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REPRESENTATIVE FUNCTIONS ON THE ALGEBRA OF POLYNOMIALS IN INFINITELY MANY VARIABLES.

0. Introduction.

It is perhaps necessary to make a few informal remarks in or-
er to justify the subject dealt with in the following sections.
he obligatory starting-point is in calling attention to the notions
f both incidence algebra $A(\mathcal{P})$ and reduced (standard) incidence
lgebra $S(\mathcal{P})$ associated with a locally finite, partially ordered
et (abbeviated, l.f. poset) $\mathcal{P}$. The importance of these concepts
s mostly due to the fact that they clarify the so frequent appear-
nce of formal series in combinatorics. Accordingly, we shall limit
urselves to the case in which the underlying vector space of $S(\mathcal{P})$
s either the space of formal power series in infinitely many vari-
bles or a suitable subspace of this.

It is to be observed that practice has pointed out the major
nterest, within $A(\mathcal{P})$, in the subspace $\mathfrak{M}(\mathcal{P})$ of the multiplicative
unctions as well as in some other special incidence functions like
, ζ, λ, η, k, ρ, μ (we are using here the notations in [2] Ch.IV).
otice that not all of them (for instance ρ) are multiplicative func-
ions. These remarks lead to the following question: among the sub-
lgebras of $A(\mathcal{P})$ (or, equivalently, of $S(\mathcal{P})$) really useful in com-
inatorics, which is the greatest? It is clear that such a question,

because of its vagueness, cannot receive a convincing final answer. Nevertheless, it is legitimate to make a proposal. In our opinion a good candidate is the subalgebra of representative functions relative to the algebra of polynomials (either in a finite number or in infinitely many variables). Let us recall that a linear form f over an infinite-dimensional algebra $\mathcal{A}=(V,m,u)$ is said to be <u>representative</u> if Ker(f) contains a cofinite ideal J of $\mathcal{A}$. (It is plain that if V is finite-dimensional then every element of V^* is representative). The underlying space $V^o \subseteq V^*$ of the coalgebra $\mathcal{A}^o=(V^o,m^o,u^o)$ dual of $\mathcal{A}$ contains precisely the representative functios over $\mathcal{A}$. Lateron, we shall give such functions a characterization and describe their usefulness in several settings.

First, however, we wish to give some further evidence in support of the plausibility of the above thesis.

In the first place, on may observe that all the foreseen incidence functions are representative. Secondly, consider that each incidence algebra $A(\mathcal{P})$ may be regarded as obtained by duality from a coalgebra $C(\mathcal{P})$, the so-called incidence coalgebra (see [11],[12]), and what is more $C(\mathcal{P})$ may be structured as a bialgebra $\mathcal{B}(\mathcal{P})$ in a large number of cases (among which the case of hereditary classes of matroids studied in [11]). The combinatorial significance of these cases, together with the fact that the elements of the dual bialgebra $\mathcal{B}^o(\mathcal{P})$ are the representative functions, appears to represent a meaningful point in support of our thesis.

§1.

Let I be a set of any cardinality, possibly containing $\mathbb{N}$, $X=\{x_\iota \mid \iota\in I\}$ a set of indeterminates indexed on I and M(X), or simply M, the free abelian monoid on X. Let us denote an arbitrary element of M(X) with $\underline{x}^{\underline{a}} = x_1^{a_1}\dots x_n^{a_n}$. Here $\underline{a}$ is a map $I \to \mathbb{N}$, $\iota \leadsto a_\iota$ such that Supp($\underline{a}$) is finite. Obviously, $\underline{x}^{\underline{a}}\underline{x}^{\underline{b}}=\underline{x}^{\underline{a}+\underline{b}}=x_1^{a_1+b_1}\dots x_n^{a_n+b_n}$.

Consider the monoid algebra $\mathcal{A}=(K[M],m,u)$ over a field K of characteristic zero as well as its dual coalgebra $\mathcal{A}^o=(K[M]^o,m^o,u^o)$.

The addition in $\mathcal{A}$ is defined formally and the multiplication $m: K[M]\otimes K[M] \longrightarrow K[M]$ is obtained by extending the law of composition of M by linearity. Moreover:

$$u: K \longrightarrow K[M] \qquad \lambda \rightsquigarrow \lambda\cdot 1 \qquad \text{(unit map)}$$

$$m^{o}: K[M]^{o} \longrightarrow K[M]^{o}\otimes K[M]^{o} \qquad f \rightsquigarrow m^{o}(f):=f\circ m \qquad \text{(comultiplication or diagonalization)}$$

and

$$u^{o}: K[M]^{o} \longrightarrow K \qquad f \rightsquigarrow u^{o}(f):=f\circ u \qquad \text{(counit map)}$$

The elements of the underlying space $K[M]^{o}\subseteq K[M]^{*}$ of $\mathcal{A}^{o}$ are the representative functions on $\mathcal{A}$.

In the case of $X=\{x\}$, Peterson and Taft [15] proved that $K[x]^{o}\subseteq K[[x]]$ is the space of all the linearly recursive sequences. Our intention is to obtain an analogous characterization in the general case. To this aim, we need first the following definitions.

Def. 1. *A polynomial $\gamma(x)\in K[x]$ is said to be dependent on the set $\Gamma=\{\gamma_1(x),\dots,\gamma_t(x)\}$, $\gamma_\iota(x)\in K[x]$, via a polynomial $Z(x_1,\dots,x_t)$, if $\gamma(Z(\rho_1,\dots,\rho_t))=0$ whenever $\gamma_\iota(\rho_\iota)=0$ for every $\iota\in\{1,\dots,t\}$.*

Def. 2. *A family $G=\{\gamma_\iota(x)\mid\iota\in I\}$ is said to be of finite type if there exists a finite subset Γ of G such that every $\gamma_\iota\in G$ is dependent on the set Γ.*

With this terminology we can state the following proposition.

Prop. 1. *A linear form $f\in K[M]^{*}$ is representative (i.e. $f\in K[M]^{o}$) if and only if the following two statements hold:*

i) for every $x_\iota\in X$ the sequence $f_{\iota,n} := f(x_\iota^{n})$, $n\in\mathbb{N}$, is a linearly recursive sequence, whose characteristic polynomial will be denoted with $\gamma_\iota(x)$;

ii) the family $G=\{\gamma_\iota(x)\mid\iota\in I\}$ is of finite type. ■

Def. 3. *The family G in Prop.1. is said to be associated with the representative function f.*

The foregoing characterization may be made more precise. In fact we have:

Prop. 2. *Let f be a representative function on $K[M]$, J a cofinite ideal contained in $Ker(f)$ and $x_1,\dots,x_t$ the variables involved in a given basis of $K[M]/J$. For every element $\underline{\alpha}\in K[M]$ let $Z(x_1,\dots,x_t)$ be such that*

$$\underline{\alpha} \equiv Z(x_1,\dots x_t) \qquad \text{mod. } J$$

and let us put

$$(Z(x_1,\dots,x_t))^n \equiv \sum_{a_1,\dots a_t} Z^{(n)}_{a_1,\dots,a_t}\, x_1^{a_1}\cdots x_t^{a_t} \qquad \text{mod. } J$$

Then, for every $\underline{\beta}\in K[M]$, the sequence $f(\underline{\beta}\cdot\underline{\alpha}^n)$, $n\in N$, is a linearly recursive sequence for which the following explicit formula holds:

$$f(\underline{\beta}\cdot\underline{\alpha}^n) = \sum_{a_1\dots a_t} Z^{(n)}_{a_1\dots a_t}\, f(\underline{\beta}\cdot x_1^{a_1}\cdots x_t^{a_t}) .$$

Moreover, the characteristic polynomial $\gamma_{\underline{\alpha}}(\underline{\alpha})\in K[\underline{\alpha}]$ of such a linearly recursive sequence has the value $Z(\rho_1,\dots,\rho_t)$ as a root of multiplicity less than or equal to $r_1(\rho_1)+\dots+r_t(\rho_t)-t+1$, where ρ_ι is a root of multiplicity $r_\iota(\rho_\iota)$ of the characteristic polynomial γ_ι of the linearly recursive sequence $f(x_\iota^n)$. ∎

Let $J^\perp$ denote the subspace of $K[M]^o$ of all the representative functions whose kernel contains J and let $B\subseteq M(x_1,\dots,x_t)$ be the finite set of the monomials $\underline{x}^{\underline{b}}$ such that $[\underline{x}^{\underline{b}}]_{mod.J}$ is an element of the given basis of $K[M]/J$. Notice that we can always choose B in such a way that $\underline{x}^{\underline{b}}\in B$ implies $b_\iota<\deg(\gamma_\iota)$ if $\iota\in\{1,\dots,t\}$ and $b_\iota=0$ otherwise. In view of Prop.2., it is clear that each element f of $J^\perp$ is univocally determined by its values on B; such values will hereafter be referred to as the initial values of f. For every element $\underline{x}^{\underline{b}}\in B$, consider the representative functions $f^{(\underline{x}^{\underline{b}})}\in J^\perp$ defined by their initial values as follows:

$$f^{(\underline{x}^{\underline{b}})}(\underline{x}^{\underline{a}}) = \begin{cases} 1 & \text{if } \underline{x}^{\underline{b}}=\underline{x}^{\underline{a}} \\ 0 & \text{if } \underline{x}^{\underline{b}}\neq\underline{x}^{\underline{a}}\in B \end{cases}$$

Such functions will be called the fundamental recursive functions relative to the cofinite ideal J. They form a basis for $J^\perp$. It follows that the value of any representative function $f \in J^{\perp}$ on an arbitrary polynomial $\underline{\alpha} \in K[M]$ can be expressed in terms of both the initial values of f and the values of the fundamental recursive functions on $\underline{\alpha}$. In fact we have:

Prop. 3. *Let f, J, $\underline{\alpha}$ and Z be as in Prop.2. Then*

$$Z^{(1)}_{a_1,\dots,a_t} = f^{(x_1^{a_1}\dots x_t^{a_t})}(\underline{\alpha}) \ . \qquad \blacksquare$$

§2.

2.1. The reader interested in the proofs of the above propositions is referred to [9]. However, we emphasize here that the main step towards this goal is accomplished by the following lemma, which also has some other consequences of algebraic as well as combinatorial interest.

Lemma. *Let*

$$\begin{aligned} \zeta : K[M] &\longrightarrow K[M] \\ x_i &\rightsquigarrow Z_i(x_1,\dots,x_t) \end{aligned}$$

be a morphism of algebras and let

$$\begin{aligned} \zeta^0 : K[M]^0 &\longrightarrow K[M]^0 \\ f &\rightsquigarrow \zeta^0(f) := f\circ\zeta \end{aligned}$$

be its dual. Moreover, let us denote by G and G' the families of polynomials associated (see Def.3) with f and $f'=\zeta^0(f)$ respectively. Then each $\gamma_i \in G'$ depends, via $Z_i(x_1,\dots,x_t)$, on the set $\{\gamma_1,\dots,\gamma_t\} \subseteq G$ and we have

$$f'(x_i^n) = f(Z_i^n) = \sum_{a_1\dots a_t} Z^{(n)}_{i;a_1,\dots,a_t} f(x_1^{a_1}\dots x_t^{a_t}) \ . \qquad \blacksquare$$

2.2. Let us now examine the foreseen consequences of the above Lemma. We shall need the following definition which generalizes Def.1.

Def. 4. *A polynomial $\gamma(z) \in K[z]$ is said to be* <u>*dependent on a couple of sets*</u> *(G_1, G_2), with $G_1=\{\eta_i(x)\in K[x] \mid 1\le i\le t_1\}$ and $G_2=\{\theta_i(y)\in K[y] \mid 1\le i\le t_2\}$, via the polynomial $Z(x_1,\dots x_{t_1}, y_1,\dots,y_{t_2})$, if $\gamma(Z(\rho_1,\dots,\rho_{t_1},\sigma_1,\dots,\sigma_{t_2}))=0$*

whenever $\eta_\iota(\rho_\iota)=0$ *and* $\theta_\iota(\sigma_\iota)=0$.

Def. 5. *A set* $G=\{\gamma_\iota(z)\in K[z]\,|\,\iota\in I\}$ *is said to be dependent on the couple* (G_1,G_2) *if every* $\gamma_\iota\in G$ *depends on such a couple.*

Consider first of all a bialgebra $\mathcal{B}=(K[M],m,u,\Delta,\varepsilon)$ obtained by adding to the algebra $\mathcal{A}=(K[M],m,u)$ a comultiplication Δ and a counit ε that are also morphisms of algebra. Thus, it is clear that the identities

$$(\#) \qquad \Delta x_\iota = Z_\iota(x_1,\dots,x_{t_1},y_1,\dots,y_{t_2})$$

(where $\iota\in I$, Z_ι is a polynomial and the identification $x_\iota\otimes x_\lambda = x_\iota\cdot y_\lambda$ is used) determine univocally Δ.

For instance, we have:

a) $\Delta x_\iota = x_\iota + y_\iota$ for the binomial bialgebra;

b) $\Delta x_\iota = x_\iota\cdot y_\iota$ for the bialgebra of semigroup;

c) $\Delta x_\iota = \sum_{\|\underline{a}\|=\iota} \dfrac{\iota!}{\underline{a}!\;1!^{a_1}\cdots(\iota!)^{a_\iota}}\, \underline{x}^{\underline{a}}\, y_{|\underline{a}|}$

(where $\underline{a}=(a_1,\dots,a_\iota)$, $|\underline{a}|:=a_1+\dots+a_\iota$, $\|\underline{a}\|:=a_1+2a_2+\dots+\iota a_\iota$ and $\underline{a}!:=a_1!\,a_2!\dots a_\iota!$) for the bialgebra of Faà di Bruno.

In the bialgebra $\mathcal{B}^o=(K[M]^o,\Delta^o,\varepsilon^o,m^o,u^o)$ the multiplication Δ^o is given by $\Delta^o(f_1\otimes f_2)(\underline{\alpha}) = \Sigma f_1(\underline{\alpha}')f_2(\underline{\alpha}'')$ with $\Delta\underline{\alpha}=\Sigma\underline{\alpha}'\otimes\underline{\alpha}''$. Bearing in mind formula (#), from the Lemma follows:

Prop. 4. *Let* $f_1,f_2\in\mathcal{B}^o$ *and* $f=\Delta^o(f_1\otimes f_2)$ *and let* G_1,G_2 *and* G *be the sets of polynomials associated, in the sense of Def.3., with* f_1,f_2 *and* f *respectively. Then* G *depends, via* $Z_\iota(x_1,\dots,x_{t_1},y_2,\dots,y_{t_2})$, *on the couple* (G_1,G_2). ■

This proposition enables us to calculate G in terms of G_1 and G_2. In particular, if ρ_ι,σ_ι and τ_ι run over the roots of $\eta_\iota\in G_1$, $\theta_\iota\in G_2$ and $\gamma_\iota\in G$ respectively, we have:

a) $\tau_\iota = \rho_\iota + \sigma_\iota$ for the binomial bialgebra;

b) $\tau_\iota = \rho_\iota\cdot\sigma_\iota$ for the bialgebra of semigroup;

c) $\tau_\iota = \sum_{\|\underline{a}\|=\iota} \frac{\iota!}{\underline{a}!\ 1!^{a_1} \cdots (\iota!)^{a_\iota}}\ \rho_1^{a_1} \cdots \rho_\iota^{a_\iota}\ \sigma_{|\underline{a}|}$ for the bialgebra of Faà di Bruno.

2.3. Another consequence of the Lemma is relative to the following elimination problems.

Problem 1. Given the polynomials $\gamma_\iota(x)$, $\iota=0,1,\dots,n$, and $Z(x_0,x_1,\dots,x_n)$, determine, rationally on the coefficients of γ_ι and Z, a polynomial $\gamma(x)$ depending, via $Z(x_0,\dots,x_n)$, on the set $\Gamma=\{\gamma_\iota \mid \iota=0,\dots,n\}$.

Algorithm 1. The algorithm we propose consists of the following steps:

a) construct $n+1$ linearly recursive sequences $u_\iota=(u_{\iota,p})$, $p\in\mathbb{N}$, admitting γ_ι as (minimal) characteristic polynomial;

b) construct the linearly recursive sequence $w=(w_p)$, $p\in\mathbb{N}$, given by

$$w_p = \sum_{p_0\dots p_n} Z^{(p)}_{p_0\dots p_n} \cdot u_{0,p_0} \cdot u_{1,p_1} \cdots u_{n,p_n}$$

c) the polynomial searched for is the characteristic polynomial of w, that is

$$\gamma(x) = \begin{vmatrix} 1 & x & \dots & x^h \\ w_0 & w_1 & \dots & w_h \\ \cdot & \cdot & \cdot & \cdot \\ w_{h-1} & w_h & \dots & w_{2h-1} \end{vmatrix}$$

Proof. From the Lemma with $\zeta : x \rightsquigarrow Z(x_0,\dots,x_n)$ and $f(x_0^{p_0}\dots x_n^{p_n}) = u_{0,p_0} \cdots u_{n,p_n}$. ∎

The following problem is a generalization of the previous one.

Problem 2. Given the polynomials $\gamma_\iota(x)$, $\iota=1,2,\dots,n$, and $Z(x_0,\dots,x_n)$ determine rationally a polynomial $h(z)$ such that $Z(\sigma,\rho_1,\dots,\rho_n)=0$ whenever ρ_ι and σ run over the roots of γ_ι and h respectively.

Algorithm 2.

a) Put $\gamma_0(x)=x-z$ and use Algoritm 1 to compute a polynomial $\gamma(x)$;

b) the constant term of such a $\gamma(x)$ is the polynomial $h(z)$ required.

Proof. We have

$$\gamma(x) = \prod_{\rho_1 \dots \rho_n} (x - Z(z,\rho_1,\dots,\rho_n)) .$$

Hence

$$h(z) = \gamma(0) = \prod_{\rho_1 \dots \rho_n} Z(z,\rho_1,\dots,\rho_n) \qquad \blacksquare$$

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