OPERATIONS OF THE ADDITIVE GROUP ON THE AFFINE LINE AND UMBRAL CALCULUS

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(a generalization of the "umbral calculus" of S.M. Roman and G.C. Rota which contains the calculus of finite differences and the q-analysis as special cases)

We consider operations of the additive group G_a on the affine line $\mathbb{A}$, not in the sense of algebraic geometry ("polynomials"), but in the sense of formal geometry ("formal power series"). Let k be a field of characteristic 0 (e.g. the real or complex numbers). An <u>operation of G_a on $\mathbb{A}$</u> is given by a formal power series $p \in k[[x,t]]$ such that

$$p(x,0)=x \quad \text{and} \quad p(x,t_1+t_2) = p(p(x,t_1),t_2) \quad .$$

A derivation of $k[[x]]$ is a linear operator D on $k[[x]]$ which obeys the product rule $D(bc) = D(b)c+bD(c)$. Then $D= d\frac{\partial}{\partial x}$ for some $d \in k[[x]]$.

It can be shown that the operations of G on $\mathbb{A}$ are in one-to-one correspondence with the derivations of $k[[x]]$ by

$$D= \frac{\partial p}{\partial t}(x,0)\frac{\partial}{\partial x} \quad \text{and} \quad p= \exp(tD)(x) \quad .$$

Here $\exp(tD) = \Sigma_{i=0}^{\infty} t^i/i! \, (d\frac{\partial}{\partial x})^i$ is a Lie series ([Gr]).

In the sequel let p be a non-trivial operation of G_a on $\mathbb{A}$. We can write $p= x+tr$ for some $r \in k[[x,t]]$, $r \neq 0$. Let

$$\Delta: k[[x,t]] \to k[[x,t]] \ , \ f \to f(p,t) \ .$$

Then

$$\Delta= \exp(tD) \ , \ D= r(x,0)\frac{\partial}{\partial x} \ .$$

DEFINITION: The <u>quotient operator</u> Q on $k[[x,t]]$ is given by

$$Qf = \frac{\Delta f - f}{\Delta x - x} = \frac{f(x+tr,t)-f(x,t)}{tr} \quad , \quad f \in k[[x,t]]$$

$(t=0: \ Q= \frac{d}{dx})$.

PROPOSITION: Q is surjective with kernel K=k[[t]].

Hence there exists a sequence

$$g_0, g_1, g_2, \ldots \quad \text{in } k[[x,t]]$$

such that

$$Qg_0=0 \quad , \quad Qg_n=g_{n-1} \text{ for } n\geq 1 \quad \text{and} \quad g_n(0,t)=\delta_{n,0} \quad .$$

In particular, the uniquely determined Q-invariant power series $e\in k[[x,t]]$ with $e(0,t)=1$ is $e=\Sigma_{n=0}^{\infty} g_n$.

($t=0$: $g_n= x^n/n!$, $e=\exp(x)$) .

EXAMPLES:

(a) G_a acts on $\mathbb{A}$ by addition:

$$p(x,t)= x+t \quad , \quad D=\frac{\partial}{\partial x}$$

$$Qf = \frac{f(x+t)-f(x)}{t} \quad , \quad f\in k[[x]] \qquad \text{(difference quotient)}$$

$$g_n= (x)_{n,t}/n! = x(x-t)..(x-(n-1)t)/n!$$

$$e= \Sigma_{n=0}^{\infty} g_n = \exp(\frac{\log(1+t)}{t} x) = (1+t)^{x/t} \quad .$$

(b) G_a acts on $\mathbb{A}$ by multiplication:

$$p(x,t)= xe^t = qx \quad (q=e^t) \quad , \quad D= x\frac{\partial}{\partial x}$$

$$Qf = \frac{f(qx)-f(x)}{qx-x} = D_q f \quad , \quad f\in k[[x]] \qquad \text{(q-derivative,[Ci])}$$

$$g_n= x^n/[n]! \qquad ([n]!=[n][n-1]..[1], \ [n]=(q^n-1)/(q-1) \ )$$

$$e= \Sigma_{n=0}^{\infty} x^n/[n]! = \exp_q(x) \qquad \text{(q-exponential function)} \quad .$$

Similarly, for $0\neq\alpha\in k$,

$$p(x,t)= xe^{\alpha t} = q^{\alpha}x$$

$$Qf = \frac{f(q^{\alpha}x)-f(x)}{q^{\alpha}x-x} \quad , \quad f\in k[[x]] \quad .$$

(c) a mixture of (a) and (b):

$$p(x,t)= xe^t+e^t-1 = qx+q-1 \quad (q=e^t) \quad , \quad D= (1+x)\frac{\partial}{\partial x}$$

$$Qf = \frac{f(qx+q-1)-f(x)}{(q-1)(x+1)} \quad , \quad f\in k[[x]]$$

$$Q= \exp(\frac{\partial}{\partial x})D_q\exp(-\frac{\partial}{\partial x}) \quad \text{on } k[x]$$

$$g_n= x(x+1-q)(x+1-q^2)..(x+1-q^{n-1})/[n]!$$

$$e= \Sigma_{n=0}^{\infty} g_n = \exp_q(x+1)/\exp_q(1) \quad .$$

(d) another operation: $D= x^2\frac{\partial}{\partial x}$, $p= \exp(tx^2\frac{\partial}{\partial x})x = \frac{x}{1-tx}$.

We return to the general situation.
The "umbral calculus" of Roman and Rota is based on the duality between polynomials and formal power series by the continuous bilinear form

$$< \ , \ >: k[Y]\times k[[x]] \to k \ , \quad <\pi,f> = \pi(\tfrac{d}{dx})(f)|_{x=0} \ .$$

Since (Y^m) and $(x^n/n!)$ are dual (topological) bases, we have

$$f = \Sigma_{n=0}^{\infty} <Y^n,f> x^n/n! \qquad \text{in } k[[x]]$$

and

$$\pi = \Sigma_{m=0}^{\infty} <\pi,x^m/m!> Y^m \qquad \text{in } k[Y] \ .$$

The customary adjointness relation for linear (continuous) operators yields the bijections

$$\begin{array}{ccccc} Op(k[[x]]) & \overset{*}{\longleftrightarrow} & Op(k[Y]) & \longleftrightarrow & k[Y]^{\mathbb{N}_0} \\ F & \longleftrightarrow & G & \longleftrightarrow & (p_m) \ . \\ & <\pi,Ff>=<G\pi,f> & & p_m=G(Y^m) & \end{array}$$

(p_m) is called the polynomial sequence associated with F.
Dual to the multiplication $\mu: k[[x]]\hat{\otimes}k[[x]] \to k[[x]]$ is the comultiplication $\mu^*: k[Y] \to k[Y]\otimes k[Y]=k[Y_1,Y_2]$,

$$\mu^*(Y^m) = (Y_1+Y_2)^m = \Sigma_{i=0}^{m} \binom{m}{i} Y_1^{\,i} \, Y_2^{\,m-i} \ .$$

Using μ^* "binomial sequences" and "Sheffer sequences" can be defined in such a way that

F algebra-automorphism $\Leftrightarrow$ (p_m) binomial sequence

F semilinear automorphism $\Leftrightarrow$ (p_m) Sheffer sequence

etc. (see [RR]).
All this can be generalized by choosing the quotient operator Q instead of the derivative $\frac{d}{dx}$.

THEOREM: Let K=k[[t]]. Then the continuous K-bilinear form

$$< \ , \ >: k[Y][[t]]\times k[[x,t]] \to K \ ,$$

$$<\pi,f> = \Sigma_{i=0}^{\infty} \pi_i(Q)(f)|_{x=0} \, t^i \quad \text{where} \quad \pi = \Sigma_{i=0}^{\infty} \pi_i(Y)t^i \ ,$$

is a duality of topological K-modules with dual topological bases (Y^m) and (g_n). Hence

$$f = \Sigma_{n=0}^{\infty} <Y^n,f> g_n \qquad \text{in } k[[x,t]]$$

and

$$\pi = \Sigma_{m=0}^{\infty} <\pi,g_m> Y^m \qquad \text{in } k[Y][[t]] \ .$$

By adjointness we have bijections

$$\begin{array}{ccccc} Op(k[[x,t]]) & \overset{*}{\longleftrightarrow} & Op(k[Y][[t]]) & \longleftrightarrow & k[Y][[t]]^{\mathbb{N}_0} \\ F & \longleftrightarrow & G & \longleftrightarrow & (p_m) \;. \\ & <\pi,Ff>=<G\pi,f> & & p_m=G(Y^m) & \end{array}$$

If F is invertible, then (p_m) is a topological basis of $k[Y][[t]]$ and

$$\pi = \Sigma_{m=0}^{\infty} <\pi,F^{-1}(g_m)> p_m \quad .$$

Using the comultiplication $\mu^*:k[Y][[t]] \to k[Y_1,Y_2][[t]]$, binomial and Sheffer sequences can be introduced.

EXAMPLES:

(a) G_a acts on $\mathbb{A}$ by addition: $p(x,t)= x+t$

$<Y^m,f> = Q^m(f)|_{x=0}$ = the m-th divided difference of f

$f = \Sigma_{n=0}^{\infty} <Y^n,f>/n!\ (x)_{n,t}$, $f\in k[[t]]$ (Newton series)

$$\begin{array}{ccc} Op(k[[x,t]]) & \overset{*}{\longleftrightarrow} & Op(k[Y][[t]]) \\ Q & \longleftrightarrow & \text{multiplication with } Y \\ Z & \longleftrightarrow & \text{multiplication with } Y/\sqrt{1+tY} \end{array}$$

Z = central difference operator

$$Zf = \frac{f(x+t/2)-f(x-t/2)}{t} \quad , \quad f\in k[[x]]$$

It follows that

$$Zf = \Sigma_{n=0}^{\infty} <Y^{n+1}/\sqrt{1+tY},f>/n!\ (x)_{n,t} =$$
$$= \Sigma_{n=0}^{\infty} \left(\frac{1}{n!} \Sigma_{m=0}^{\infty} \binom{-1/2}{m} t^m <Y^{m+n-1},f> \right) (x)_{n,t} \quad .$$

(b) G_a acts on $\mathbb{A}$ by multiplication: $p(x,t)= xe^t = qx$

Associated with F= multiplication with $\exp_q$ is the sequence of Rogers-Szegö polynomials

$$H_m(Y) = \Sigma_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} Y^i \quad .$$

Thus the expansion of a polynomial $\pi\in k[Y]$ by Rogers-Szegö polynomials is

$$\pi = \Sigma_{n=0}^{\infty} <\pi,x^n/\exp_q(x)>/[n]!\ H_n(Y) \quad .$$

Binomial and Sheffer sequences have been studied in [Ki].

(c) G_a acts on $\mathbb{A}$ by $p(x,t)= xe^t+e^t-1 = qx+q-1$.

The expansion of a power series $f\in k[[x]]$ by the polynomials $g_n= x(x+1-q)..(x+1-q^{n-1})/[n]!$ is

$$f = \Sigma_{n=0}^{\infty} <Y^n,f>\ g_n \quad .$$

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