ENUMERATING REGULAR MAPS AND NORMAL SUBGROUPS OF THE MODULAR GROUP

Gareth A. Jones

1. Introduction

The icosahedron is a regular orientable triangular map with rotation group isomorphic to $PSL_2(q)$ for $q = 4$ and $q = 5$. We shall consider, for each finite group G, the number N_G of regular orientable triangular (= r.o.t.) maps with orientation-preserving automorphism group G. The method used is quite general, though here we will concentrate on the groups $G = PSL_2(q)$; thus we are enumerating the 'q-analogues' of the icosahedron.

The first step is to use the algebraic theory of maps developed by David Singerman and the author [9, see also 5, 6, 10] to show that N_G is equal to the number of normal subgroups M of the modular group $\Gamma = PSL_2(\mathbb{Z})$ with quotient-group $\Gamma/M \cong G$, or equivalently the number of orbits of $\mathrm{Aut}\, G$ on pairs of elements of order 2 and 3 which generate G. In the case where $G = PSL_2(p)$, with p prime, this has already been calculated by Philip Hall [3], using his extension of the Möbius inversion formula to arbitrary finite groups:

<u>Theorem A</u> (Hall [3]). Let $G = PSL_2(p)$, where p is prime. Then $N_G = 1$ for $p \leqslant 5$, whereas for $p > 5$ we have $N_G = \frac{1}{2}(p-c)$ where

$$c = \begin{cases} 3 & \text{if} \quad p \equiv \pm 43 \text{ or } \pm 53 \bmod 120, \\ 5 & \text{if} \quad p \equiv \pm 7, \pm 13, \pm 17 \text{ or } \pm 37 \bmod 120, \\ 7 & \text{if} \quad p \equiv \pm 19, \pm 23, \pm 29 \text{ or } \pm 47 \bmod 120, \\ 9 & \text{if} \quad p \equiv \pm 11, \pm 31, \pm 41 \text{ or } \pm 59 \bmod 120, \\ 11 & \text{if} \quad p \equiv \pm 1 \text{ or } \pm 49 \bmod 120. \end{cases}$$

(Sinkov [12] rediscovered this result, with a different proof.) In [8], Hall's method was used to enumerate certain non-congruence subgroups of Γ; we shall use the same approach here to extend Theorem A to the case where $q = 2^e$. Let μ denote the Möbius function, and let $\sum_f$ denote summation over all positive divisors f of e.

Theorem B. If $G = PSL_2(2^e)$ then $N_G = \frac{1}{e}\sum_f \mu(\frac{e}{f})(2^f - 1)$. (Thus $N_G = \frac{1}{e}\sum_f \mu(\frac{e}{f})2^f$ if $e > 1$.)

This formula for N_G also gives the number of irreducible monic polynomials of degree e over $GF(2)$, or equivalently the number of orbits of length e in the action of the cyclic group C_e on its subsets; it would be interesting to exhibit natural bijections between the maps and the polynomials or orbits.

Martin Downs [2] has considered odd prime-powers q:

Theorem C (Downs). Let $G = PSL_2(p^e)$ where p is an odd prime.

i) if $e = 2$ then $N_G = \begin{cases} \frac{1}{4}(p+1)(p-3) & \text{if } p \equiv \pm 2 \bmod 5, \\ \frac{1}{4}(p-1)^2 & \text{otherwise}; \end{cases}$

ii) for all odd $e > 1$, $N_G = \frac{1}{2e}\sum_f \mu(\frac{e}{f})p^f$;

iii) for all even $e > 2$, $N_G = \frac{1}{2e}\sum_f^* \mu(\frac{e}{f})(p^{\frac{1}{2}f} - 1)^2$.

(Here $\sum_f^*$ denotes summation over all $f | e$ with e/f odd.)

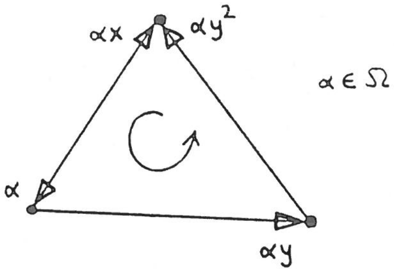
2. Algebraic theory of maps (see [5,6,9])

A map $\mathcal{M}$ consists of a graph $\mathcal{G}$ imbedded in a surface $\mathcal{S}$, so that the faces (connected components of $\mathcal{S} \setminus \mathcal{G}$) are simply connected. We will assume that $\mathcal{M}$ is orientable and is triangular (every face meets 3 edges). A dart (= brin) of $\mathcal{M}$ is an incidence between an edge and a vertex; we say that $\mathcal{M}$ is regular if its orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M}$

acts transitively on the set Ω of all darts of $\mathcal{M}$.

We define two permutations x, y of Ω : x sends each dart to the other dart on the same edge, while the cycles (of length 3) of y are formed by following a chosen orientation of $\mathcal{S}$ around each face. Clearly

$$x^2 = y^3 = 1 .$$



Let G be the group of permutations of Ω generated by x and y. Since the faces are simply connected, $\mathcal{G}$ is connected and hence G acts transitively on Ω. The modular group $\Gamma = PSL_2(\mathbb{Z})$ has a presentation

$$\Gamma = \langle X , Y \mid X^2 = Y^3 = 1 \rangle$$

(see [11]), so there is an epimorphism $\theta : \Gamma \rightarrow G$, $X \mapsto x$, $Y \mapsto y$, that is, a transitive permutation representation of Γ. Conversely, given such a representation we can reconstruct $\mathcal{M}$ (as a combinatorial map): the vertices, edges and faces correspond to the cycles of $(XY)^{-1}$, X and Y , with incidence corresponding to non-empty intersection. Thus orientable triangular maps correpond to transitive permutation representations of Γ , and hence to conjugacy classes of subgroups $M \leqslant \Gamma$ (the stabilizers of darts). This gives a 'dictionary' relating combinatorial and topological properties of $\mathcal{M}$ to algebraic properties of M . For example :

i) two maps are isomorphic if and only if they correspond to the same conjugacy class ;

ii) $\mathcal{M}$ is compact if and only if $|\Gamma : M|$ is finite ;

iii) $\mathcal{M}$ is regular if and only if $M \trianglelefteq \Gamma$, in which case

$$\mathrm{Aut}^+ \mathcal{M} \cong \Gamma / M \cong G .$$

Using this dictionary, one easily proves:

Theorem. Every orientable triangular map has the form $\mathcal{M}/A$ where $\mathcal{M}$ is an r.o.t. map and $A \leqslant \mathrm{Aut}^+ \mathcal{M}$.

This gives us an 'Erlangen program' for maps: study r.o.t. maps and their automorphisms.

3. Generating pairs

For a given group G, r.o.t. maps $\mathcal{M}$ with $\mathrm{Aut}^+ \mathcal{M} \cong G$ correspond to normal subgroups $M \trianglelefteq \Gamma$ with $\Gamma/M \cong G$, and thus to pairs $x = X\theta$ and $y = Y\theta$ which generate G and satisfy $x^2 = y^3 = 1$; if $|G| > 3$ then this implies that x, y have orders $2, 3$ respectively.

Define $(x,y) \in G \times G$ to be a pair if x, y have orders 2, 3; it is a generating pair if x, y generate G. Two generating pairs (x,y) and (x',y') determine isomorphic maps if and only if they correspond to the same normal subgroup $M = \ker \theta$, that is, if and only if $x' = x\sigma$ and $y' = y\sigma$ for some $\sigma \in \mathrm{Aut}\, G$. Thus N_G (the number of maps $\mathcal{M}$, or normal subgroups M, corresponding to G) is the number of orbits of $\mathrm{Aut}\, G$ on generating pairs in G.

Only the identity automorphism can fix a generating pair, so $\mathrm{Aut}\, G$ acts semi-regularly (i.e. freely) on generating pairs; thus if $3 < |G| < \infty$ then

$$N_G = \frac{n_G}{|\mathrm{Aut}\, G|}, \qquad (3.1)$$

where n_G is the number of generating pairs in G. In most cases, it is easy to find $|\mathrm{Aut}\, G|$ and the number of pairs in G; the difficulty is to eliminate those pairs which generate proper subgroups of G, and for this one needs detailed knowledge of the subgroup structure of G.

4. Proof of Theorem B

Let $G = G_e := PSL_2(q)$, where $q = 2^e$. The structure of G_e is described in [1, 4]. If e is *odd*, then there are $q^2 - 1$ elements of order 2, and $q^2 - q$ of order 3, so there are

$$m_G = (q^2-1)(q^2-q) = (q-1)\omega_e \tag{4.1}$$

pairs (x,y) in G_e, where $\omega_e := |G_e| = q(q^2-1)$. Every pair generates a unique subgroup $H \leqslant G$, and each H is generated by n_H pairs, so

$$m_G = \sum_{H \leqslant G} n_H . \tag{4.2}$$

By inspection of the list of subgroups $H \leqslant G$, one sees that the only subgroups H generated by a pair (x,y) are those isomorphic to G_f where $f|e$; there are $|G:H| = \omega_e/\omega_f$ such subgroups for each f, so if n_f denotes n_{G_f} then (4.2) becomes

$$m_G = \sum_f \frac{\omega_e}{\omega_f} . n_f . \tag{4.3}$$

Now (4.1) and (4.3) give

$$2^e - 1 = \sum_f \frac{n_f}{\omega_f} ; \tag{4.4}$$

applying the Möbius inversion formula to this, we obtain

$$\frac{n_e}{\omega_e} = \sum_f \mu(\tfrac{e}{f})(2^f - 1) . \tag{4.5}$$

Since $\mathrm{Aut}\, G = P\Gamma L_2(q)$ has order $e\omega_e$, (3.1) gives

$$N_G = \frac{n_e}{e\omega_e} = \frac{1}{e} \sum_f \mu(\tfrac{e}{f})(2^f - 1) .$$

If $e > 1$ then $\sum_f \mu(\frac{e}{f}) = 0$, so $N_G = \frac{1}{e}\sum_f \mu(\frac{e}{f})2^f$.

For *even* e, the only changes are that there are $q^2 + q$ elements of order 3, giving $(q+1)\omega_e$ pairs in G_e, and that there are $\omega_e/12$ subgroups $H \cong A_4$, each of which can be generated by $n_H = 24$ pairs. Thus extra terms $2\omega_e$ and $\frac{\omega_e}{12}.24 = 2\omega_e$ must be added to the right-hand sides of (4.1) and (4.3); these cancel in (4.4), so the final result is the same as for odd e.

5. Generalisations

1) We can apply Hall's method to any finite group G for which we know $|\mathrm{Aut}\, G|$ and the subgroup structure of G. We define $\mu_G(H)$, for each subgroup $H \leqslant G$, by

$$\left.\begin{aligned} &\mu_G(G) = 1 , \\ &\sum_{K \geqslant H} \mu_G(K) = 0 \quad \text{if} \quad H < G . \end{aligned}\right\} \tag{5.1}$$

This function μ_G is effectively computable if we know the subgroup lattice of G. For any function ϕ defined on the subgroups of G, let ψ be defined by

$$\psi(G) = \sum_{H \leqslant G} \phi(H) . \tag{5.2}$$

Then

$$\phi(G) = \sum_{H \leqslant G} \mu_G(H)\psi(H) , \tag{5.3}$$

as can be seen by applying (5.1) and (5.2) to the right-hand side.

As a simple example, if $G = C_n$ then $H = C_d$ $(d\,|\,n)$ and $\mu_G(H) = \mu(n/d)$, so (5.3) is the Möbius inversion formula; in particular, if we take $\phi(H)$ to be the number $\phi(d)$ of elements which are cyclic generators for H, then (5.2) and (5.3) become the classical equations $n = \sum_{d|n} \phi(d)$ and $\phi(n) = \sum_{d|n} \mu(\frac{n}{d})d$.

In our case, if we take $\phi(H) = n_H$ then $\psi(H) = m_H$ (the product of the numbers of elements of orders 2 and 3 in H), and (5.3) gives

$$n_G = \sum_{H \leqslant G} \mu_G(H) m_H ,$$

the analogue of (4.5). Thus we can calculate n_G, and hence obtain N_G from (3.1).

2) One can enumerate regular objects, having a given automorphism group, in other combinatorial categories if one replaces Γ with other suitably-chosen groups. Thus the extended modular group $PGL_2(\mathbb{Z})$ corresponds to all (not necessarily orientable) triangular maps [10], while the free product $V_4 * C_2$ corresponds to all maps [10]; for example, the number of reflexible (i.e. chiral) maps $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} = PSL_2(q)$, $q = 2^e > 2$, is

$$\frac{1}{e} \sum_f \mu(\tfrac{e}{f})(2^f - 1)(2^f - 2) ,$$

the number of orbits of length e in the action of C_e on distinct pairs of non-empty subsets. Similarly, the free group $F_2 = C_\infty * C_\infty$ and the free product $C_2 * C_2 * C_2$ correspond to orientable hypermaps and all hypermaps [7], so Hall's results in §§ 4.2 and 4.3 of [3] give formulae for the numbers of regular objects in these categories with automorphism group $PSL_2(p)$ — asymptotically $p^3/4$ and $p^3/8$ respectively.

References

1. L.E.Dickson, Linear groups, Dover, New York (1958).

2. M.L.N.Downs, private communication.

3. P.Hall, The Eulerian functions of a group, Quarterly J. Math. Oxford 7 (1936), 134 - 151.

4. B.Huppert, Endliche Gruppen I, Springer-Verlag, Berlin - Heidelberg - New York (1967).

5. G.A.Jones, Graph imbeddings, groups, and Riemann surfaces, in Algebraic Methods in Graph Theory, Szeged, 1978 (L.Lovász and V.T.Sós, Eds.), Colloq. Math. Soc. János Bolyai Vol. 25, North-Holland, Amsterdam (1981).

6. G.A.Jones, Triangular maps and non-congruence subgroups of the modular group, Bull. London Math. Soc. 11 (1979), 117-123.

7. G.A.Jones, Operations on maps and hypermaps, Actes du 13ème Séminaire Lotharingien de Combinatoire, Bologna, 1985 (G. Nicoletti, Ed.).

8. G.A.Jones, Some maximal normal subgroups of the modular group, Proc. Edinburgh Math. Soc. (to appear).

9. G.A.Jones and D.Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. (3) 37 (1978), 273-307.

10. G.A.Jones and J.S.Thornton, Operations on maps, and outer automorphisms, J. Combin. Theory Ser. B 35 (1984), 93-103.

11. M.Newman, Integral Matrices, Academic Press, New York (1972).

12. A.Sinkov, The number of abstract definitions of LF(2,p) as a quotient group of (2,3,n), J. Algebra 12 (1969), 525-532.

Department of Mathematics
University of Southampton
Southampton S09 5NH
United Kingdom