A q-analoque of the representation theory of S_n

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Our talk outlined some recent work done jointly with Richard Dipper.

Let S_n denote the group of permutations of $\{1,2,\ldots,n\}$, and let $\mathcal{B}$ be the set of *basic transpositions* in S_n (that is, $\mathcal{B}$ consists of those $v = (i,i+1)$, with $1 \leqslant i \leqslant n-1$). Every element $w \in S_n$ may be written as

$$w = v_1 v_2 \ldots v_k \quad (v_i \in \mathcal{B}),$$

and the *length of* w, lt(w), is the minimal k in such an expression.

One way of studying the representation theory of S_n over an *arbitrary* field K is to consider the *group algebra* KS_n, which is the vector space over K having basis $\{w \in S_n\}$, the multiplication being determined by the product of elements in S_n.

Definition. Suppose that K is a field and $0 \neq q \in K$. The *Hecke algebra* $\mathcal{H}$ $(=\mathcal{H}(K,q,n))$ is defined as follows:

i) $\mathcal{H}$ is an associative algebra over K, with basis $\{T_w \mid w \in S_n\}$

ii) If $w \in S_n$ and $v \in \mathcal{S}$

$$T_w T_v = \begin{cases} T_{wv} & \text{if } \operatorname{lt}(wv) = \operatorname{lt}(w)+1 \\ qT_{wv}+(q-1)T_w & \text{if } \operatorname{lt}(wv) = \operatorname{lt}(w)-1. \end{cases}$$

Note that if $u \in S_n$, $\operatorname{lt}(u) = k$ and $u = v_1 v_2 \ldots v_k$ ($v_i \in \mathcal{S}$), then ii) implies that $T_u = T_{v_1} T_{v_2} \ldots T_{v_k}$. Hence, using the fact that $\mathcal{H}$ is associative, one may calculate $T_w T_u$ for $w,u \in S_n$.

When q is a prime power, $\mathcal{H}$ occurs naturally in the representation theory of the general linear group $GL_n(q)$, and this enables one to prove that the multiplication in $\mathcal{H}$ is well-defined. This special case, with $K = \mathbb{C}$ has been extensively studied, but the more general situation - in particular, when K has arbitrary characteristic - has not previously been investigated in detail. It emerges that the analogy with KS_n is remarkable, and there are significant applications to the modular representation theory of $GL_n(q)$.

Here are some facts about $\mathcal{H}$ (see [1]):

i) If $q = 1$, then $\mathcal{H} \cong KS_n$ (Thus, our theory contains the representation theory of S_n)

ii) If $q \neq 1$, then the multiplication in $\mathcal{H}$ is horrible! (Anyone with doubts should calculate $T^2_{(13)}$).

iii) Even if char $K = 0$, $\mathcal{H}$ need not be isomorphic to KS_n. For example, if $K = C$, $1+q = 0$ and $n = 2$, then $\mathcal{H} \ncong \mathbb{C}S_2$. In fact,

iv) $\mathcal{H} \not\cong KS_n$ if and only if there exists m, $2 \leqslant m \leqslant n$ such that $[m]_q = 0$, where $[m]_q := 1+q+q^2+\ldots+q^{m-1}$. Even when $\mathcal{H} \cong KS_n$, the isomorphism is not natural.

v) $\mathcal{H}$ is a symmetric algebra. This means that $\mathcal{H}$ enjoys many of the properties of group algebras. In particular, every irreducible right $\mathcal{H}$-module is isomorphic to a simple right ideal of $\mathcal{H}$.

vi) For each partition λ of n, there exists $b_\lambda \in \mathcal{H}$ such that $S_\lambda := b_\lambda \mathcal{H}$ is a q-analogue of a Specht module. For example, the dimension of S_λ equals the number of standard λ-tableaux.

<u>Definition.</u> Let e be the least natural number m such that $[m]_q = 0$ (or $e = \infty$ if no such m exists). Thus if char K = p, then e = p if q = 1, and e is the multiplicative order of q, otherwise.

vii) For each partition λ of n, there exists $a_\lambda \in \mathcal{H}$ such that $a_\lambda b_\lambda \mathcal{H}$ is a simple right ideal of $\mathcal{H}$ if λ is e-regular, $a_\lambda b_\lambda \mathcal{H} = 0$ if λ is not e-regular. Furthermore, $\{a_\lambda b_\lambda \mathcal{H} \mid \lambda$ is e-regular$\}$ is a complete set of irreducible $\mathcal{H}$-modules. When q = 1, this translates directly into the familiar construction of the irreducible KS_n-modules.

Next we discuss the centre of $\mathcal{H}$. Recall that, of course,

$$\{ \sum_{w \in b} w \mid b \text{ is a conjugacy class of } S_n \}$$

is a basis for the centre of KS_n. (In particular, the centre has dimension equal to the number of partitions of n). For example,

$$1,\ (12)+(23)+(13),(123)+(132)$$

gives a basis for the centre of KS_3. Contrary, perhaps, to expectation $T_{(12)}+T_{(23)}+T_{(13)}$ and $T_{(123)}+T_{(132)}$ are not in the centre of $\mathcal{H}$ when $q \neq 1$. The correct q-analoque is inspired by the following Theorem of Gene Murphy:

<u>Theorem</u> (Murphy [3]): For $2 \leqslant m \leqslant n$, define

$$R_m = (m-1,m)+(m-2,m)+\ldots+(2,m)+(1,m) \in KS_n.$$

Then the R_m's commute and the centre of KS_n equals the set of elements which can be written as a symmetric polynomial in $R_2,R_3,\ldots,R_n$.

e.g. If $n = 3$, then $R_2 = (12)$, $R_3 = (23)+(13)$, and the centre is spanned by 1, $R_2+R_3 = (12)+(23)+(13)$, and $R_2R_3 = (132)+(123)$.

<u>Theorem [2]:</u> For $2 \leqslant m \leqslant n$, define

$$R_m = \frac{1}{q} T_{(m-1,m)} + \frac{1}{q^2} T_{(m-2,m)} + \ldots + \frac{1}{q^{m-1}} T_{(1,m)} \in \mathcal{H}.$$

Then the R_m's commute. The centre of $\mathcal{H}$ has dimensions equal to the number of partitions of n, and equals the set of elements which can be written as a symmetric polynomial in $R_2,R_3,\ldots,R_m$.

e.g. If $n = 3$, then $R_2 = \frac{1}{q} T_{(12)}$ and $R_3 = \frac{1}{q} T_{(23)} + \frac{1}{q^2} T_{(13)}$. The centre of $\mathcal{H}$ has basis 1, $R_2 + R_3$ and R_2R_3, where

$$R_2+R_3 = \frac{1}{q} T_{(12)} + \frac{1}{q} T_{(23)} + \frac{1}{q^2} T_{(13)}.$$

$$R_2R_3 = \frac{1}{q^2} T_{(132)} + \frac{1}{q^2} T_{(123)} + \frac{q-1}{q^3} T_{(13)}.$$

Using this Theorem, we have been able to translate Murphy's proof [3] of the Nakayama Conjecture into its q-analogue. We say that S_λ and S_μ are *linked* if and only if there exist partitions of n, $\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(s)} = \mu$ such that $S_{\lambda^{(i)}}$ and $S_{\lambda^{(i+1)}}$ have a common composition factor for each i. (Thus, for char K = p and q = 1, the Specht modules S_λ and S_μ are linked if they belong to the same p-block of KS_n. The Nakayama Conjecture states that in this case S_λ and S_μ are linked if and only if λ and μ have the same p-core.)

<u>Theorem [2]:</u> The $\mathcal{H}$ -modules S_λ and S_μ are linked if and only if λ and μ have the same e-core.

<u>References.</u>

1. R. Dipper and G.D. James: Representations of Hecke Algebras, to appear.
2. R. Dipper and G.D. James: Representations of Hecke Algebras II, to appear.
3. G.E. Murphy: The idempotents of the symmetric group and Nakayama's conjecture, J. Algebra 81 (1983), 258-265.