On the evolution of finite affine and projective spaces

Bernd Voigt, Bielefeld

My talk reported about results from [1]. Typically, the following question has been investigated:

Let F be a finite field. By $F\binom{n}{m}$ we denote the set of m-dimensional linear subspaces of the n-dimensional vector space F^n. Clearly, $|F\binom{n}{m}| = \binom{n}{m}_q$, where $q = |F|$, are the Gaussian binomial coefficients.

Let $\alpha = \alpha(n)$ be a probability, depending on n.

Let $S \subseteq F\binom{n}{k}$ be a random subset such that for each single $B \in F\binom{n}{k}$ we have $\alpha = \mathrm{Prob}(B \in S)$.

Let $P_{k,m,n}(\alpha) := \mathrm{Prob}(\exists A \in F\binom{n}{m} \mid \forall B \in F\binom{n}{k} : B \subseteq A \Rightarrow B \in S)$, so this is the probability that there exists an m-dimensional subspace A with all its k-dimensional subspaces belonging to S.

Moreover, let $\mathrm{Num}(k,m,n,\alpha) := |\{\forall \in F\binom{n}{m} \mid \forall B \in F\binom{n}{k} : B \subseteq A \Rightarrow B \in S\}|$ be the number of such A's.

We show that

$$\alpha^*(k,m,n) := \left(\sqrt[\binom{m}{k}_q]{\binom{n}{m}_q}\right)^{-1} = \binom{n}{m}_q^{-\frac{1}{\binom{m}{k}_q}}$$

is a <u>critical probability</u> in the following sense:

Theorem

(1) If $\lim_{n\to\infty} \alpha(n)/_{\alpha^*(k,m,n)} \to \infty$

then $\lim_{n\to\infty} P_{k,m,n}(\alpha) = 0$

(2) If $\lim_{n\to\infty} \alpha(n)/_{\alpha^*(k,m,n)} = R < \infty$

then $\lim_{n\to\infty} P_{k,m,n}(\alpha) = 1-e^{-\lambda}$,

where $\lambda = R^{\binom{m}{k}} q$

(3) If $\lim_{n\to\infty} \alpha(n)/_{\alpha^*(k,m,n)} = R < \infty$

then $\lim_{n\to\infty} \mathrm{Prob}(\mathrm{Num}(k,m,n,\alpha) = j) = \frac{\lambda^j}{j!} e^{-\lambda}$,

where again $\lambda = R^{\binom{m}{k}} q$.

Reference

[1] B. Voigt, *On the evolution of finite affine and projective spaces*, to appear in Proceedings of 9th SOR, Osnabrück 1984.