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NUMBER OF PERMUTATIONS WITH GIVEN DESCENT SET AND CYCLE STRUCTURE

BY

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1. Introduction

The descent set of a permutation $\sigma \in S_n$ is the subset $\mathrm{Des}(\sigma) = \{i, 1 \le i \le n-1, \sigma(i) > \sigma(i+1)\}$ of $\{1, \dots, n-1\}$; an equivalent concept is the up-down sequence σ which is a sequence of length n - 1 of - and + , with the - in the positions determined by Des(σ). Foulkes [4] has studied representations of S_n indexed by subsets of $\{1, \dots, n-1\}$, in connection with the enumeration of permutations having a precribed descent set (see also Kerber-Thürlings [9]). We shall call these representations the Foulkes representations of S_n.

Recently, certain representations of S_n, related to the free Lie algebra, have been intensively studied; they arise from the canonical decomposition of the free associative algebra which comes from the theorem of Poincaré-Birkhoff-Witt (see Reutenauer [13], Garsia [5], Garsia-Reutenauer [6], Bergeron-Bergeron-Garsia [1]). These representations are indexed by partitions λ of n. We shall call them the Lie representations of S_n.

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Recall that the cycle-structure (or cycle-type) of a permutation $\sigma \in S_n$ is the partition

$$1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n}$$

of n, where for each i, σ has α_i cycles of length i.

The main result of this article (th. 4) is that the number of permutations of S_n having descent set D and cycle-structure λ is equal to the scalar product (or intertwinning number) of the Foulkes representation indexed by D and the Lie representation indexed by λ.

The proof uses symmetric functions.

The characteristic symmetric function S_D of the Foulkes representation indexed by D is the skew Schur function whose shape is the skew hook determined by D.

On the other hand, the characteristic symmetric function of the Lie representation indexed by λ is the enumerator of all multi-sets of necklaces of type λ.

We use the idea of quasi-symmetric functions of Gessel [7]; he gives a formula for each symmetric function as a linear combination of certain basic quasi-symmetric functions. We also need a bijection between words and multi-sets of necklaces, which preserves type and evaluation. An analoguous bijection was found independantly by Gessel (unpublished), and is stated and used by Désarménien-Wachs [2]. This bijection is also related to the cyclotomic identity of Metropolis-Rota [12] and is in some sense a particular case of the bijection of Dress-Siebeneicher [3].

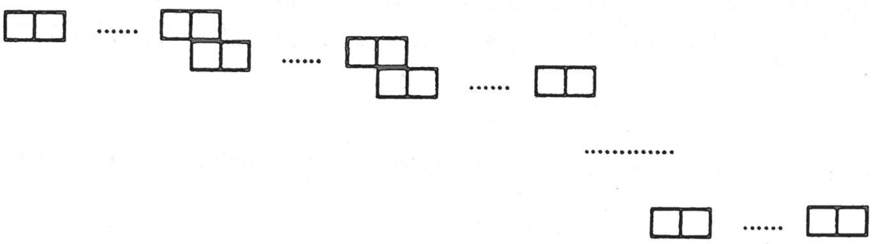
In a complementary section, we show the existence of a curious bijection between circular permutations with descent set D and permutations with the same descent set and inverse major index equal to 1 modulo n. In the final section, we show that the sum of all the Lie representations indexed by partitions having parts 1 or 2 is equal to the following analytic functor: envelopping algebra of the free rank 2 nilpotent Lie algebra. This representation contains each irreducible representation of S_n with multiplicity 1. This is a functorial interpretation of the celebrated identity of Littlewood:

$$\prod_{x} (1 - x)^{-1} \prod_{x<y} (1 - xy)^{-1} = \sum_{\lambda} s_\lambda .$$

2. Foulkes representations

To each subset D of $\{1, \dots, n-1\}$ is associated the composition $C = (d_1, d_2 - d_1, \dots, d_k - d_{k-1}, n - d_k)$ of n, where $D = \{d_1 < d_2 < \dots < d_k\}$. In this way, compositions of n and subsets of $\{1, \dots, n-1\}$ are in one-to-one correspondence. We denote C(D) the composition associated to D, and D(C) the subset associated to C. We denote also $C(\sigma)$ for $C(\mathrm{Des}(\sigma))$, i.e. the descent composition of $\sigma \in S_n$.

Given a subset S of $\{1, \dots, n-1\}$, with corresponding composition C, we associate to it the following skew hook (or border strip)



where the lengths of the successive rows are $c_1, \dots, c_k, c_{k+1}$ with $C = (c_1, \dots, c_k, c_{k+1})$. To this skew hook corresponds a skew Schur function, which we denote by S_C.

The representations associated to these skew hooks have been studied by Foulkes [4], for the enumeration of permutations with precribed up-down sequence, and the study of eulerian numbers.

3. Lie representations

Let A be an alphabet, $\mathbb{Q}\langle A\rangle$ the free associative algebra, and $\mathcal{L}(A)$ the sub-Lie-algebra of $\mathbb{Q}\langle A\rangle$ generated by A; the latter is well-known to be the free Lie algebra generated by A over $\mathbb{Q}$. Its elements are called Lie polynomials.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition, and define a subspace $\mathcal{L}_\lambda$ of $\mathbb{Q}\langle A\rangle$ in the following way: $\mathcal{L}_\lambda$ is the linear span of the polynomials of the form

$$(P_1, \dots, P_k) = \frac{1}{k!} \sum_{\alpha \in S_k} P_{\alpha(1)} \cdots P_{\alpha(k)}$$

where for $i = 1, \dots, k$, P_i is an homogeneous Lie polynomial of degree λ_i.

Denote by GL(A) the linear group of the space having A as a basis. Then GL(A) acts on $\mathbb{Q}\langle A\rangle$ in the usual way, and this action leaves $\mathcal{L}(A)$ invariant and preserves degree and homogeneity. Hence each subspace $\mathcal{L}_\lambda$ is invariant under this action. In this way, we obtain for each partition λ a representation of the linear group, and a corresponding representation of S_n, which we call the Lie representation of S_n.

4. Main result

Let C be a composition of n, with associated subset D of $\{1, \dots, n-1\}$, and λ be a partition of n.

Theorem 1. The number of permutations in S_n having descent set D and cycle structure λ is equal to the scalar product of the Foulkes representation indexed by D and the Lie representation indexed by λ.

The proof of this theorem requires some more notions, which we introduce in the following sections.

5. Necklaces

Let A^* denote the free monoid generated by A. Two words (elements of A^*) x,y are *conjugate* if for some words u, v, one has $x = uv$, $y = vu$. A word x is *primitive* if $x = y^n$ implies $n = 1$ or $y = 1$ (the empty word). A *necklace* is a conjugation class of a primitive word (then all the words of the class are primitive). A necklace may be viewed as a circular word without period, that is, a regular oriented n-gon with the vertices labelled in A, which is not left fixed by any nontrivial rotation.

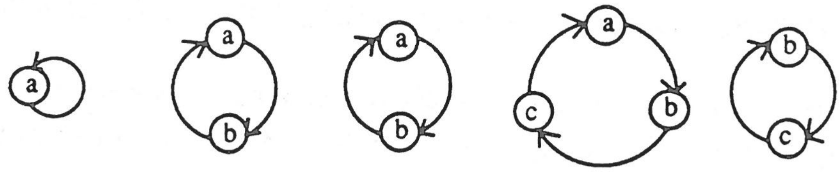
A multi-set of necklaces is a collection of necklaces, with repetitions allowed. Its type is the partition

$$1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n}$$

if there are for each $i = 1, \dots, n$, exactly α_i necklaces of length i.

The mapping ev: $\mathbb{Q}\langle A\rangle \longrightarrow \mathbb{Q}[A]$ is the canonical mapping, called *evaluation*. For each necklace, its evaluation is well defined, because two conjugate words have the same evaluation.

The evaluation of a multi-set of necklaces is the product of the evaluation of the necklaces which occur (see the figure).



a multi-set of necklaces of type 32221 and evaluation $a^4 b^4 c^2$

We are now ready to characterize the symmetric functions of the Lie representations. The following result is an easy consequence of the theorem of Poincaré-Birckhoff-Witt and the fact that the free Lie algebra has a basis which is in one-to-one correspondence with necklaces. It is implicitely in [5] and [1].

Theorem 2. The characteristic symmetric function of the Lie representation indexed by λ is equal to the sum of all evaluations of the multi-sets of necklaces of type λ.

6. Quasi-symmetric functions

In this section, we follow Gessel [7]. We take an infinite totally ordered set A of variables, which will be either commutative or non-commutative.

A quasi-symmetric function is a function F in $\mathbb{Q}[A]$ such that for any $a_1, \dots, a_n, b_1, \dots, b_n$ in A with $a_1 < \dots < a_n$, $b_1 < \dots < b_n$ and $k_1, \dots, k_n$ in $\mathbb{N}$, the coefficients of $a_1^{k_1} \dots a_n^{k_n}$ and $b_1^{k_1} \dots b_n^{k_n}$ in F are equal.

The algebra QSym of quasi-symmetric functions admits a basis F_C indexed by compositions which we describe now: let $D = D(C)$ the subset of $\{1, \dots, n-1\}$ associated to the composition C of n. Then F_C is the sum of all increasing monomials $a_1 \dots a_n$ such that $a_i \leq a_{i+1}$ for any i and $a_i < a_{i+1}$ if $i \in D$. Example: $F_{23} = \sum_{a \leq b < c \leq d \leq e} abcde$.

The following result is due to Gessel ([7] th. 3).

Theorem 3. Let g be a symmetric function. Then

$$g = \sum_C (g, S_C)\ F_C$$

(where (,) is the usual scalar product of the symmetric functions; S_C is defined in Sect. 2).

We need a variant of Gessel's definition of the quasi-symmetric functions F_C. Given a word $w = a_1 \ldots a_n$ in A^*, we define, following Lascoux-Schützenberger [11], its standard permutation ("standardisé de w") by

$$st(w) = \sigma \in S_n$$

if $\sigma(i)$ = number of letters in w which are < a_i + number of a_i in the word $a_1 \ldots a_i$. In other words, σ is the numbering of the letters of w, from left to right, starting by the smallest letter, then the next one, ... etc.

Example: w = baabdaec
σ = 41257386

Lemma 1. Let σ in S_n such that the descent composition of σ^{-1} is C. Then F_C is the sum of the evaluations of all words whose standard permutation is σ.

7. Necklaces and words

In this section, we describe a natural bijection and its inverse between words and multi-sets of necklaces. It is a variant (discovered independantly) of an unpublished bijection of Gessel; Gessel's bijection is described and used in Désarménien-Wachs [2]. Another related bijection has been described by Dress-Siebeneicher [3], and it is also related to the cyclotomic identity of Rota and Metropolis [12].

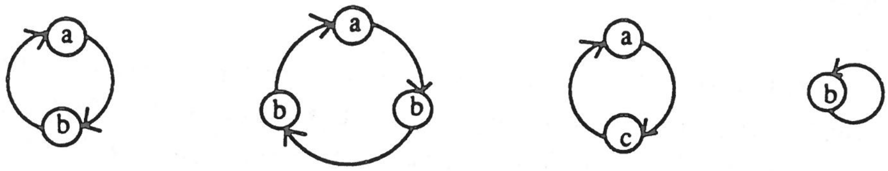
Let $w = a_1 \ldots a_n$ be a word and $\sigma \in S_n$ its standard permutation. For each cycle $\alpha = (i_1\, i_2 \ldots i_k)$ of σ, define the circular word $\overline{\alpha}$ to be the conjugation class of the word

$$a_{i_1}\, a_{i_2} \ldots a_{i_k}$$

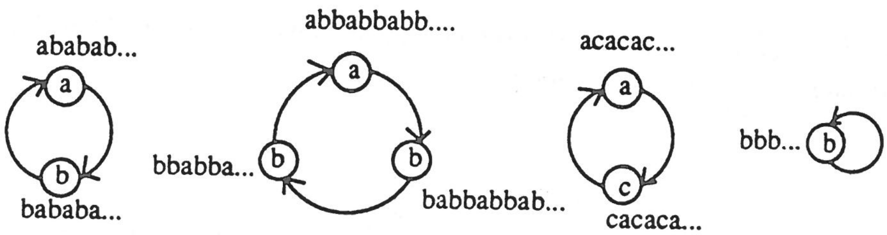
Then $\Phi(w)$ is the collection of all these $\overline{\alpha}$.

Theorem 4. The mapping Φ is an evaluation-preserving bijection between words whose standard permutation has cycle structure λ and multi-sets of necklaces of type λ.

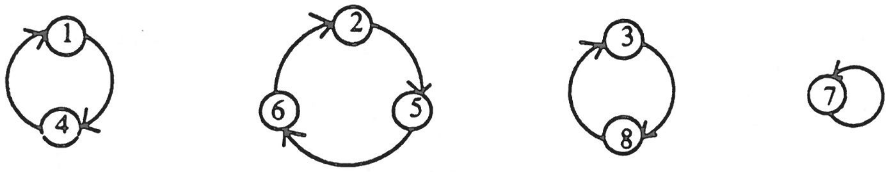
We describe on an example the inverse of Φ. Take the following multi-set of necklaces:



Label each occurence of a letter by the infinite sequence obtained by reading the necklace counter-clockwise:



Number these sequences from 1 to 8 (= total length of the multi-set) according to the lexicographical ordering:



We obtain a permutation σ in cycle form. Write σ in linear form and replace each digit by the original label of the necklaces:

$$\sigma = 45816273$$
$$w = bbcababa$$

Then σ is the standard permutation of w and w is the inverse image under Φ of the original multi-sets of necklaces.

8. Proof of theorem 1

Let P_λ denote the characteristic symmetric function of the Lie representation corresponding to λ. We have to show that (P_λ, S_C) = number of permutations with cycle structure λ and descent composition C.

By theorem 2, P_λ is the sum of all evaluations of the multi-sets of necklaces of type λ. By theorem 4, it is therefore equal to the sum of the evaluations of the words whose standard permutation is of cycle structure λ. As σ and σ^{-1} have the same cycle structure, we obtain by lemma 1

$$P_\lambda = \sum_\sigma F_{C(\sigma)}$$

where the sum is extended to all permutations of cycle structure λ. Now, P_λ is a symmetric function, hence we have by theorem 3

$$P_\lambda = \sum_C (P_\lambda, S_C)\, F_C$$

Comparing these two equations, we deduce that for any composition C

$$(P_\lambda, S_C) = \sum_\sigma 1$$

where the sum runs over all σ of cycle structure λ and descent composition C. This proves theorem 1.

9. Circular permutations and major index

Call descent class a subset of S_n consisting of permutations having the same descent set. Recall that the inverse major index of σ is

$$\mathrm{imaj}(\sigma) = \sum_{\substack{1 \le i \le n-1 \\ \sigma^{-1}(i) > \sigma^{-1}(i+1)}} i$$

Theorem 5. Let q and n be relatively prime. In each descent class of S_n, there are as many circular permutations as permutations whose inverse major index is equal to q modulo n.

Proof. The multiplicity of the irreducible representation corresponding to λ in the Lie representation $\mathcal{L}_n$ is equal to the number n_λ of standard Young tableau of shape λ and of major index $\equiv$ q mod. n: this is a result of Kraskiewicz-Weyman [10], see Garsia [5], and Stembridge [14]. In other words, the symmetric function P_n satisfies

$$P_n = \sum_{\lambda \vdash n} n_\lambda s_\lambda$$

where s_λ is the Schur function.

This may be rewritten as

$$P_n = \sum_{(P,Q)} ev(P)$$

where the sum runs over all semi-standard tableaux P and standard tableaux Q of the same shape (partition of n), where Q has major index $\equiv$ q mod. n; here, ev(P) stands for the usual content of P.

By Robinson-Schensted algorithm, this is equal to the sum of all evaluations of the words whose right tableau Q has major index congruent to q mod. n. By Lascoux-Schützenberger [11], a word and its standard permutation have the same right tableau. Moreover, the major index of the right tableau of $\sigma \in S_n$ is equal to the major index of σ (see Thomas [15] sect.II). Hence, we obtain that P_n is the sum of the evaluation of the words whose standard permutation has a major index $\equiv$ q mod. n. This may be written, by lemma 1, as

$$P_n = \sum_{maj(\sigma) \equiv q \bmod . n} F_{C(\sigma^{-1})}$$

On the other hand, we have by theorem 2 that P_n is the sum of all evaluations of necklaces of length n; this is equal by theorem 4 to the sum of all evaluations of words whose standard permutation is circular, hence by lemma 1 it is

$$P_n = \sum_{\alpha \text{ circular}} F_{C(\alpha^{-1})}$$

Let C be a fixed composition. Recall that the functions F_C are linearly independant. Then by comparing the previous equations, we obtain that the number of permutations σ such that $C(\sigma^{-1})$ =

C and maj(σ) $\equiv$ q mod. n is equal to the number of circular permutations α such that $C(\alpha^{-1}) = C$. This implies the theorem. □

It would be interesting to find a bijection for theorem 5, or a bijective proof of it. Note that, by the Foata-Schützenberger bijection, th.5 remains true when "major index" is replaced by "number of inversions".

10. Involutions

The well-known identity of Littlewood

$$\prod_a \frac{1}{1-a} \prod_{a<b} \frac{1}{1-ab} = \sum_\lambda s_\lambda$$

has several interpretations in the free Lie algebra. The left hand side is equal to the sum of the evaluations of all the multi-sets of necklaces of length 1 or 2. Hence, by theorem 4 and lemma 1, it is

$$\sum_{\sigma \text{ involution}} F_{C(\sigma)}$$

By theorem 2, it is also equal to the characteristic symmetric function of $\oplus_\lambda \mathcal{L}_\lambda$ where the sum is extended over all partitions having only the parts 1 or 2.

This analytic functor (see Joyal [8]) has several equivalent descriptions: let $N_2(A)$ be the free rank 2 nilpotent Lie algebra over A, that is, the quotient of $\mathcal{L}(A)$ by the relations [a, [b, c]] = 0. Then $N_2(A)$ admits as a basis the elements a ($a \in A$), and [a, b] ($a, b \in A$, $a < b$). Thus, by the theorem of Poincaré-Birckhoff-Witt, its envelopping algebra $EN_2(A)$ has asz basis the elements

$$a_1 \dots a_p [b_1, c_1] \dots [b_q, c_q]$$

where $a_1 \leq \dots \leq a_p$, $(b_1, c_1) \leq \dots \leq (b_q, c_q)$ (lexicogaphic), $p, q \geq 0$, $b_1 < c_1, \dots, b_q < c_q$. This shows that the generating function of $EN_2(A)$ is the left-hand side of Littlewood's identity. Hence, as an S_n-space, $EN_2(A)$ contains each irreducible representaiton of S_n once and only once.

The dual of $EN_2(A)$ is canonically embedded in $\mathbb{Q}\langle A\rangle$: it is the sub-shuffle-algebra of $\mathbb{Q}\langle A\rangle$ generated by the words of length ≤ 2. This space, as $EN_2(A)$, contains each irreducible representation exactly once. It would be interesting to determine exactly the irreducible components, therefore giving an alternative construction of the irreducible representations of the symmetric group.

Similarly, EN_k (A) (= envelopping algebra of the free rank k nilpotent Lie algebra) has as generating function the symmetric function

$$\prod \frac{1}{1 - ev(C)}$$

where the product runs over all necklaces c of length $\leq k$.

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