot (\exp(x)-1)^{-1} \cdot x = \exp((B+1)x);$$

equating coefficients of $x^n/n!$ in identities 7) and 8) we get:

9) $(B+1)^n - B^n = \delta_1^n$, $(B+1)^n = (-B)^n$

and, by recurrence, we derive

10) $(b_i) = (1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, \dots)$

|THE EULER NUMBERS| (see [4]) The Euler numbers are the elements of the sequence (e_i) such that

11) $\exp(Ex) = \operatorname{sech} x = 2(\exp(x)+\exp(-x))^{-1}$, $E^i := e_i$;

since the function sech is even, we have

12) $e_{2n+1} = 0$ for $n \geq 0$;

moreover, from 11) we get:

13) $\exp((E+1)x) + \exp((E-1)x) = 2$

and equating coefficients of $x^n/n!$ we derive the recurrence formula

14) $(E+1)^n + (E-1)^n = 2\,\delta_o^n$

which yields

15) $(e_i) = (1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, \dots)$.

|INVERSE RELATIONS| (see [4])

Theorem 1.- *Let (a_i), (b_i) be two sequences, then*

16) $a_n = \sum_k (-1)^k \binom{n}{k} b_k$ *for every* $n \in N$

if and only if

17) $b_n = \sum_k (-1)^k \binom{n}{k} a_k$ *for every* $n \in N$.

Proof. In umbral form, 16) and 17) can be written respectively as:

18) $A^n = (1-B)^n$, $A^n := a_n$, $B^n := b_n$

19) $B^n = (1-A)^n$

that is

20) $A = 1-B$

21) $B = 1-A$

which are clearly equivalent. ■

More generally, we have:

Theorem 2.- *Let (a_i), (b_i), (c_i), (d_i) be four given sequences; if two of the following identities hold*

22) $a_n = \sum_h \binom{n}{h} c_h b_{n-h}$ *for every* $n \in N$

23) $$b_n = \sum_h \binom{n}{h} d_h a_{n-h} \quad \text{for every } n \in N$$

24) $$\delta_o^n = \sum_h \binom{n}{h} c_h d_{n-h} \quad \text{for every } n \in N$$

then the third one also holds.

<u>Proof.</u> In umbral form, identities 22), 23), and 24) are

25) $$A = B+C$$

26) $$B = A+D$$

27) $$C+D = 0. \qquad \blacksquare$$

|CAUTION !|

28) $$1+1 = 2$$

because, for every $n \in N$:

29) $$(1+1)^n := {}_i \binom{n}{i} = 2^n ,$$

but

30) $$A+A \neq 2A$$

because, for every $n \in N$:

31) $$(A+A)^n := \sum_i \binom{n}{i} a_i a_{n-i} \neq 2A =: (2A)^n .$$

Moreover, for every $x \in N$:

32) $$\underbrace{(1+1+\ldots+1)}_{x \text{ times}} = x1$$

that is:

33) $$\underbrace{(1+1+\ldots+1)}_{x \text{ times}}{}^n = x^n$$

What about $\underbrace{(A+A+\ldots+A)}_{x \text{ times}}{}^n$? The answer lies in

|THE NOTION OF BINOMIAL SEQUENCE|

<u>Theorem 3.-</u> *Let (a_n) be a given sequence, and, for every $x, n \in N$, set:*

34) $$f_n(x) := \underbrace{(A+A+\ldots+A)}_{x \text{ times}}{}^n$$

then, for every $x, y, n \in N$:

35) $$f_n(x+y) = \sum_i \binom{n}{i} f_i(x) f_{n-i}(y) \qquad \underline{\text{binomial law}} .$$

Moreover, if $a_o=1$, then all the f_n are polynomials in x, with

36) $$\deg f_n \leq n \qquad \text{for every } n \in N;$$

if, in addition, $a_1 \neq 0$, then

37) $$\deg f_n = n \qquad \text{for every } n \in N .$$

Proof. In umbral form, we have

$$f_n(x+y) := \underbrace{(A+A+\ldots+A)}_{(x+y)\text{ times}}{}^n = ((\underbrace{A+A+\ldots+A}_{x\text{ times}})+(\underbrace{A+A+\ldots+A}_{y\text{ times}}))^n =$$

$$= \sum_i \binom{n}{i} (\underbrace{A+A+\ldots+A}_{x\text{ times}})^i (\underbrace{A+A+\ldots+A}_{y\text{ times}})^{n-i} = \sum_i \binom{n}{i} f_i(x)\, f_{n-i}(y) .$$

If $a_o=1$, then $f_o(x)=1$ for every $x\varepsilon N$, and the second part follows by induction on n. ∎

A sequence $f_n(x)$ satisfying identity 35) will be called a *binomial sequence;* a binomial sequence of polynomials satisfying 37) is usually called a *polynomial sequence of binomial type*.

UMBRAL NOTATION FOR POLYNOMIALS As a simple instance of umbral notation for polynomials, we give the umbral form of binomial law:

38) $$\underline{\underline{p}}(x+y)^n = (\underline{\underline{p}}(x) + \underline{\underline{p}}(y))^n \qquad \underline{\underline{p}}^n := p_n .$$

If $\underline{\underline{p}} :=(p_n)$ is a polynomial sequence of binomial type, for every polynomial

39) $$q(x) := \sum_i a_i x^i$$

we will set

40) $$q(\underline{\underline{p}}) := \sum_i a_i p_i .$$

The map

41) $$T: q(x) \longmapsto q(\underline{\underline{p}})$$

is clearly an automorphism of the vector space $R[x]$; such a map will be called the *umbral representation* of the sequence $\underline{\underline{p}}$.

CHAPTER 2. UMBRAL ALGEBRA.

Umbral methods become an effective calculus by regarding sequences as elements of $(R[x])^*$, that is, linear functionals over polynomials, and by structuring $(R[x])^*$ as a complete topological algebra which is, in fact, the algebra of exponential formal series. This algebra is the so-called umbral algebra.

The endomorphisms of this algebra are compositions with infinitesimal functionals. They are represented by means of recursive matrices.

LINEAR ALGEBRA MACHINERY As usual, we will denote by $R[x]$ the vector space of all real polynomials, and by $R[[x]]$ the linear dual of $R[x]$.

For every $n \in N$, we will write x_n instead of $x^n \in R[x]$; in fact, no use will be made of multiplication in $R[x]$, and n in x^n is nothing but an index.

If $\alpha \in R[[x]]$ and $p \in R[x]$, we will set

1) $\langle \alpha | p \rangle := \alpha(p)$.

A linear functional $\alpha \in R[[x]]$ will be represented by the sequence (a_n), where

2) $a_n := \langle \alpha | x_n \rangle$;

conversely, each sequence (a_n) represents - in such a way - an element $\alpha \in R[[x]]$.

A TOPOLOGY OVER R[[x]] The *order* of a linear functional $\alpha := (a_i)$ is

3) $\sigma(\alpha) := \min\{ i \in N : a_i \neq 0\}$

if $\alpha \neq 0$ and

4) $\sigma(0) := +\infty$.

The *distance* between $\alpha, \beta \in R[[x]]$ is defined to be

5) $d(\alpha, \beta) := 2^{-\sigma(\alpha-\beta)}$

with the convention

6) $2^{-\infty} := 0$.

$R[[x]]$, endowed with the map $d: R[[x]] \times R[[x]] \to R$ turns out to be a complete linear metric space. A sequence (α^n) of elements of $R[[x]]$

converges to $\alpha \in R[[x]]$ whenever for every $m \in N$ there exists $n(m) \in N$ such that, for every $n \geq n(m)$:

7) $$\langle \alpha^n | x_m \rangle = \langle \alpha | x_m \rangle .$$

Note that the elements of a sequence in $R[[x]]$ are indexed as powers, but they are not powers.

A PSEUDOBASIS FOR $R[[x]]$ For every $m \in N$, the linear functional ξ^m is defined by

8) $$\langle \xi^m | x_n \rangle := \delta_n^m ;$$

for every $\alpha := (a_i) \in R[[x]]$ we have

9) $$\sum_{n \in N} a_n \xi^n := \lim_{n \in N} \sum_i a_i \xi^i = \alpha .$$

Then, the sequence (ξ^n) is a pseudobasis for $R[[x]]$.

EVALUATIONS For every $a \in R$, the map $\varepsilon_a : R[x] \longrightarrow R$

10) $$\varepsilon_a : p(x) \longmapsto p(a)$$

is a linear functional called *evaluation at a;* its expansion is:

11) $$\varepsilon_a = \sum_n a^n \xi^n .$$

We will write ε instead of ε_o; we have:

12) $$\varepsilon := \varepsilon_o = \xi^o .$$

$R[[x]]$ BECOMES AN ALGEBRA For every $i,j \in N$, set

13) $$\xi^i \cdot \xi^j := \binom{i+j}{i} \xi^{i+j} ;$$

by linearity and contituity, we can extend 13) to the multiplication over $R[[x]]$ such that, if $\alpha := (a_i)$ and $\beta := (b_i)$ then $\alpha \cdot \beta = \gamma = (c_n)$, with

14) $$c_n := \sum_i \binom{n}{i} a_i b_{n-i} .$$

This multiplication structures $R[[x]]$ as an associative, commutative topological algebra, with ε as identity. We have:

15) $$\varepsilon_a \cdot \varepsilon_b = \varepsilon_{a+b} \quad \text{for every } a,b \in R$$

and

16) $$\xi^i = \frac{1}{i!} (\xi^1)^i \qquad \text{for every } i \in N$$

(here, the exponent in the right hand side really denotes a power in the algebra just defined: in the following the correct interpretation of exponents will be suggested by the context).

For every $\alpha, \beta \in R[[x]]$ we have

17) $$\sigma(\alpha\beta) = \sigma(\alpha) + \sigma(\beta) .$$

PSEUDOGENERATORS By identities 9) and 16), each linear functional can be expanded as an exponential series in ξ^1. More precisely, if $\alpha=(a_i)$, then

18) $$\alpha = \sum_i a_i \frac{(\xi^1)^i}{i!} \quad ;$$

we will say that ξ^1 is a *pseudogenerator* for the algebra R[[x]].

A *delta-functional* is a functional $\alpha:=(a_i)$ such that

19) $$a_o = 0 \neq a_1 \quad ;$$

a functional α is a pseudogenerator for the algebra R[[x]] if and only if it is a delta-functional.

UMBRAL METHOD MADE RIGOROUS (cfr. [6])

Umbral method described in the preceding chapter can be completely explained by using the algebra structure just defined over R[[x]]. For instance, if we set $(A+B)^n:=\langle\alpha\beta|x_n\rangle$ and $x:=\xi^1$, then identities 3) and 4) of Chap.1 become identities 14) and 18) of the present one. Consequently, the algebra R[[x]] can be rightly called the *umbral algebra*.

LOCALLY FINITE MATRICES Define a *locally finite matrix* to be an N×N matrix each column of which has only a finite number of non-zero entries.

Let us represent every linear functional $\alpha=\sum a_i \xi^i$ as a row-vector with entries (a_i).

For any given linear operator T over R[[x]], let us define the *representing matrix* M(T)

20) $$M(T) := (\tau^n)$$

to be the N×N matrix whose n-th row is

21) $$\tau^n := T(\xi^n) .$$

Then M(T) is a locally finite matrix if and only if T is continuous. If such is the case, the action of T on $\alpha=\sum a_i \xi^i$ is the row-by-column product of matrices:

22) $$T(\alpha) = (a_i)\times M(T) .$$

Moreover, if S and T are both continuous linear operators, the product $M(S)\times M(T)$ can be performed, and

23) $$M(T\circ S) = M(S)\times M(T) \ .$$

RECURSIVE MATRICES (see [1])

Let T be a continuous linear operator over $R[[x]]$ and let $M(T)=(\tau^n)$: then T is an algebra morphism if and only if for every $n\varepsilon N$:

24) $$\tau^n := T(\frac{(\xi^1)^n}{n!}) = \frac{(T\xi^1)^n}{n!} = \frac{(\tau^1)^n}{n!}$$

and

25) $$\langle \tau^1 | x_0 \rangle = 0 \ .$$

A sequence of functionals (σ^n) satisfying 24) and 25) will be called a *recursive sequence* and σ^1 will be its *recursive rule*. A *recursive matrix* will be a locally finite matrix whose rows are a recursive sequence of functionals; the recursive rule of such a matrix will be the recurrence rule of the sequence of its rows.

Thus, we can conclude that, because of 23), *the multiplicative monoid of recursive matrices is anti-isomorphic to the monoid of continuous endomorphisms of the umbral algebra.*

COMPOSITIONS Let us define an *infinitesimal functional* to be a $\beta\varepsilon R[[x]]$ such that

26) $$\langle \beta | x_0 \rangle = 0 \ .$$

By the preceding arguments, a continuous endomorphism T of the umbral algebra is completely determined by the infinitesimal functional

27) $$\tau^1 := T(\xi^1) \ .$$

For every $\alpha:=(a_i)\varepsilon R[[x]]$, we have - because of 24) -

28) $$T(\alpha) = T(\Sigma a_n \frac{(\xi^1)^n}{n!}) = \Sigma a_n \frac{(\tau^1)^n}{n!}$$

The map T is usually called the *composition with* τ^1. We will write, for every $\alpha\varepsilon R[[x]]$

29) $$\alpha\circ\tau^1 := \alpha(\tau^1) := T(\alpha) \ .$$

Conversely, compositions with infinitesimal functionals are continuous endomorphisms of the umbral algebra. Obviously, the composition with ξ^1 is the identity map. In conclusion: *the monoid of infinitesimal functionals, under composition, is anti-isomorphic to the monoid of continuous endomorphisms of the umbral algebra.*

AUTOMORPHISMS A continuous automorphism of the umbral algebra will map ξ^1 in another pseudogenerator: it follows that *continuous automorphisms of* $R[[x]]$ *are precisely compositions with delta-functionals.*

Let us consider the following groups:

AU := the group of all automorphisms of the umbral algebra;

DF := the group of all delta-functionals, under composition of formal series;

RM := the group of all invertible recursive matrices, under matrix-product;

AU^{op} := the opposite of the group AU.

Theorem 4.- *The maps*

$$\begin{array}{ccc} AU^{op} & \longrightarrow & DF \\ T & \longmapsto & T(\xi^1) \end{array}$$

and

$$\begin{array}{ccc} AU^{op} & \longrightarrow & RM \\ T & \longmapsto & M(T) \end{array}$$

are group isomorphisms. ∎

CHAPTER 3. UMBRAL COALGEBRA.

The structure of topological algebra we gave to $(R[x])^*$ is the dual counterpart of a coalgebra structure defined over $R[x]$ by the assignment $p(x) \mapsto p(y+z)$. In this chapter we study in detail this "umbral coalgebra".

LINEAR OPERATORS OVER POLYNOMIALS ... Let us represent every polynomial $p(x)=\sum a^i x_i$ as a column-vector with entries (a^i): then the pairing $\langle \beta | p \rangle$, $\beta = \sum b_i \xi^i$ is the usual row-by-column product of matrices

1) $$\langle \beta | p \rangle = (b_i) \times (a^i)$$

For any given linear operator T over $R[x]$ let us define the *representing matrix* $M(T)$

2) $$M(T) := (\tau_n)$$

to be the $N \times N$ matrix whose n-th column is

3) $$\tau_n := T(x_n) .$$

By definition, $M(T)$ is a locally finite matrix. Conversely, every locally finite matrix is the representing matrix of a suitable linear operator over $R[x]$.

... AND THEIR DUALS Let T be a linear operator over $R[x]$, with dual operator T^*, and let $M(T)$, $M(T^*)$ be the representing matrices of T, T^*, respectively. Obviously:

4) $$M(T) = M(T^*) .$$

Moreover, because of locally finiteness of $M(T)$, we can conclude that *the dual of a linear operator over* $R[x]$ *is continuous, and conversely, every continuous linear operator over* $R[[x]]$ *is the dual of a suitable linear operator over polynomials.*

UMBRAL OPERATORS For every infinitesimal functional α, the composition C_α

5) $$\pi \xmapsto{C_\alpha} \pi \circ \alpha$$

is a continuous linear operator over $R[[x]]$: then there exists a linear operator T_α over polynomials, such that

6) $$T_\alpha^* = C_\alpha .$$

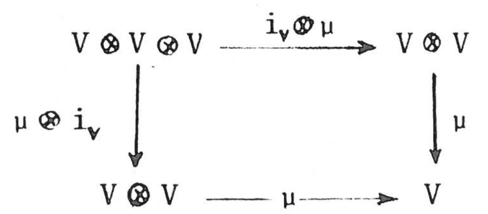
If α is a delta-functional, T_α is invertible and it will be called an *umbral operator*.

THE UMBRAL GROUP What we said above suggests us to give the same name of umbral group to the group of all umbral operators and to the group AU^{op}, the opposite of the group of all continuous automorphisms of the umbral algebra.

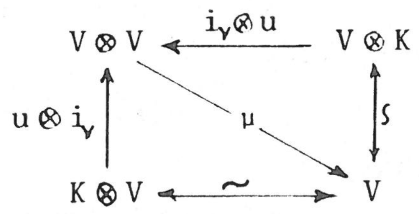
In fact, by Theorem 4 and identity 4), these groups can be regarded as the same group of invertible recursive matrices, acting both over polynomials and linear functionals.

AUTOMORPHISMS OF WHAT? The umbral group acts over $R[[x]]$ as the group of all automorphisms of the umbral algebra. Now, we look for an additional structure over $R[x]$, whose automorphism group is the umbral group. Clearly, such a structure, if exists, must have the umbral algebra as its dual algebra.

ALGEBRAS First of all, recall that an *associative unitary K-algebra* is a triple (V,μ,u), where V is a K-vector space, $\mu: V\otimes V \longrightarrow V$ and $u: K \longrightarrow V$ are linear maps, called *multiplication* and *unit map* respectively, such that the following diagrams commute:

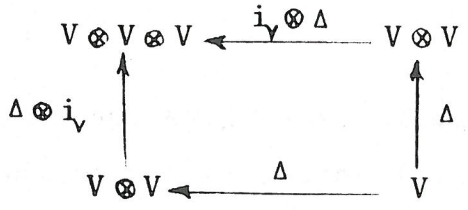


associativity

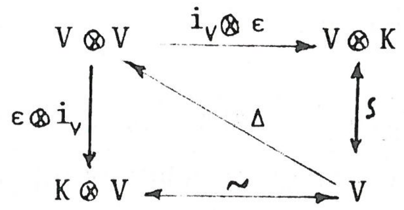


unitary property

COALGEBRAS Dualizing the notion of algebra we define a *coassociative, counitary K-coalgebra* to be a triple (V,Δ,ε), where V is a K-vector space, $\Delta: V \longrightarrow V\otimes V$ and $\varepsilon: V \longrightarrow K$ are linear maps, called *comultiplication* or *diagonalization*, and *counit* or *augmentation*, respectively, such that the following two diagrams are commutative:



coassociativity



counitary property

DUAL ALGEBRA Suppose $C:=(V,\Delta,\varepsilon)$ is a K-coalgebra and set

7) $$u := \varepsilon^*$$

and

8) $$\mu := \Delta^* \circ j$$

9) $$V^* \otimes V^* \xrightarrow{j} (V \otimes V)^* \xrightarrow{\Delta^*} V^*$$

where j is the natural injection.

Then, (V^*,μ,u) is a K-algebra, called the dual algebra of the coalgebra C.

A VERY NATURAL COALGEBRA The umbral algebra can be formally defined as the triple

10) $$A := (R[[x]],\mu,u)$$

where, for every $i,j \in N$:

11) $$\mu(\xi^i \otimes \xi^j) := \binom{i+j}{i} \xi^{i+j}$$

and, for every $k \in R$:

12) $$u(k) := k\xi^0 .$$

Recall that

13) $$R[x] \otimes R[x] \cong R|y,z|$$

and set, for every $n \in N$

14) $$\Delta x_n := \sum \binom{n}{i} y_i z_{n-i} , \qquad y_i := y^i, \; z_i := z^i$$

15) $$\varepsilon(x_n) := \delta_n^0$$

then, for every $p \in R[x]$:

16) $$\Delta p(x) = p(y+z)$$

17) $$\varepsilon p(x) = p(0)$$

and the triple

18) $$C := (R[x],\Delta,\varepsilon)$$

turns out to be a coalgebra, whose dual algebra - because of 11), 12), 14), 15) - is easily shown to be the umbral algebra, as desired.

This coalgebra will be called the *umbral coalgebra*.

COALGEBRA AUTOMORPHISMS An endomorphism of a coalgebra (V,Δ,ε) is a linear map $T: V \longrightarrow V$ such that

19) $$\Delta \circ T = (T \otimes T) \circ \Delta$$

20) $$\varepsilon \circ T = \varepsilon$$

By construction, the automorphisms of the umbral coalgebra are precisely the umbral operators:

Theorem 5.- *The umbral group is the automorphism group of the umbral coalgebra.* ∎

THE BINOMIAL LAW A polynomial sequence (p_n) satisfying

21) $$p_n(y+z) = \sum_i \binom{n}{i} p_i(y)\, p_{n-i}(z)$$

for every $n \in N$, will be called a *polynomial sequence with binomial law*. If, in addition, for every $n \in N$

22) $$\deg p_n = n$$

the sequence (p_n) will be said to be *of binomial type*.

Theorem 6.- *Let T be a linear operator over $R[x]$, and set*

23) $$p_n(x) := Tx_n$$

Then, T is an endomorphism of the umbral coalgebra if and only if (p_n) is a polynomial sequence with binomial law. Moreover, T is an umbral operator if and only if (p_n) is a polynomial sequence of binomial type.

Proof. Suppose T is an endomorphism of the umbral coalgebra: then, for every $n \in N$

24) $$p_n(y+z) = \Delta p_n = \Delta \circ T x_n = (T \otimes T) \circ \Delta x_n = (T \otimes T) \sum \binom{n}{i} y_i z_{n-i} =$$
$$= \sum \binom{n}{i} p_i(y)\, p_{n-i}(z)$$

which proves that (p_n) satisfies the binomial law. This implies

25) $$\deg p_n \leqslant n .$$

Conversely, suppose (p_n) is a polynomial sequence with binomial law; then, for every $n \in N$:

26) $$\Delta T x_n = \Delta p_n = p_n(y+z) = \sum \binom{n}{i} p_i(y)\, p_{n-i}(z) = \\ = \sum \binom{n}{i} (Tx_i) \otimes (Tx_{n-i}) = (T \otimes T)\Delta x_n$$

which proves that T in an endomorphism of the umbral coalgebra. Moreover, if (p_n) is of binomial type, then

27) $$\deg p_n = n$$

and T is invertible; conversely, if T is invertible, then 27) holds and (p_n) is of binomial type. ∎

ASSOCIATIONS For any given endomorphism T of the umbral coalgebra, set

28) $$t_n := Tx_n$$

29) $$\tau := T^{*}\xi^{1} ;$$

the endomorphism T, the sequence (t_n) with binomial law and the infinitesimal functional τ will be said to be reciprocally associated. T is an umbral operator if and only if (t_n) is of binomial type and if and only if τ is a delta-functional.

WARNING The present notion of associated delta-functional correspods to Rota's notion of conjugate delta-functional. The compositional inverse of our associated delta-functionals is called associated delta-functional in Rota's papers.

UMBRAL NOTATION REVISITED By the preceding result, the umbral notation of Chap.1 can be translated in terms of umbral operators. More precisely, let T be an umbral operator, with associated sequence $\underline{\underline{p}} := (p_n)$ and associated delta-functional α. Then, for every polynomial q we have

30) $$q(\underline{\underline{p}}) = Tq .$$

COEFFICIENTS OF A POLYN. SEQ. OF BIN. TYPE The coefficients of polynomials in a sequence of binomial type $\underline{\underline{p}} = (p_n)$ are the column-entries in the representing matrix M(T) of the associated umbral operator T. Such a matrix, as we have seen, is the recursive matrix whose recurrence rule is the associated delta-functional α. This proves the following result:

<u>Theorem 7.-</u> *Let (p_n) be a polynomial sequence of binomial type, with associated delta-functional α, and let*

31) $$p_n = \sum a_n^i x_i$$

then, for every $i,n \in N$

32) $$a_n^i = \langle \frac{\alpha^i}{i!} | x_n \rangle$$ ■

We explicitely note that 32), for i=1, gives

33) $$a_n^1 = \langle \alpha | x_n \rangle$$

that is, the pseudocomponents of the associated delta-functional α are precisely the coefficients of x_1 in p_n.

<u>Theorem 8.-</u> *A polynomial sequence (p_n), with*

34) $$p_n = \sum a_n^i x_i$$

is of binomial type if and only if

35) $$a_o^1 = 0 \neq a_1^1$$

and, for every $n,i,j \in N$

36) $$\binom{i+j}{i} a_n^{i+j} = \sum_k \binom{n}{k} a_k^i a_{n-k}^j .$$

<u>Proof.</u> Condition 36) is equivalent to recursivity of the representing matrix M(T) of the associated umbral operator T, and condition 35) is equivalent to say that the recurrence rule of M(T) is a delta-functional. ■

<u>CONNECTION CONSTANTS</u> Theorem 7 allows us to compute the components, with respect to (x_n), of a given polynomial sequence (p_n) of binomial type. Is it possible to compute the components of (p_n) with respect to a different sequence of binomial type? This problem is known as the problem of connection constants. The following result gives us a complete answer:

<u>Theorem 9.-</u> *Let (p_n) and (s_n) be two given polynomial sequences of binomial type, with associated delta-functional π and σ respectively. The components of (p_n) with respect to (s_n) are the column-entries of the recursive matrix M whose recurrence rule ρ is*

37) $$\rho = \tilde{\sigma}\circ\pi$$

where $\tilde{\sigma}$ *denotes the compositional inverse of* σ.

Proof. Let P and S be the associated umbral operators of (p_n) and (s_n), respectively. Then:

38) $$s_i \overset{PS^{-1}}{\longmapsto} p_i$$

and the recurrence rule of the recursive matrix $M(PS^{-1})$ is

39) $$\tau = \pi\circ\tilde{\sigma}$$

On the other hand, the desired matrix M represents PS^{-1} with respect to the basis (s_n), so

40) $$M = M(S^{-1})M(PS^{-1})M(S) = M(S^{-1}P)$$

and the assertion is proved. ∎

GENERATING FUNCTIONS The *a-th generating function* $(a\in R)$ of a given polynomial sequence $\underline{p}:=(p_n)$ is the functional $\Phi_a(\underline{p})$ defined by

41) $$\Phi_a(\underline{p}) := \sum_i \langle\varepsilon_a|p_i\rangle\xi^i = \sum_{i,j}\langle\varepsilon_a|x_j\rangle\langle\xi^j|p_i\rangle\xi^i = \sum_{i,j} a^j\langle\xi^j|p_i\rangle\xi^i .$$

We recall that the functional ε_1 is frequently denoted by "exp" :

42) $$\exp := \varepsilon_1 = \sum_i \xi^i = \sum_i \frac{1}{i!}(\xi^1)^i .$$

Polynomial sequences of binomial type can be characterized by means of their generating functions:

Theorem 10.- *A polynomial sequence* $\underline{p}:=(p_n)$ *satisfies the binomial law if and only if there exist an infinitesimal functional* π *and a countable subset A of R such that, for every* $a\in A$

43) $$\Phi_a(\underline{p}) = \exp(a\pi) = \varepsilon_1\circ a\pi .$$

If this is the case, π *is the associated infinitesimal functional of* $\underline{p}$, *and identity 43) holds for every* $a\in R$.

Proof. Suppose $\underline{p}$ satisfies the binomial law, with associated infinitesimal functional π: then, for every $a\in R$:

44) $$\Phi_a(\underline{p}) = \sum_{i,j} a^j\langle\xi^j|p_i\rangle\xi^i = \sum_{i,j}\frac{a^j}{j!}\langle(\xi^1)^j|p_i\rangle\xi^i = \sum_j\frac{a^j}{j!}\sum_i\langle(\xi^1)^j|p_i\rangle\xi^i =$$
$$= \sum_j \frac{a^j\pi^j}{j!} = \exp(a\pi) .$$

Conversely, suppose 43) holds for an infinitesimal functional and for every $a \in A \subseteq R$, A countable. Let us denote by $\underline{p}'$ the polynomial sequence with binomial law associated with π: then, for every $a \in A$:

45) $$\Phi_a(\underline{p}) = \Phi_a(\underline{p}')$$

which implies

46) $$\underline{p} = \underline{p}' \ .$$ ■

|ORTHOGONALITY| A functional sequence (σ^i) and a polynomial sequence (p_i) are said to be a pair of orthogonal sequences if for every $i,j \in N$:

47) $$\langle \sigma^i | p_j \rangle = \delta^i_j \ .$$

Theorem 11.- *Let (σ^i), (p_i) be a pair of orthogonal sequences: then (σ^i) is a recursive sequence and its recurrence rule is a delta-functional if and only if (p_i) is a polynomial sequence of binomial type. If this is the case,*

48) $$\sigma^1 = \tilde{\pi}$$

where $\tilde{\pi}$ denotes the compositional inverse of the associated delta-functional π of (p_i).

Proof. Suppose (σ^i) is a recursive sequence, whose recurrence rule σ^1 is a delta-functional. Let S be the umbral operator associated with the compositional inverse $\tilde{\sigma}^1$ of σ^1: then, for every $n \in N$:

49) $$\delta^i_j = \langle \xi^i | x_j \rangle = \langle \xi^i | S^{-1} S x_j \rangle = \langle (S^{-1})^* \xi^i | S x_j \rangle = \langle \sigma^i | S x_j \rangle$$

then

50) $$S x_i = p_i$$

and (p_i) is of binomial type with associated delta-functional $\tilde{\sigma}^1$. Similar arguments prove the converse. ■

If π is the associated delta-functional of the polynomial sequence of binomial type (p_n), the compositional inverse $\tilde{\pi}$ will be called the *inverse associated delta-functional of (p_n)*. Such functional is called associated in Rota's works.

EXPANSION THEOREM By the preceding result we get:

Theorem 12.- *Let (p_i) be a polynomial sequence of binomial type, with inverse associated delta-fanctional $\tilde{\pi}$; the for every $q \in R[x]$ and every $\sigma \in R[[x]]$*

51) $$q = \sum_i \frac{\langle \tilde{\pi}^i | q \rangle}{i!} p_i$$

52) $$\sigma = \sum_i \frac{\langle \sigma | p_i \rangle}{i!} \tilde{\pi}^i$$

CHAPTER 4. SHIFT INVARIANT OPERATORS.

The derivative D, with its powers, spans a commutative topological algebra of linear operators over R[x], which is isomorphic to the umbral algebra.

For every pseudogenerator S of this new algebra there exists a polynomial sequence of binomial type (p_n) such that, for every $n \in N$, $Sp_n = np_{n-1}$.

HEMIMORPHISMS Recall that a morphism of an algebra $A:=(V,\mu)$ is a linear operator $T: V \longrightarrow V$ such that

1) $$T \circ \mu = \mu \circ (T \otimes T)$$

and, dually, a morphism of a coalgebra $C:=(V,\Delta)$ is a linear operator $T: V \longrightarrow V$ such that

2) $$\Delta \circ T = (T \otimes T) \circ \Delta$$,

Now, we define a notion which is "half" the notion of morphism. A *left hemimorfism* of the algebra A is a linear operator $T: V \longrightarrow V$ such that

3) $$T \circ \mu = \mu \circ (T \otimes i_V)$$

and a left hemimorphism of the coalgebra C is a linear operator $T: V \rightarrow V$ such that

4) $$\Delta \circ T = (T \otimes i_V) \circ \Delta$$

Right hemimorphisms are defined in a similar way. Left hemimorphisms of a commutative algebra (or coalgebra) are also right hemimorphisms, and conversely.

The set of all hemimorphisms of a commutative algebra (coalgebra) is closed under linear combination and functional composition; thus, it is an algebra.

We will denote by Hem(A), Hem(C) the algebra of all hemimorphisms of the commutative algebra A, and the algebra of all hemimorphisms of the cocommutative coalgebra C.

TWO BASIC RESULTS Suppose the commutative algebra A has unit 1. If $T: V \longrightarrow V$ is a hemimorphism of A, we have, for every $v \in V$

5) $T(v) = T(v \cdot 1) = v \cdot T(1)$.

Thus, *every $T \in Hem(A)$ is the multiplication by $T(1)$*. Conversely, if $w \in W$, the map T

6) $v \overset{T}{\longmapsto} v \cdot w$

is in Hem(A), and $T(1)=w$.

The *indicator* of $T \in Hem(A)$ will be

7) $ind(T) := T(1)$.

Theorem 13.- *If A is a commutative algebra with unit, the map*

8) $$ind: Hem(A) \longrightarrow A, \quad T \longmapsto ind(T)$$

is an isomorphism of algebras. ∎

Now, let C be a cocommutative coalgebra with counit ε, and let $A:=C^*$ be its dual algebra (which is commutative, with unit 1). If $T \in Hem(C)$, then $T^* \in Hem(A)$: the dual of T is the multiplication by $T^*(1)$; but not every multiplication is the dual of a $T \in Hem(C)$.

The *indicator* of $T \in Hem(C)$ will be

9) $ind(T) := T^*(1)$.

Theorem 14.- *If C is a cocommutative coalgebra with counit ε, the map*

10) $$ind: Hem(C) \longrightarrow C^*, \quad T \longmapsto ind(T)$$

is a monomorphism of algebras. ∎

$Hem(R[x]) \simeq R[[x]]$ Theorem 14 can be strengthened if C is the umbral coalgebra and A the umbral algebra. Let us denote by $Hem(R[[x]])$ the algebra of all continuous hemimorphisms of the umbral algebra: then, every $S \in Hem(R[[x]])$ is the dual operator of a $T \in Hem(R[x])$, and $Hem(R[[x]])$ is still isomorphic to the umbral algebra. Then:

Theorem 15.- *The map*

11) $$ind: Hem(R[x]) \longrightarrow R[[x]], \quad T \longmapsto ind(T)$$

is an isomorphism of algebras. ∎

We shall endow the algebras Hem(R[x]) and Hem(R[[x]]) with the topologies which make them isomorphic to R[[x]] as topological algebras.

AUTOMORPHISMS OF Hem(R[x]) The automorphism group of Hem(R[x]) clearly is isomorphic to the automorphism group of R[[x]].

Theorem 16.- *Let $S \in Hem(R[x])$ and let T be an umbral operator with associated delta-functional τ; then*

12) $$T^{-1}ST \in Hem(R[x])$$

and

13) $$ind(T^{-1}ST) = ind(S)\circ\tau \ .$$

Proof. For every $\pi \in R[[x]]$ we have

14) $$(T^{-1}ST)^{*}\pi = (ST)(T^{-1*}\pi) = T^{*}((T^{-1*}\pi)\cdot ind(S)) = $$
$$= \pi \cdot T^{*}(ind(S)) = \pi\cdot(ind(S)\circ\tau)$$ ∎

Theorem 17.- *For every umbral operator T, the map*

15) $$\hat{T}: S \longrightarrow T^{-1}ST$$

is an automorphism of Hem(R[x]), and conversely, for every automorphism Φ of Hem(R[x]) there exists an umbral operator T such that

16) $$\Phi = \hat{T} \ .$$ ∎

PSEUDOGENERATORS OF Hem(R[x]) The order of a hemimorphism will be the order of its indicator. Invertible hemimorphisms are precisely hemimorphisms of order 0. Hemimorphisms of order 1 will be called *delta-operators*. Hemimorphisms of positive order will be called *differential operators*. In fact, one can easily prove that, for every $S \in Hem(R[x])$ and for every $p \in R[x]$:

17) $$deg(Sp) = deg(p) - \sigma(S) \ .$$

Obviously, pseudogenerators of Hem(R[x]) are precisely delta-operators.

As usual, let D denote the derivative over R[x].

Theorem 18.- *D is in Hem(R[x]) and*

18) $$ind(D) = \xi^{1} \ .$$

Proof. First, one can easily prove that, for every $n \in N$

19) $$\Delta Dx_n = (D\otimes I)\Delta x_n .$$

Then, for $n \in N$:

20) $$\langle D^*(1) | x_n \rangle = \langle \xi^0 | Dx_n \rangle = \langle \xi^0 | nx_{n-1} \rangle = n\delta^0_{n-1}$$

which implies 18). ■

SHIFT-INVARIANT OPERATORS Hemimorphisms of $R[x]$ can be characterized as follows:

Theorem 19.- *Let $T \in Hem(R[x])$, with*

21) $$\langle ind(T) | x_1 \rangle \neq 0.$$

A linear operator S over $R[x]$ is a hemimorphism of $R[x]$ if and only if

22) $$ST = TS \quad .$$

Proof. If $S \in Hem(R[x])$ then 22) clearly holds. Conversely: suppose 22) holds, and set

23) $$\alpha := ind(T) \quad .$$

First, suppose

24) $$\langle \alpha | x_0 \rangle = 0$$

then α is a pseudogenerator of $R[[x]]$. For every $i \in N$:

25) $$S^*(\alpha^i) = S^*((T^*\xi^0)^i) = S^* T^{*i}(\xi^0) = T^{*i} S^*(\xi^0) = \alpha^i S^*(\xi^0)$$

hence, $S \in Hem(R[x])$. Suppose now

26) $$\langle \alpha | x_0 \rangle = a_0 \neq 0$$

and set

27) $$\bar{T} := T - a_0 I_V$$

then $\bar{T} \in Hem(R[x])$ and

28) $$ind(\bar{T}) = ind(T) - a_0 \xi^0$$

29) $$S\bar{T} = \bar{T}S$$

and the preceding arguments apply to $\bar{T}$. ■

For every $a \in R$, the *shift operator* E_a is defined as follows: for every $p \in R[x]$

30) $$p(x) \overset{E_a}{\longmapsto} p(x+a)$$

or, equivalently, for every $n \in N$:

31) $$x_n \overset{E_a}{\longmapsto} \sum_i \binom{n}{i} a^i x_{n-i} \quad .$$

We have:

Theorem 20.- *For every* $a \in R$:

32) $$E_a \in Hem(R[x])$$

33) $$ind(E_a) = \varepsilon_a = \sum_i a^i \frac{(\xi^1)^i}{i!}$$ ∎

Theorems 19 and 20 ensure that a linear operator S is a hemimorphism of R[x] if and only if

34) $$SE_a = E_a S$$

for some $a \in R$. Accordingly, the hemimorphisms of R[x] are usually called *shift-invariant operators*.

ASSOCIATED DELTA-OPERATOR Let S be a delta-operator with indicator σ, and let T be the umbral operator with associated delta-functional σ. Then

35) $$T^{-1}DT = S$$

because

36) $$ind(T^{-1}DT) = ind(D) \circ \sigma = \xi^1 \circ \sigma = \sigma$$

Set

37) $$p_n := T^{-1}x_n$$

then

38) $$Sp_n = T^{-1}DTT^{-1}x_n = T^{-1}Dx_n = T^{-1}nx_{n-1} = np_{n-1} .$$

Conversely, if (p_n) is a polynomial sequence of binomial type, with associated umbral operator T and inverse associated delta-functional σ, then the delta-operator S such that

39) $$ind(S) = \sigma$$

satisfies

40) $$Sp_n = np_{n-1} \qquad \text{for every } n \in N.$$

This proves the following result:

Theorem 21.- *Let* (p_n) *be a polynomial sequence of binomial type, with associated delta-functional* π, *and let S be a delta-operator with indicator* σ; *then:*

41) $$Sp_n = np_{n-1} \qquad \text{for every } n \in N$$

if and only if σ *is the compositional inverse of* π. ∎

The sequence (p_n) and the delta-operator S satisfying 41) will be said to be *associated*. One can easily prove:

<u>Theorem 22.-</u> *Let (p_n) be a polynomial sequence such that*

42) $$\deg(p_n) = n \qquad \text{for every } n \in N$$

and let S be the linear operator such that

43) $$Sp_n = np_{n-1} \qquad \text{for every } n \in N.$$

Then, (p_n) is a polynomial sequence of binomial type if and only if S is a delta-operator. ∎

<u>HEAVISIDE FORMULA</u> Because of Theorem 15, every shift-invariant operator can be expanded as an exponential series in any given delta operator. More precisely:

<u>Theorem 23.-</u> *Let D_P denote the associated delta-operator of the polynomial sequence (p_n) of binomial type, and let S be a linear operator. Then, S is a shift-invariant operator if and only if*

44) $$S = \sum_i \frac{c_i}{i!} D_P^i$$

where, for every $i \in N$:

45) $$c_i := \langle \varepsilon | Sp_i \rangle = \langle S^* \xi^0 | p_i \rangle.$$

<u>Proof.</u> We have:

46) $$\langle \varepsilon | S(p_i) \rangle = \langle \varepsilon | \sum_j \frac{c_j}{j!} D_P^j p_i \rangle = \langle \varepsilon | \sum_j c_j \binom{i}{j} p_{i-j} \rangle =$$

$$= \sum_j c_j \binom{i}{j} \langle \varepsilon | p_{i-j} \rangle = c_i .$$ ∎

R E F E R E N C E S .

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