Constant Term Identities

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Introduction

This article is intended as an overview of a very rapidly developing and exciting subject. The problem at hand is the evaluation of the constant term in the Laurent expansions of certain products indexed by root systems of Lie algebras. These evaluations are equivalent to computing certain multi-dimensional definite integrals which have arisen in physical problems.

The implications of this subject, however, go far beyond their physical applications. As will be discussed in the last section, there are tie-ins to representation theory and the decomposition of characters, to cyclic homology and most significantly to higher dimensional analogs of hypergeometric series which carry the symmetry of the Weyl group of the associated root system.

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1. Definite Integral Evaluations

Our subject has its origins in two definite integral evaluations. The first was found by A. Selberg and published in 1941 [28] and in 1944 [29]: (here and in the integrals to follow the exponents are complex numbers with positive real part)

$$(1.1)\qquad \int_0^1 \cdots \int_0^1 |\Delta_n(\underline{t})|^{2z} \prod_{i=1}^{n} t_i^{x-1} (1-t_i)^{y-1} dt;$$

$$= \prod_{j=1}^{n} \frac{\Gamma(x+(j-1)z)\ \Gamma(y+(j-1)z)\ \Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\ \Gamma(z+1)}$$

where $\Delta_n(\underline{t}) = \Pi(t_j - t_k)$, $1 \le j < k \le n$.

When n is 1 this becomes the beta integral

$$(1.2)\qquad \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} .$$

The second integral arose in work of F. J. Dyson in 1962 on the statistical properties of a coulomb gas [11]. He conjectured its value which was independently verified the same year by Gunson [16] and Wilson [34].

$$(1.3)\qquad (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\Delta_n(e^{i\theta})|^{2z}\ d\theta_1 \ldots d\theta_n$$

$$= \frac{\Gamma(nz+1)}{\Gamma^n(z+1)} ,$$

where $\Delta_n(e^{i\theta}) = \Pi(e^{i\theta_j} - e^{i\theta_k})$, $1 \le j < k \le n$.

The study of these constant term identities has led to new definite integral evaluations, including one of this author and I. Goulden [8] which has had direct application to a problem posed by the physicists P. Forrester and B. Jancovici [13] with regard to the anomalous Hall effect:

$$(1.4)\qquad (2\pi)^{-n_1-n_2}\int_0^{2\pi}\cdots\int_0^{2\pi} |\Delta_{n_1}(e^{i\theta})|^{2z}\,|\Delta_{n_2}(e^{i\phi})|^{2z+2}\times$$

$$\times \prod_{\substack{1\le j\le n_1\\ 1\le k\le n_2}} |e^{i\theta_j}-e^{i\phi_k}|^{2z}\, d\theta_1\ldots d\theta_{n_1}\, d\phi_1\ldots d\phi_{n_2}$$

$$=\frac{\Gamma(n_1z+n_2(z+1)+1)\ \Gamma(n_2+1)\ \Gamma\left(\frac{n_1z}{z+1}+1\right)}{\Gamma^{n_1}(z+1)\ \Gamma^{n_2}(z+2)\ \Gamma\left(\frac{n_1z}{z+1}+n_2+1\right).}\ .$$

We shall begin our motivation of what is to follow by looking at the Dyson integral (1.3) in more detail. We first observe that

$$(1.5)\qquad |\Delta_n(e^{i\theta})|^2 = \prod_{j<k}(e^{i\theta_j}-e^{i\theta_k})(e^{-i\theta_j}-e^{-i\theta_k})$$

$$=\prod_{j<k}\left(1-\frac{e^{i\theta_j}}{e^{i\theta_k}}\right)\left(1-\frac{e^{i\theta_k}}{e^{i\theta_j}}\right)\ .$$

If we now assume that z is a non-negative integer, an assumption that can be made without loss of generality because of analytic continuation, then the value of the integral in (1.3) is the constant term in the Laurent expansion of

$$\prod\left(1-\frac{x_j}{x_k}\right)^z\left(1-\frac{x_k}{x_j}\right)^z,\quad 1\le j<k\le n.$$

Letting x represent the monomial $x_1x_2\ldots x_n$, we can thus rewrite equation (1.3) as

$$(1.6)\qquad [x^0]\prod\left(1-\frac{x_j}{x_k}\right)^z\left(1-\frac{x_k}{x_j}\right)^z=\frac{nz!}{(z!)^n}\ ,$$

or, by clearing the denominator, as

$$(1.7)\qquad [x^{z(n-1)}]\ \Pi(x_j-x_k)^{2z} = (-1)^{z\binom{n}{2}}\ \frac{nz!}{(z!)^n}\ ,$$

where $[m]f(\underline{x})$ denotes the coefficient of the monomial m in the expansion of $f(\underline{x})$.

With n equal to 2, equation (1.7) follows from the binomial theorem. For n equal 3 it is equivalent to the summation identity

$$(1.8)\qquad (-1)^z \sum_n (-1)^n \binom{2z}{n}^3 = \frac{3z!}{(z!)^3}\ ,$$

a special case of the Pfaff-Saalschütz summation for a well-poised hypergeometric series with three numerator parameters and two denominator parameters (see Andrews [1], 3.3.12).

For larger n, however, the left-side of equation (1.7) becomes a multiply-indexed summation, specifically with $\binom{n-1}{2}$ independent indices of summation. In other words, one is looking at higher dimensional well-poised hypergeometric series. Virtually nothing is known about such series, and in fact the Dyson integral evaluation has given us the first nice results about them. It is to be hoped that it will provide a starting point for the study of higher dimensional well-poised hypergeometric series.

The Selberg integral evaluation is also equivalent to a constant term identity: (a,b,c non-negative integers)

$$(1.9)\qquad [x^0]\ \prod_{1\le j<k\le n} \left(1-\frac{x_j}{x_k}\right)^b \left(1-\frac{x_k}{x_j}\right)^b (1-x_jx_k)^b \left(1-\frac{1}{x_jx_k}\right)^b \times$$

$$\times \prod_{j=1}^{n} (1-x_j)^a(1-x_j^{-1})^a(1-x_j^2)^c(1-x_j^{-2})^c$$

$$= \prod_{j=1}^{n} \frac{(2a+2c+2(j-1)b)!(2c+2(j-1)b)!jb!}{b!(a+c+(j-1)b)!(c+(j-1)b)!(a+2c+(n+j-2)b)!} .$$

To grasp what is transpiring here, we consider three specializations of the parameters:

<u>a=c=0</u>:

$$(1.10) \qquad [x^0] \prod \left(1 - \frac{x_j}{x_k}\right)^b \left(1 - \frac{x_k}{x_j}\right)^b (1-x_jx_k)^b \left(1 - \frac{1}{x_jx_k}\right)^b$$

$$= \binom{2b}{b}\binom{4b}{b}\cdots\binom{2(n-1)b}{b}\binom{nb}{b},$$

<u>a=b, c=0</u>:

$$(1.11) \qquad [x^0] \prod \left(1 - \frac{x_j}{x_k}\right)^b \left(1 - \frac{x_k}{x_j}\right)^b (1-x_jx_k)^b \left(1 - \frac{1}{x_jx_k}\right)^b \times$$

$$\times \prod (1-x_j)^b(1-x_j^{-1})^b$$

$$= \binom{2b}{b}\binom{4b}{b}\cdots\binom{2nb}{b},$$

<u>a=0, c=b</u>:

$$(1.12) \qquad [x^0] \prod \left(1 - \frac{x_j}{x_k}\right)^b \left(1 - \frac{x_k}{x_j}\right)^b (1-x_jx_k)^b \left(1 - \frac{1}{x_jx_k}\right)^b \times$$

$$\times \prod (1-x_j^2)^b(1-x_j^{-2})^b$$

$$= \binom{2b}{b}\binom{4b}{b}\cdots\binom{2nb}{b}.$$

The sequence of coefficients in the product of binomial coefficients may suggest something to those familiar with Lie algebras. What is actually going on is summed up in a conjecture made by I. Macdonald [25] which, in its simplest form, states that

Conjecture: Let R be a reduced, indecomposable root system, then

$$[e^0] \prod_{\alpha \in R} (1-e^{\alpha})^b = \prod_{i=1}^{n} \begin{bmatrix} bd_i \\ b \end{bmatrix} \tag{1.13}$$

where the d_i are the degrees of the fundamental invariants of the Weyl group for R.

The terms used in this conjecture are defined below.

Definition: A reduced root system, $R \subseteq \mathbb{R}^n$, is a finite set of non-zero vectors spanning $\mathbb{R}^n$ and satisfying:

1. $\alpha \in \mathbb{R}$ implies $n\alpha \in R$ if and only if $n = \pm 1$,
2. let ω_α be reflection through the hyperplane containing 0 and perpendicular to α, then $\alpha, \beta \in R$ implies that $\omega_\alpha(\beta) \in R$,
3. $\alpha, \beta \in R$ implies that $\beta - \omega_\alpha(\beta) = n\alpha$ with $n \in \mathbb{Z}$.

Definition: If a root system is not reduced, then it satisfies all of the above conditions except 1. Note that the second condition still imples that if $\alpha \in R$ then $-\alpha \in R$.

Definition: A root system is indecomposable if it cannot be written as the disjoint union of non-empty, mutually orthogonal subsets.

Definition: The Weyl group for R is the group

$$W = \langle \omega_\alpha : \alpha \in R \rangle .$$

We refrain from defining the degrees of the fundamental invariants, suffice it to say that for a given root system in n dimensions they are n well-defined positive integers.

There are only four infinite families of reduced, indecomposable root systems:

$$A_n = \{\pm(e_i-e_j) \mid 1 \le i < j \le n+1\},$$
$$D_n = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\},$$
$$B_n = D_n \cup \{\pm e_i \mid 1 \le i \le n\}$$
$$C_n = D_n \cup \{\pm 2e_i \mid 1 \le i \le n\},$$

plus five special root systems living in low dimensions: G_2, F_4, E_6, E_7, E_8. We shall also need to refer to the non-reduced root system $BC_n = B_n \cup C_n$.

To see how the Macdonald conjecture works, let us take the root system $A_{n-1} = \{\pm(e_i-e_j) \mid 1 \le i < j \le n\}$, which has as degree sequence $d_i = i+1$, $1 \le i \le n-1$. If we let x_i denote the formal exponential e^{e_i}, then for each $\alpha \in A_{n-1}$ we have that $e^\alpha = x_i/x_j$ or $= x_j/x_i$ and so Macdonald's conjecture says that

$$(1.14) \qquad [x^0] \prod_{1\le i<j\le n} \left(1 - \frac{x_i}{x_j}\right)^b \left(1 - \frac{x_j}{x_i}\right)^b = \binom{2b}{b}\binom{3b}{b}\cdots\binom{nb}{b} = \frac{nb!}{(b!)^n} .$$

Equations (1.10), (1.11) and (1.12) are the Macdonald conjecture, equation (1.13), for the root systems D_n, B_n and C_n respectively. In fact, the entire Selberg integral evaluation, equation (1.9), arises out of a more general conjecture also made by Macdonald.

Conjecture If $|\alpha| = |\beta|$ implies that $k_\alpha = k_\beta$, then for any root system R, not necessarily reduced, we have that

$$(1.15) \qquad [e^0] \prod_{\alpha\in R} (1-e^\alpha)^{k_\alpha} = \prod_{\alpha\in k} \frac{|f(\alpha)+k_\alpha|!}{|f(\alpha)|!},$$

where $f(\alpha) = \frac{1}{2} k_{\alpha/2} + \sum k_\beta \frac{\langle\alpha,\beta\rangle}{|\alpha|^2}$, the sum being over all roots β on one side of an arbitrary, predetermined hyperplane containing the origin but none of the roots (β are the "positive roots") and $k_{\alpha/2} = 0$ if $\frac{1}{2}\alpha \notin R$.

This conjecture implies equation (1.13), the case where all k_α are equal, and is equation (1.9) when $R=BC_n$. It has thus been verified for all of the infinite families of root systems. Recently, L. Habsieger [18] and D. Zeilberger [37] have independently verified equation (1.15) for $R=G_2$. Both conjectures (1.13) and (1.15) are still open for the remaining special root systems F_4, E_6, E_7 and E_8.

While the Macdonald conjectures give us a simple, unifying formula for various integral evaluations, they have not, to date yielded a unifying proof. The nicest proof of the evaluation of the Dyson integral and the only proofs for the Selberg integral are by recursive arguments on the exponents. But these arguments are heavily dependent on the specific structure of A_n and BC_n respectively. We shall give such proofs here, I.J. Good's [14] elegant proof of equation (1.3) and Aomoto's [4] recent proof of equation (1.1). In section 3 we shall discuss attempts at a uniform proof.

Dyson actually conjectured more than equation (1.3) or (1.6). He considered a case of unequal exponents:

$$(1.16) \qquad [x^0] \prod_{1\le i<j\le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} \left(1 - \frac{x_j}{x_i}\right)^{a_j} = \frac{(a_1+\dots+a_n)!}{a_1!a_2!\dots a_n!}.$$

Proof of equation (1.16), (Good [14]):

Let $F(\underline{a};\underline{x}) = \prod \left(1 - \frac{x_i}{x_j}\right)^{a_i} \left(1 - \frac{x_j}{x_i}\right)^{a_j}$,

$$G(\underline{a}) = \frac{(a_1+\ldots+a_n)!}{a_1!\ldots a_n!} .$$

Then $G(\underline{a}\ )$ is uniquely determined by its boundary conditions

1. $G(0) = 1$,
2. $a_i = 0 \Rightarrow G(\underline{a}) = G(a_1,\ldots,a_{i-1},a_{i+1},\ldots a_n)$,

plus the recursion

3. $G(\underline{a}) = \sum_{i=1}^{n} G(\underline{a}-\delta_i)$, where δ_i is the unit vector in the ith direction.

It is thus sufficient to verify that the constant term in $F(\underline{a};\underline{x})$ also satisfies 1,2 and 3. Conditions 1 and 2 are immediate and the recursion 3 is satisfied not just by the constant term of F but by the entire function for we have, by Lagrange interpolation, that

$$1 = \sum_{i=1}^{n} \prod_{j\neq i} \frac{x-x_j}{x_i-x_j}$$
$$= \sum_{i=1}^{n} \prod_{j\neq i} \frac{1}{1-x_i/x_j} ,$$

and thus

$$F(\underline{a};\underline{x}) = \sum_{i=1}^{n} F(\underline{a}-\delta_i;\underline{x}). \blacksquare$$

Outline of Proof of Equation (1.1), (Aomoto [4]):

Let $I(\alpha,\beta,\gamma) = \int d\omega$

$$I_j(\alpha,\beta,\gamma) = \int x_1x_2\ldots x_j \, d\omega$$

where $d\omega = |\Delta_n(\underline{x})|^{2\gamma} \prod_{i=1}^{n} x_i^{\alpha-1} (1-x_i)^{\beta-1}dx_i$ and integration is over the unit cube $\{\underline{x} \mid 0 \le x_i < 1\}$. We assume that neither α nor β is one, then

(1.17)
$$0 = \int \frac{\partial}{\partial x_1} x_1\ldots x_j \, d\omega$$
$$= \alpha \int x_2\ldots x_j \, d\omega + (\beta-1) \int \frac{x_1\ldots x_j}{x_1-1} d\omega$$
$$+ 2\gamma \sum_{k=2}^{n} \int \frac{x_1\ldots x_j}{x_1-x_k} d\omega.$$

By using the symmetry in the variables we also have that

(1.18)
$$\int \frac{x_1\ldots x_j}{x_1-x_k} d\omega = \begin{cases} 0 & , \text{ if } 2 \le k \le j \\ \frac{1}{2} I_{j-1}, & \text{ if } j < k \end{cases}.$$

Taking (1.17) and (1.18) together yields

(1.19)
$$0 = (\alpha+\gamma(n-j))I_{j-1} + (\beta-1) \int \frac{x_1\ldots x_j}{x_1-1} d\omega.$$

We also have that

$$(1.20)\qquad 0 = \int \frac{\partial}{\partial x_1} x_1^2 x_2 \cdots x_j \, d\omega$$

$$= (\alpha+1) I_j + (\beta-1) I_j + (\beta-1) \int \frac{x_1 \cdots x_j}{x_1 - 1} \, d\omega$$

$$+ 2\gamma \sum_{k=2}^{n} \int \frac{x_1^2 x_2 \cdots x_j}{x_1 - x_k} \, d\omega .$$

Again by symmetry one can also prove that

$$(1.21)\qquad \int \frac{x_1^2 x_2 \cdots x_j}{x_1 - x_k} \, d\omega = \begin{cases} \frac{1}{2} I_j , & \text{if } 2 \le k \le j \\ I_j , & \text{if } j < k \end{cases}$$

Combining (1.19), (1.20) and (1.21) yields

$$(1.22)\qquad I_j(\alpha,\beta,\gamma) = \frac{\alpha+(n-j)\gamma}{(\alpha+\beta+(2n-j-1)\gamma)} I_{j-1}(\alpha,\beta,\gamma).$$

By iteration and the fact that $I_n(\alpha,\beta,\gamma) = I(\alpha+1,\beta,\gamma)$, we get

$$(1.23)\qquad I(\alpha+1,\beta,\gamma) = \frac{(\alpha)(\alpha+\gamma)\cdots(\alpha+(n-1)\gamma)}{(\alpha+\beta+(n-1)\gamma)\cdots(\alpha+\beta+(2n-2)\gamma)} I(\alpha,\beta,\gamma).$$

Equation (1.23) together with the symmetry of I in α and β implies that

$$(1.24)\qquad I(\alpha,\beta,\gamma) = \left\{ \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)} \right\} F(\gamma),$$

where $F(\gamma)$ is an unknown function depending only on γ which can be found by specializing α and β. ∎

The Aomoto proof thus gives us a curious generalization of the BC_n constant term identity whose effect is to perturb some of the exponents by one. Let S,T be disjoint subsets of $\{1,\ldots,n\}$, σ, τ their respective cardinalities. Then the Aomoto proof implies

$$(1.25)\qquad [x^0] \prod_{i<j} \left\{\left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{x_j}{x_i}\right)(1-x_i x_j)\left(1 - \frac{1}{x_i x_j}\right)\right\}^b \times$$

$$\times \prod_j \{(1-x_j)(1-x_j^{-1})\}^{a+\chi(j\in S)-\chi(j\in T)} \times$$

$$\times \prod_j \{(1-x_j^2)(1-x_j^{-2})\}^{c+\chi(j\in T)}$$

$$= \prod_j \frac{(2a+2c+2(j-1)b+2\chi(j\geq n+1-\sigma))!}{(a+c+(j-1)b+\chi(j\geq n+1-\sigma))!} \times$$

$$\times \frac{(2c+2(j-1)b+2\chi(j\geq n+1-\tau))!}{(c+(j-1)b+\chi(j\geq n+1-\tau))!} \times$$

$$\times \frac{jb!}{(a+2c+(n+j-2)b+\chi(j\geq n+1-\sigma-\tau))!\,b!},$$

where $\chi(A) = 1$ if A is true, $= 0$ otherwise.

This shows us that we have more free parameters in the exponents than Macdonald's conjectured equation (1.15) implies. How much freedom and in what directions is still an open problem.

2. Generalizations of Jacobi's Triple Product Identity

This section gives a build-up and motivation to Macdonald's identities published in 1972 [24]. It largely follows the observations made by D. Stanton [30].

We begin with the Jacobi triple product identity and the one-line proof given by G. Andrews at Oberwolfach in 1982. In all that follows, we have convergence provided $|q| < 1$.

$$(2.1) \qquad \prod_{n=1}^{\infty} (1+zq^{n-1})(1+z^{-1}q^n) = \frac{1}{(q)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m(m-1)/2},$$

where $(q)_\infty = (1-q)(1-q^2)(1-q^3)\dots$.

Proof of (2.1) (Andrews): It is obvious! ∎

Why it is obvious needs some explanation. We observe that the right side is simply the Laurent expansion in z of the infinite product. If we let $f(z)$ be this infinite product and $\Sigma a_m z^m$ be its Laurent expansion, with the a_m to be determined, then we see that

$$(2.2) \qquad f(zq) = \frac{1+z^{-1}}{1+z} f(z) = z^{-1}f(z),$$

and thus

$$(2.3) \qquad a_m q^m = a_{m+1},$$

(2.4) $$a_m = a_0 q^{m(m-1)/2}.$$

The entire problem reduces to finding the constant term in

$$\Pi(1+zq^{n-1})(1+zq^n), \ n \geq 1.$$

The constant term, a_0, will be a function of q,

(2.5) $$a_0 = \Sigma\, b_n q^n, \ n \geq 0$$

where b_n is the number of pairs of partitions (π_1, π_2) into distinct parts such that

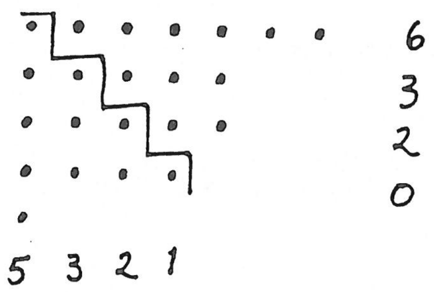
a. n is the sum of all parts in π_1 and π_2,

b. 0 is permitted in π_1 but not in π_2,

c. the number of parts in π_1 equals the number of parts in π_2.

But there is a natural bijection between such pairs of partitions and the set of ordinary partitions of n, best illustrated by an example:

$$\pi_1 = 6+3+2+0, \quad \pi_2 = 5+3+2+1$$



$$\pi = 7+5+5+4+1$$

The parts of π_1 are the horizontal lines to the right of the staircase, the parts of π_2 are the vertical columns below the staircase.

The generating function for ordinary partitions is well known to be $(q)_\infty^{-1}$, and thus this is a_0.

Andrews interest was in the argument used for finding a_0. Our interest is in the fact that a knowledge of a_0 suffices to find all coefficients. We shall now be looking for other infinite products for which finding the Laurent expansion reduces to finding the constant term. It should be emphasized that this approach is not new. See, for example, Hardy and Wright [21], chapter 19. The reduction of the expansion of the triple product to a constant term evaluation was used by Jacobi.

In 1929, G. N. Watson [33] discovered the quintuple product identity. Starting with the infinite product, we shall rediscover it for ourselves. Let

$$(2.6)\qquad \begin{aligned} f(z) &= \Pi(1-zq^{n-1})(1-z^{-1}q^{n})(1-z^{2}q^{2n-1})(1-z^{-2}q^{2n-1}),\ n \geq 1, \\ &= \Sigma\, a_m z^m,\ -\infty < z < \infty. \end{aligned}$$

Again we see that

$$(2.7)\qquad \begin{aligned} f(zq) &= \frac{1-z^{-1}}{1-z}\,\frac{1-z^{-2}q^{-1}}{1-z^{2}q}\, f(z) \\ &= z^{-3}q^{-1}f(z), \end{aligned}$$

$$(2.8)\qquad a_m q^m = q^{-1}a_{m+3}.$$

This reduces the expansion to the computation of three coefficients, but we can use the symmetry in z and z^{-1} to do better.

$$(2.9)\qquad f(z^{-1}) = \frac{1-z^{-1}}{1-z}\, f(z) = -z^{-1}f(z),$$

(2.10) $$a_{-m} = -a_{m+1}.$$

Equation (2.10) implies that

(2.11.a) $a_1 = -a_0$,

.b) $a_2 = -a_{-1}$.

Since equation (2.8) implies that

$$a_2 = a_{-1},$$

a_2 must be zero, and we thus have that

(2.12) $$f(z) = a_0 \sum z^{3n} q^{(3n^2-n)/2} (1-zq^n), \quad -\infty < n < \infty.$$

Letting $z = e^{2\pi i/3}$ shows that a_0 must be $(q)_\infty^{-1}$.

The next identity of this type was discovered by Winquist [35] in 1969. We now move to two variables. Let

(2.13) $$\begin{aligned} f(y,z) &= \prod (1-yq^{n-1})(1-y^{-1}q^n)(1-zq^{n-1})(1-z^{-1}q^n) \times \\ &\quad \times (1-yz^{-1}q^{n-1})(1-y^{-1}zq^n)(1-yzq^{n-1})(1-y^{-1}z^{-1}q^n), \quad n \geq 1 \\ &= \sum a(m,n) y^m z^n, \quad -\infty < m,n < \infty. \end{aligned}$$

Substituting as before, we get the relations

(2.14.a) $f(yq,z) = -y^{-3} f(y,z)$,

.b) $f(y,zq) = -z^{-3} q^{-1} f(y,z)$,

.c) $f(y^{-1},z) = -y^{-3}f(y,z)$,

.d) $f(y,z^{-1}) = -z^{-1}f(y,z)$,

which imply

(2.15.a) $a(m,n)q^{m} = -a(m+3,n)$,

.b) $a(m,n)q^{n} = -q^{-1}a(m,n+3)$,

.c) $a(-m,n) = -a(m+3,n)$,

.d) $a(m,-n) = -a(m,n+1)$.

These relations are not enough to reduce our problem to finding $a(0,0)$, but we also have the symmetry between y and z.

2.14.e) $f(z,y) = -y^{-1}zf(y,z)$,

2.15.e) $a(n,m) = -a(m+1,n-1)$.

Using equations 2.15.a-.e yields that

2.16) $a(0,0) = -a(0,1) = a(1,2) = -a(2,2)$,

2.17) $0 = a(0,2) = a(2,1) = a(2,0) = a(1,1) = a(1,0)$.

The evaluation of the constant term is again straightforward by letting y and z be primitive cube roots of unity. One obtains that

(2.18) $f(y,z)$

$$= \frac{1}{(q)_\infty^2} \sum (-1)^{i+j} y^{3i} z^{3j} q^{3\binom{i}{2}+3\binom{j}{2}+j} \times$$

$$\times \{1 - zq^{j} + yz^2q^{i+2j} - y^2z^2q^{2i+2j}\}, \quad -\infty < i,j < \infty.$$

What is happening here is most easily seen if we regard y and z as formal exponentials in the perpendicular unit vectors spanning the plane, $y=e^{e_1}$, $z=e^{e_2}$. The substitutions giving rise to equations 2.14.c, .d and .e are symmetries of the square, and the terms in the product (2.13) involve formal exponentials in the roots of $B_2 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$. This suggests that we may be able to do the same for any infinite product associated with a root system, the variables in the product being the formal exponentials in the roots, and the substitutions coming from the Weyl group. This realization came to both F.J. Dyson [12] and I. G. Macdonald [24] at approximately the same time, though only Macdonald published it in full generality. Let us illustrate what happens with the case $C_n = \{\pm 2e_i, \pm e_i \pm e_j \mid 1 \le i<j \le n\}$. To simplify our products, we introduce the notation

$$(a;q)_\infty = (a)_\infty = (1-a)(1-aq)(1-aq^2)\dots .$$

Let

(2.19)
$$\begin{aligned} F(\underline{x}) &= \prod_1^n (x_i^2)_\infty (qx_i^{-2})_\infty \times \\ &\times \prod_{i<j} (x_i x_j)_\infty (qx_i^{-1}x_j^{-1})_\infty (x_i x_j^{-1})_\infty (x_i x_j^{-1})_\infty (qx_i^{-1}x_j)_\infty \\ &= \sum a(\underline{m})\underline{x}^{\underline{m}}. \end{aligned}$$

If we replace x_i by $x_i q$, we obtain

(2.20)
$$F(x_1,\dots,x_i q,\dots,x_n) = x_i^{-2(n+1)} q^{-i} F(x_1,\dots,x_i,\dots,x_n),$$

which implies that

(2.21) $$q^{m_i+i}\, a(\underline{m}) = a(\underline{m} + (2n+2)\delta_i).$$

As before, δ_i is the unit vector in the ith direction. The problem has been reduced to finding $(2n+2)^n$ coefficients. We now use the symmetries of the Weyl group of C_n which is isomorphic to the semi-direct product of S_n and $\mathbb{Z}_2^n$. That is to say that each element of the Weyl group is a permutation on the co-ordinates, followed by possible changes of sign in arbitrarily many co-ordinates. The generators of this group are the transpositions of adjacent co-ordinates, $\omega_{e_i - e_{i+1}}$, together with change of sign in the last co-ordinate, $\omega_{e_{2n}}$. These have the following effects:

(2.22.a) $$F(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)$$
$$= -x_i^{-1}x_{i+1}\, F(x_1,\ldots,x_i,x_{i+1},\ldots,x_n),$$

.b) $$F(x_1,\ldots,x_n^{-1}) = -x_n^{-2}\, F(x_1,\ldots,x_n),$$

which imply that

2.23.a) $$a(\underline{m}) = -a(\underline{m} + (m_{i+1}+1-m_i)\delta_i + (m_i-1-m_{i+1})\delta_{i+1}),$$

.b) $$a(\underline{m}) = -a(\underline{m} + (2-2m_n)\delta_n).$$

The actions described in equation 2.23:

$$\underline{m} \longrightarrow \underline{m} + (m_{i+1}+1-m_i)\delta_i + (m_i-1-m_{i+1})\delta_{i+1}$$

$$\underline{m} \longrightarrow \underline{m} + (2-2m_n)\delta_n$$

generate a group of actions corresponding to elements of the Weyl group. The ith co-ordinate of the image of $\underline{m}$ under $\omega = (\sigma,\pi)$ is $n+1-i+\pi(i)(m_{\sigma(i)}-(n+1-\sigma(i)))$. If we define

$$\omega(a_1,\ldots,a_n) = (\pi(1)a_{\sigma(1)},\ldots,\pi(n)a_{\sigma(n)}),$$
$$\rho = (n,n-1,\ldots,1)$$

then the image of $\underline{m}$ under the action corresponding to ω can be written as

$$\rho+\omega(\underline{m}-\rho).$$

We now consider the lattice Λ of points all of whose co-ordinates are integral multiples of $2n+2$. We can prove the following result on the orbits of these points.

Proposition:

1. The elements of the W-orbit of a given lattice point are distinct modulo the lattice.
2. The W-orbits of distinct lattice points are pairwise disjoint.
3. $a(\underline{m})$ is non-zero if and only if $\underline{m}$ is in the W-orbit of a lattice point.

We thus get the expansion of $F(\underline{x})$,

$$F(\underline{x}) = a_0 \sum_{\underline{m}} q^{\sum_i (2n+2)\binom{m_i}{2}+im_i} \times \tag{2.24}$$
$$\times \sum_{\omega\in W} \operatorname{sgn}(\omega)\, \underline{x}^{\rho+\omega((2n+2)\underline{m}-\rho)}.$$

The value of a_0 is easily computed to be $(q)_\infty^{-n}$. Replacing x_i by e^{e_i}, the identity becomes

$$(2.25) \qquad \sum_{\alpha\in C_n^+} (e^\alpha)_\infty (qe^{-\alpha})_\infty$$

$$= \frac{1}{(q)_\infty^n} \sum_{\mu\in\Lambda} q^{(\|\mu-\rho\|^2-\|\rho\|^2)/2(2n+2)} \times$$

$$\times \sum_{\omega\in W} \operatorname{sgn}(\omega)\ e^{\rho+\omega(\mu-\rho)} .$$

Our product in equation (2.25) is actually a product over roots in an affine root system associated to C_n:

$$\{e_i + a\delta \mid 1 \le i \le n,\ a \ge 0\}$$
$$\cup \{e_i \pm e_j + a\delta \mid 1 \le i < j \le n,\ a \ge 0\}$$
$$\cup \{-e_i + a\delta \mid 1 \le i \le n,\ a \ge 1\}$$
$$\cup \{-e_i \pm e_j + a\delta \mid 1 \le i < j \le n,\ a \ge 1\},$$

where δ is a unit vector perpendicular to each of the e_i and where q is the formal exponential e^δ.

In general, Macdonald has shown that for an arbitrary affine root system, R, there exists a lattice Λ, a constant g, and a function $P(q)$ such that

$$(2.26) \qquad \prod_{\alpha\in R} (1-e^\alpha)$$

$$= \frac{1}{P(q)} \sum_{\mu\in\Lambda} q^{(\|\mu-\rho\|^2-\|\rho\|^2)/2g} \times$$

$$\times \sum_{\omega \in W} \operatorname{sgn}(\omega)\, e^{\rho+\omega(\mu-\rho)},$$

where ρ is the half sum of the positive roots of the underlying ordinary root system.

Equation (2.26) is an analog for affine root systems of the Weyl denominator formula, and in fact reduces to the Weyl denominator formula when q is 0. The identities with which we started this section are the special cases of the affine systems corresponding to A_1 (Jacobi triple product), BC_1 (quintuple product) and B_2 (Winquist).

But the story doesn't end here. As anyone familiar with hypergeometric series knows, the Jacobi triple product identity is simply the limiting case of a far more useful result, the q-binomial theorem:

$$\text{(2.27)} \qquad \prod_1^a (1+yq^{i-1}) \prod_1^b (1+y^{-1}q^i)$$

$$= \sum_j y^j q^{j(j-1)/2} \frac{(q;q)_{a+b}}{(q;q)_{a-j}(q;q)_{b+j}},$$

where $(x;q)_a = \dfrac{(x;q)_\infty}{(xq^a;q)_\infty}$.

It should, therefore, not be surprising that finite forms of the Macdonald identities have arisen in different problems. In fact, the expansions of the products for truncated affine root systems, products of the form

$$\sum_{\alpha \in R^+} (e^\alpha;q)_k (qe^{-\alpha};q)_{k-1}$$

should play the role of the q-binomial theorem in providing the basic identities from which a theory of higher dimensional hypergeometric series can be constructed.

Two questions naturally pose themselves at this point:

Q1: Can the problem of expanding such truncated products be reduced to finding the constant term?

Q2: Can this constant term be evaluated?

The answers to these questions will be discussed in the next section. Question 1 is only known for the root system A_n where the answer is Yes. Question 2 has been answered with evaluations for A_n and G_2 and conjectured values for the other root systems. We note that the Macdonald conjectures of section 1 are the limiting case $q=1$ of the solution to question 2.

3. Toward a Theory of Higher Dimensional Hypergeometric Series

We begin by considering the second question of the last section, that of finding the constant term in

$$\prod_{\alpha\in R^+} (e^{\alpha};q)_k(qe^{-\alpha};q)_{k-1}.$$

Numerous conjectures have been made about the value of the constant term in this and related products. G. Andrews [2] conjectured that

$$(3.1)\qquad [x^0]\prod_{i<j}\left(\frac{x_i}{x_j};q\right)_{a_i}\left(q\frac{x_j}{x_i};q\right)_{a_j} = \frac{(q)_{a_1+\dots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}.$$

This was proved by D. Zeilberger and the author [38]. K. Kadell [23] conjectured that

$$(3.2)\qquad [x^0]\prod_{i<j}\left(\frac{x_i}{x_j};q\right)_{a_i}\left(q\frac{x_j}{x_i};q\right)_{a_j-1}$$

$$= \frac{(q)_{a_1+\dots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_1^n \frac{1-q^{a_i}}{1-q^{a_1+\dots+a_i}}.$$

This was proved by I. Goulden and the author [7]. I. Macdonald [25] gave the following conjecture for an arbitrary root system

$$(3.3)\qquad [e^0]\prod_{\alpha\in R^+}(e^{\alpha};q)_k(qe^{-\alpha};q)_k$$

$$= \prod_1^n \frac{(q)_{kd_i}}{(q)_k(q)_{kd_i-k}},$$

where as before the d_i are the degrees of the fundamental invariants of the

Weyl group. W. Morris [27] has refined the Macdonald conjecture to allow the subscript k to depend on the length of the associated root. The proof of equation (3.1) implies that equation (3.3) is valid for $R=A_n$. L. Habsieger [18] and D. Zeilberger [37] have recently verified Morris' conjecture (and thus also equation (3.3)) for $R=G_2$. Finally, J. Stembridge [32] has shown how to pass between a knowledge of $[e^0]\prod_{\alpha\in R^+}(e^{\alpha};q)_k(qe^{-\alpha};q)_k$ and the value of $[e^0]\prod_{\alpha\in R^+}(e^{\alpha};q)_k(qe^{-\alpha};q)_{k-1}$ for an arbitrary root system. Neither value is actually known, however, unless R is A_n or G_2.

In order to sketch the proof of equation (3.1), we present here D. Zeilberger's [36] proof of the case $q=1$, a combinatorial proof of equation (1.16)

<u>Second Proof of Equation (1.16)</u> (Zeilberger [36]):

For convenience, we replace each x_i by its inverse. The product whose constant term we seek is

$$(3.4)\qquad \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)^{a_j}\left(1-\frac{x_j}{x_i}\right)^{a_i}$$
$$= (-1)^{\Sigma(i-1)a_i}\left(\prod x_i^{-(n-1)a_i}\right)\prod_{i<j}(x_i-x_j)^{a_i+a_j}.$$

Thus equation (1.16) can be rewritten as

$$(3.5)\qquad [\underline{x}^{(n-1)\underline{a}}]\prod_{i<j}(x_i-x_j)^{a_i+a_j}$$
$$= (-1)^{\Sigma(i-1)a_i}\begin{bmatrix}a_1+\dots+a_n\\ a_1,\dots,a_n\end{bmatrix},$$

where the multi-nomial coefficient, equal to

$$\frac{(a_1+\ldots+a_n)!}{a_1!\ldots a_n!},$$

appears on the right-hand side. This is well-known to count the number of words constructed from an n letter alphabet with the ith letter appearing a_i times in each word.

The left-hand side counts weighted <u>multi-tournaments</u>. We consider the formal expansion of

$$\prod_{i<j} (x_i-x_j)^{a_i+a_j}.$$

For each pair $i < j$ there are a_i+a_j choices of either the factor x_i or the factor $-x_j$. We can record the choices as a binary word in i's and j's of length a_i+a_j, which can be viewed as a record of the outcomes of games between players i and j. If we call such a word w_{ij}, then each term in the expansion of the product corresponds to a set of such binary words, $M = \{w_{ij} \mid 1 \le i < j \le n\}$. If we restrict our attention to the coefficient of $\underline{x}^{(n-1)\underline{a}}$, we are looking at only those sets of binary words where for each i, player i wins a total of $(n-1)a_i$ games. We call such a set a multi-tournament. The corresponding monomials carry a sign which is (-1) to the total number of upsets, that is to say the sum over all words w_{ij} in the tournament of the number of times j appears in w_{ij}.

The proof of equation (3.5) falls into two parts:

1. Providing a bijection between the words counted by $\begin{Bmatrix} a_1+\ldots+a_n \\ a_1,\ldots,a_n \end{Bmatrix}$ and some subset of the multi-tournaments which have sign $(-1)^{\Sigma(i-1)a_i}$.

2. Providing an involution on the remaining multi-tournaments which pairs a positive multi-tournament with a negative.

The first bijection is accomplished by considering the 2-letter subwords of the word counted by $\begin{Bmatrix} a_1+\ldots+a_n \\ a_1,\ldots,a_n \end{Bmatrix}$. For example, let $n=4$, $a_1=a_2=a_3=a_4=2$. The word

$$3\ 2\ 1\ 1\ 4\ 2\ 4\ 3$$

has as its 2-letter subwords:

2112

3113

1144

3223

2424

3443.

This gives a multi-tournament in which each pair of players, $i<j$, play a_i+a_j games, player i wins a total of $(n-1)a_i$ games, and the number of upsets is $\sum (i-1)a_i$. The original word is uniquely reconstructable from its 2-letter subwords.

The involution of part 2 is accomplished by trying to reconstruct the word of which our multi-tournament consists of the 2-letter subwords. As an example, we take the multi-tournament

2111

3133

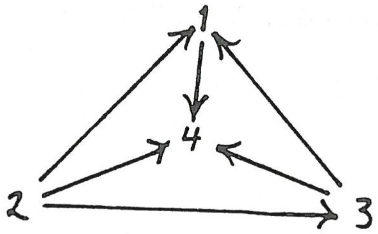
1144

2232

2424

3443

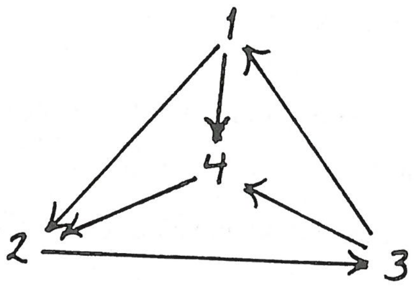
The leading terms of the 2-letter words together define a tournament.



If our multi-tournament does correspond to a single word, then this tournament cannot have any cycles (is <u>transitive</u>) and the player who beats all others must be the first letter of the single word, here that letter is 2.

We record that first letter, remove the three 2's which it records and repeat the procedure with the leading entries that remain.

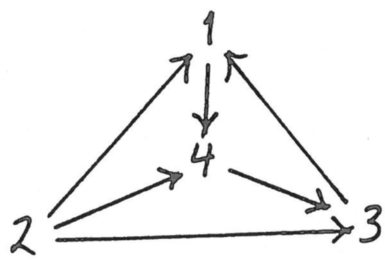
111
3133
1144
232
424
3443



If we started with a multi-tournament that does not consist of the 2-letter subwords of a single word, then we will eventually arrive at a non-transitive tournament (at least one cycle). In the case above we have two cycles of length three: $1 \to 2 \to 3 \to 1$, $2 \to 3 \to 4 \to 2$. Reversing the arrows in a single 3-cycle will not change the total number of games won by each player, but it will change the parity of the number of upsets and thus the sign of the multi-tournament. Since we have two 3-cycles in the example, we conceivably have two multi-tournaments with which we could pair the original:

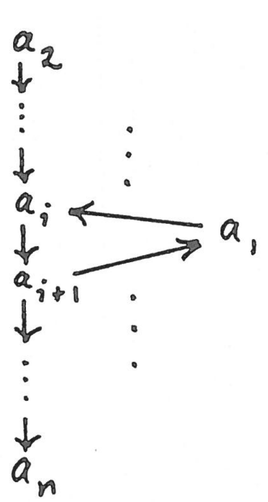
reverse 2-3-4		reverse 1-2-3
2111		$2\underline{2}11$
3133		$\underline{1}133$
1144	or	1144
$2\underline{3}32$		$2\underline{3}32$
$2\underline{2}24$		2424
$\underline{4}443$		3443

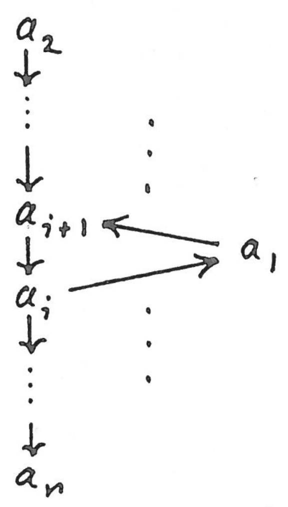
But if we now apply our algorithm to each of these possibilities, we see that only the second returns us to our original multi-tournament because the initial tournament defined on the left is



which contains the cycle $1 \to 4 \to 3 \to 1$.

It is possible to choose which 3-cycle to reverse in such a way that our algorithm is self-inverse. If any letters have been recorded, then from the previous tournament which was transitive to the present which is non-transitive, the only directed edges which have been changed are all incident to the vertex corresponding to the last letter recorded. Thus, all cycles pass through this vertex which we'll call a_1. The non-transitive tournament can be represented as on the left below.





Since there is a cycle through a_1, there must be a 3-cycle of the form $a_1 \longrightarrow a_i \longrightarrow a_{i+1} \longrightarrow a_1$. choose the smallest i for which such a 3-cycle exists and reverse that cycle. This gives us the desired involution.

If no letters have been recorded, then we have much more freedom in choosing our involution. We can, for example, put a total order on all unordered pairs of vertices and then switch the labels of the smallest pair with the same out-degrees. ∎

The proof of equation (3.1) is similar in outline if more complicated in detail. The constant term in

$$\prod_{i<j} \left(\frac{x_i}{x_j} ; q\right)_{a_j} \left(q\,\frac{x_j}{x_i} ; q\right)_{a_i}$$

counts the multi-tournaments as before, but now each carries the weight

$$(-1)^{\Sigma(i-1)a_i + \#\text{ of upsets}}\; q^{\sum_{i<j} \text{MAJ}'(w_{ij})}$$

where, if

$$a_i + k_{ij} = \#\text{ of } i\text{'s in } w_{ij}, \tag{3.6}$$

then

(3.7) $$MAJ'(w_{ij}) = MAJ(w_{ij}) + k_{ij}(k_{ij}-1)/2,$$

$MAJ(w)$ being the sum of the positions where descents occur in the word w.

If a multi-tournament consists of the 2-letter subwords of a single word w, then all k_{ij} are zero and, letting

(3.8) $$z(w) = \sum_{i<j} MAJ(w_{ij}),$$

the weight of the multi-tournament is $q^{z(w)}$.

Equation (3.1) follows from the following results.

<u>Lemma 1</u>:

(3.9) $$\sum_{w\in M(a_1,\ldots,a_n)} q^{z(w)} = \frac{(q)_{a_1+\ldots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} .$$

<u>Lemma 2</u>: There is an involution on the remaining multi-tournaments which pairs multi-tournaments whose weights have equal absolute value but opposite sign.

Proofs of these lemmas can be found in Zeilberger and Bressoud [38]. J. Greene [15] has come up with the first independent proof of Lemma 1, actually proving by explicit bijection that

(3.10) $$\frac{1}{(q)_{a_1+\ldots+a_n}} \sum q^{z(w)} = \frac{1}{(q)_{a_1}\cdots(q)_{a_n}} .$$

One advantage of Zeilberger's proof is that it is not recursive and thus holds some promise of being applicable to arbitrary root systems. In general, the expression

$$[e^0] \prod_{\alpha\in R^+} (e^{\alpha};q)_{k(\alpha)} (qe^{-\alpha};q)_{k(-\alpha)}$$

counts multi-choice sets

$$M = \{w(\alpha) : \alpha \in R^+\}$$

where $w(\alpha)$ is a binary word in 1's and 0's of length $k(\alpha)+k(-\alpha)$ which records choices being made of either the positive or negative of a given root $\alpha \in R^+$. If we set

(3.11) $$k(\alpha) + \pi(\alpha) = \#\text{ of } 1\text{'s in } w(\alpha),$$

then M must satisfy

(3.12) $$\sum_{\alpha\in R^+} \alpha\, \pi(\alpha) = 0.$$

The weight of M is given by

$$(-1)^{\Sigma\, \pi(\alpha)}\, q^{\Sigma(MAJ(w(\alpha))+\pi(\alpha)(\pi(\alpha)-1)/2)}$$

where both sums are over all $\alpha\in R^+$. A discussion of what can be done along these lines has been given by the author [6].

In a very different direction, P. Hanlon [19,20] has reduced the constant term problem for the infinite families of root systems to a problem of computing a certain cohomology, and then further reduced it to proving that a certain map is surjective.

The first question posed in section 2, that of reducing the expansion of

$$\prod_{\alpha\in R^+} (e^{\alpha};q)_k (qe^{-\alpha};q)_{k-1}$$

to a constant term computation, has been answered for A_n by J. Stembridge. For convenience, he considers

$$(3.13) \qquad [\underline{x}^{\underline{a}}] \prod_{i<j} \frac{(x_i/x_j)_\infty}{(zx_i/x_j)_\infty} \frac{(qx_j/x_i)_\infty}{(zx_j/x_i)_\infty} \equiv C^n[a](z,q).$$

Since the product is anti-symmetric in the x_i, it is sufficient to consider $\underline{a}$ for which $a_1 \geq a_2 \geq \ldots \geq a_n$. We note that each monomial in the expansion is of total degree 0:

$$(3.14) \qquad \Sigma a_i = 0.$$

We put a complete order on all such $\underline{a}$ by using lexicographic ordering read right to left so that 0 is the unique largest vector. We then have the following result.

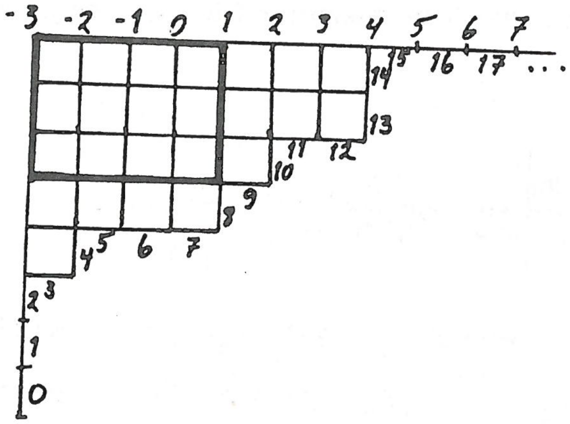
Theorem (Stembridge [28]) Each coefficient $C^n[\underline{a}](z,q)$, $\underline{a} \neq \underline{0}$, is a linear combination of coefficients whose vectors are strictly above $\underline{a}$. Specifically, this recursion is given by

$$(3.15) \qquad 0 = \sum_{k=0}^{r} (-1)^k (z^{h_k} - q^{-a_n} z^{n-h_{k+1}})(1-z^{h_{k+1}-h_k}) \times$$
$$\times \sum_{\underline{b}\in B_k(\underline{a})} C^n[\underline{b}](z,q),$$

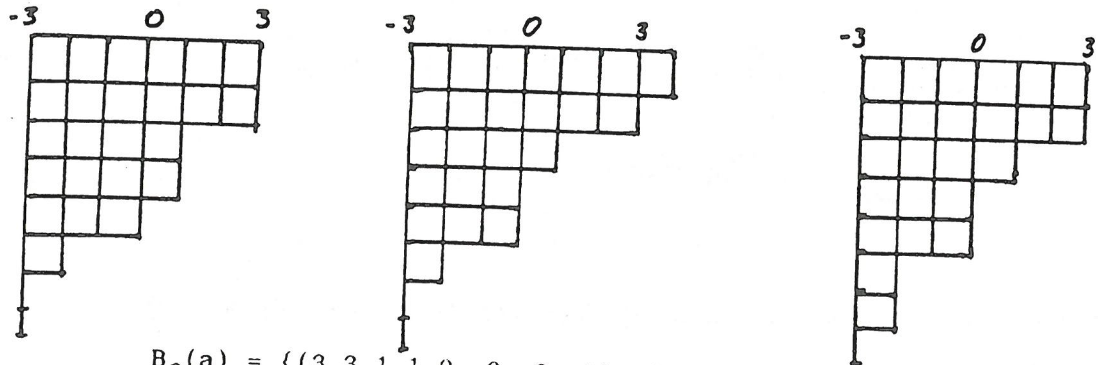
where $B_0(\underline{a}) = \{\underline{a}\}$ and the remainder of the notation is explained below.

We use an example to explain the notation of equation (3.15). Let

$\underline{a} = (4,4,2,1,-2,-3,-3,-3)$ which can be graphically represented as follows, giving us a correspondance with a partition into fewer than n parts.



The parameter r is the largest rectangle of height r and length $r+1$ in the upper left-hand corner. Here $r=3$. We label the outer edge as shown above, then $h_0=0$ and $(h_1,h_2,h_3,\ldots)$ is the sequence of integers appearing on horizontal strips, $(3,5,6,7,9,11,12,15,16,17,\ldots)$. $B_k(\underline{a})$ contains those configurations corresponding to a partition into at most n parts obtained from $\underline{a}$ by adding one square to each of the first k columns and then an arbitrary number of squares to the first column. If $k+j$ squares have been added, then $k+j$ squares must be removed, one from each of the ends of $k+j$ consecutive rows in such a way that nothing is removed from the first k columns and the resulting configuration is still a partition. Thus $B_3(\underline{a})$ consists of



$B_3(\underline{a}) = \{(3,3,1,1,0,-2,-3,-3),\ (4,3,1,0,0,-2,-3,-3),$

$(3,3,1,0,0,-2,-2,-3)\}$.

For $a_n = -1$, Stembridge's recursion yields

(3.16)
$$c^n[\underline{a}](z,q) = c^n[\underline{0}](z,q) \times$$
$$\times \frac{(z;z)_n}{(q;z)_n} \prod_{i,j\in\lambda} \frac{z^{j-1}-qz^{i-1}}{1-z^{h(i,j)}},$$

where λ is the partition given by $\lambda_i = a_i+1$ and $h(i,j)$ is the hook-length at position (i,j).

Equation (3.2) implies that

(3.17)
$$c^n[\underline{0}](q^k;q) = \frac{(q;q)_{kn}}{(q,q)_k^n} \frac{(1-q^k)^n}{(q^k;q^k)_n},$$

and thus for $a_n = -1$, these two equations yield

(3.18)
$$c^n[\underline{a}](z,q) = \frac{(z;q)_\infty^n}{(q;q)_\infty^{n-1}(qz^n;q)_\infty(q;z)_\infty} \times$$
$$\times \prod_{(i,j)\in\lambda} \frac{z^{j-1}-qz^{i-1}}{1-z^{h(i,j)}}.$$

There are still many unanswered questions. What is the general form of $c^n[\underline{a}](z,q)$? Can the constant term be computed, once the general form is known, by suitably specializing the variables? How does this extend to other root systems? Stembridge was studying these coefficients in the light of their connection to characters of $SL(n,\mathbb{C})$ so this last question is actually one of extending his results on character decomposition to the other classical groups.

Finally, we have the problem of trying to see how the expansion of $\Pi(x_i/x_j;q)_k(qx_j/x_i;q)_{k-1}$ generalizes the q-binomial thoerem and how it is going to fit into a theory of higher dimensional basic hypergeometric series. S. Milne [26] has been able to get the infinite Macdonald identity for A_n as a limiting case of his own multi-dimensional analog of the q-binomial theorem, but his identity does not appear to be equivalent to an expansion of the finite product given above.

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