Publ. I.R.M.A. Strasbourg, 1993, 461/S–24
Actes 24e Séminaire Lotharingien, p. 87–97

A NOTE ON THE EULER AND GENOCCHI NUMBERS

BY

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Abstract. — We characterize the ordinary generating functions of the Genocchi and median Genocchi numbers as unique solutions of some functional equations and give a direct algebraic proof of several continued fraction expansions for these functions. New relations between these numbers are also obtained.

1. Introduction. — The *Euler numbers* E_n are the coefficients occurring in the Taylor series expansion of $\tan x + \sec x$, *i.e*,

$$\tan x + \sec x = \sum_{n\geq 0} E_n \frac{x^n}{n!}, \tag{1.1}$$

while the *Genocchi numbers* G_{2n} can be defined by

$$t + \sum_{n\geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!} = \frac{2t}{e^t+1}. \tag{1.2}$$

Let $g_{2n}^0 = (-1)^n G_{2n}$ and $g_{2n+1}^0 = 0$, and define the associated Seidel matrix $(g_n^k)_{k,n\geq 0}$ by $g_n^k = g_n^{k-1} + g_{n+1}^{k-1}$ $(k \geq 1, n \geq 1)$, or, equivalently by $g_n^k = \sum_{i=0}^k \binom{k}{i} g_{n+i}^0$ (cf. [11, 6, 12]). Then the *Seidel identity* on Genocchi numbers (cf. [6, 8]) reads $g_n^n = 0$ and the *median Genocchi numbers* H_{2n+1} are defined by

$$H_{2n+1} = (-1)^n g_n^{n+1} \qquad (n \geq 0). \tag{1.3}$$

AMS (1991) subject classification : 11B75, 11B68, 39B99.

The first values of these numbers are given in Table 1.

n	1	2	3	4	5	6	7
E_{2n}	1	5	61	1385	50521	2702765	199360981
E_{2n-1}	1	2	16	272	7936	353792	22368256
G_{2n}	1	1	3	17	155	2073	38227
H_{2n-1}	1	1	2	8	56	608	9440

Table 1

The literature dealing with these numbers is very extensive, see [1, 2, 3, 4, 6, 7, 13, 14] and the references cited in these papers.

This paper was originally motivated by the desire to give a short proof of the two following formulas :

$$1+\sum_{n=1}^{\infty} G_{2n+2}\, x^n = \cfrac{1}{1-\cfrac{1\cdot 1\,x}{1-\cfrac{1\cdot 2\,x}{1-\cfrac{2\cdot 2\,x}{1-\cfrac{2\cdot 3\,x}{1-\cfrac{3\cdot 3\,x}{1-\cfrac{3\cdot 4\,x}{\cdots}}}}}}}, \tag{1.4}$$

$$1+\sum_{n=1}^{\infty} H_{2n+1}\, x^n = \cfrac{1}{1-\cfrac{1^2\,x}{1-\cfrac{1^2\,x}{1-\cfrac{2^2\,x}{1-\cfrac{2^2\,x}{\cdots}}}}}. \tag{1.5}$$

These formulas were first established combinatorially by VIENNOT [12, 13]. DUMONT [4] has later shown how to derive (1.5) from a formula due to BARSKY and DUMONT [3]. However, as to (1.4), VIENNOT's combinatorial proof seems to be the only one known. Since VIENNOT's proof was based on a sequence of combinatorial interpretations of these numbers, an altenative proof will make possible to reverse VIENNOT's procedure : First establish the continued fraction expansions and then derive their combinatorial interpretations. Our method was inspired by those used by PREECE [9] and ROGERS [10] and need not appeal to the Ganhdi generations of these numbers (cf. [3, 12]).

We first prove some results about the ordinary generating functions of Genocchi and median Genocchi numbers in Section 2 from scratch, especially we characterize them as the unique solutions of some functional equations, and then apply these results to derive the continued fraction expansion of these ordinary generating functions in Section 3.

2. Some functional relations. — Given any formal series of exponential type

$$U(x) = \sum_{n\geq 0} u_n \frac{x^n}{n!},$$

we define its formal Laplace tranform as the formal series

$$u(x) = \sum_{n\geq 0} u_n x^{n+1}.$$

The analytic formula

$$\int_0^{+\infty} e^{-t/x} t^n \, dt = n! x^{n+1},$$

where x is assumed to be a complex number such that $Re(x) > 0$, leads to the formula $u(x) = \int_0^{+\infty} e^{-t/x} U(t) dt$, which has a precise analytical sense in the theory of Laplace transforms, but will have a purely formal sense in what follows.

Let

$$g(x) = x^2 - x^3 + x^5 - 3x^7 + \cdots = x^2 - \sum_{n\geq 1} (-1)^{n-1} G_{2n}\, x^{2n+1}, \tag{2.1}$$

$$h(x) = x - x^2 + 2x^3 - 8x^4 + \cdots = \sum_{n\geq 1} (-1)^{n-1} H_{2n-1}\, x^n. \tag{2.2}$$

Note in passing that the first series is the Laplace transform of $2x/(e^x+1)$, for $g(x) = \int_0^{+\infty} e^{-t/x} \frac{2t}{e^t+1} dt$.

THEOREM 1. — *We have*

$$g(x) = h\Big(\frac{x^2}{1+x}\Big). \tag{2.3}$$

PROOF. — Let $y = \frac{x^2}{1+x}$, or $x^2 = yx + y$. In general, we have $x^n = F_n(y)x + y\,F_{n-1}(y)$, where $F_n(y)$ is the Fibonacci polynomial defined as follows :

$$\begin{cases} F_0(y) = 0, & F_1(y) = 1, \\ F_n(y) = y\,(F_{n-1}(y) + F_{n-2}(y)). \end{cases} \tag{2.4}$$

It is readily seen that

$$F_n(y) = \sum_{k=[n/2]}^{n-1} \binom{k}{n-1-k} y^k. \tag{2.5}$$

Now, let $f(x) = \sum_{n=1}^{\infty} F_n(y)\, x^n$. The conditions (2.4) then imply

$$f(x) = x + y \sum_{n=2}^{\infty} (F_{n-1}(y) + F_{n-2}(y))\, x^n = x + xy\, f(x) + x^2 y\, f(x).$$

So $f(x) = x/(1 - xy - x^2 y) = x \sum_{k,i\geq 0} \binom{k}{i} y^k x^{k+i}$. Extracting the coefficient of x^n yields (2.5). Therefore,

$$g(x) = x \sum_{n\geq 2} (-1)^n G_{2n}\, F_{2n+1}(y) + y + y \sum_{n\geq 1} (-1)^{n-1} G_{2n}\, F_{2n}(y). \tag{2.6}$$

But the coefficient of y^k in $\sum_{n\geq 2}(-1)^n G_{2n}\, F_{2n+1}(y)$ is

$$\sum_{n\geq 2} (-1)^n G_{2n} \binom{k}{2n-k} = \sum_{i\geq 0} (-1)^{k-i} G_{2k-2i} \binom{k}{2i} = g_k^k = 0.$$

So the first term of the sum in (2.6) is null. On the other hand, the coefficient of y^{k-1} in $\sum_{n\geq 1}(-1)^{n-1} G_{2n}\, F_{2n}(y)$ is

$$\sum_{i\geq 0} (-1)^{k-i-1} G_{2k-2i} \binom{k}{2i+1} = g_k^{k+1} = (-1)^k H_{2k+1}.$$

Thus we have proved $g(x) = h(y)$. $\square$

Corollary 1. — *We have*

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{2n-i}{i-1} H_{2i-1} = G_{2n},$$

$$\sum_{i=1}^{n} (-1)^{i-1} \binom{2n-i-1}{i-1} H_{2i-1} = \delta_{n1}.$$

Proof. — This follows by extracting the coefficient of x^n in (2.3). $\square$

Theorem 2. — *Let $\phi(x)$ be a formal power series, then the following three assertions are equivalent:*

(1) $\phi(x) = \int_0^{+\infty} e^{-t/x} \frac{t}{\cosh t} dt.$

(2) $\phi(x)$ *is even and* $\phi(x/(1+x)) - x^2$ *odd.*
(3) $\phi(x/(1+x)) + \phi(x/(1-x)) = 2x^2$.

PROOF. — ((1) $\Longrightarrow$ (2)) Let $f(x) = \int_0^{+\infty} e^{-t/x} F(t)dt$. Recall that $\int_0^{+\infty} e^{-t/x}\, t^n dt = n!\, x^{n+1}$, so $f(x)$ is even (resp. odd), if and only if $F(x)$ is odd (resp. even). Note that

$$\phi(\frac{x}{1+x}) - x^2 = -\int_0^{+\infty} e^{-t/x}\, t \tan t dt.$$

Since $t/\cosh t$ is odd and $t\tan t$ is even, the assertion (2) follows immediately.

((2) $\Longrightarrow$ (3)) Since $\varphi(x) = \phi(\frac{x}{1+x}) - x^2$ is odd, we have

$$\phi\Big(\frac{-x}{1-x}\Big) - x^2 = -\phi\Big(\frac{x}{1+x}\Big) + x^2.$$

Assertion (3) follows from the fact that $\phi(x)$ is even.

((3) $\Longrightarrow$ (1)) Let $\phi(x) = \int_0^{+\infty} e^{-t/x} F(t)dt$. Then

$$\phi\Big(\frac{x}{1+x}\Big) + \phi\Big(\frac{x}{1-x}\Big) = \int_0^{+\infty} e^{-t/x} F(t)(e^{-t} + e^t)dt.$$

But $2x^2 = \int_0^{+\infty} 2te^{-t/x}dt$, hence $F(t)(e^{-t} + e^t) = 2t$, and $F(t) = t/\cosh t$. □

THEOREM 3. — *The following identities hold*

$$g(x) = 4\phi\Big(\frac{x}{2+x}\Big), \tag{2.7}$$

$$\phi(x) = \frac{1}{4} h\Big(\frac{4x^2}{1-x^2}\Big). \tag{2.8}$$

PROOF. — According to Theorem 1, it suffices to prove only (2.7). Note that

$$\frac{e^{-t}}{\cosh t} = \frac{2}{e^{2t}+1} = 1 + \sum_{m=1}^{\infty} (-1)^m G_{2m} \frac{(2t)^{2m-1}}{(2m)!}.$$

So

$$\phi\Big(\frac{x}{x+1}\Big) = x^2 + \sum_{m=1}^{\infty} (-1)^m 2^{2m-1} G_{2m} x^{2m+1}, \tag{2.9}$$

which clearly implies (2.7). □

The next corollary is an immediate consequence of Theorems 2 and 3.

COROLLARY 2. — *The formal series $g(x)$ and $h(x)$ are respectively the unique solutions of the functional equations*

$$g(x)+g\Big(\frac{x}{1-x}\Big)=2x^2 \quad \textit{and} \quad h\Big(\frac{x^2}{1+x}\Big)+h\Big(\frac{x^2}{1-x}\Big)=2x^2.$$

COROLLARY 3. — *We have*

$$2^{2n}\,H_{2n+1}=\sum_{m=0}^{n}(2m+1)\binom{n}{m}\,E_{2m}. \tag{2.10}$$

PROOF. — Recall that

$$\frac{t}{\cosh t}=\sum_{m=0}^{\infty}(-1)^m\,E_{2m}\frac{t^{2m+1}}{(2m)!}.$$

So, by applying the formal Laplace transform, we get

$$\phi(x)=\sum_{m=0}^{\infty}(-1)^m(2m+1)\,E_{2m}x^{2m+2}. \tag{2.11}$$

Upon substituting this in (2.8) and replacing $4x^2/(1-x^2)$ by $-y$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}H_{2n+1}\,y^n&=\sum_{m\geq 0}(2m+1)\,E_{2m}\,\Big(\frac{y}{4}\Big)^m\Big(1-\frac{y}{4}\Big)^{-m-1}\\&=\sum_{n=0}^{\infty}4^{-n}\sum_{m=0}^{n}(2m+1)\binom{n}{m}\,E_{2m}\,y^n.\end{aligned}$$

Comparing the coefficients of y^n on the two sides yields then (2.10). □

Formula (2.10) relating the median Genocchi numbers H_{2n+1} with the secant numbers E_{2n} is comparable with the well-known formula relating the Genocchi numbers G_{2n} with the tangent numbers E_{2n+1} :

$$2^{2n}\,G_{2n+2}=(n+1)\,E_{2n+1}. \tag{2.12}$$

No combinatorial interpretation has been given to (2.12). Similarly, we have no combinatorial proof of (2.10) to offer at this time.

3. Continued fraction expansions. — Our aim is first to give a complete proof of the ROGERS continued fraction for $\varphi(x)$ [10], as ROGERS' original proof seems to us rather obscure, then to derive the VIENNOT continued fractions for $g(x)$ and $h(x)$ and further continued fractions for the series.

LEMMA 1. — *Let $A(x)$ be any invertible formal power series, then*

$$\cfrac{a_0}{1+\cfrac{a_1\,x}{A(x)}} = a_0 - \frac{a_0 a_1\,x}{a_1 x + A(x)}.$$

The following lemmas are well-known in the context of analytical theory of continued fractions (*cf.* [10, 9]). Here we consider them just as formal power series.

LEMMA 2. — *Let $\{c_n\}_{n\geq 0}$ be a sequence of complex numbers, then*

$$\cfrac{c_0}{1-\cfrac{c_1\,x}{1-\cfrac{c_2\,x}{1-\cfrac{c_3\,x}{1-\cfrac{c_4\,x}{\cdots}}}}} = c_0 + \cfrac{c_0 c_1\,x}{1-(c_1+c_2)\,x-\cfrac{c_2 c_3\,x^2}{1-(c_3+c_4)\,x-\cfrac{c_4 c_5\,x^2}{\cdots}}}$$

$$= \cfrac{c_0}{1-c_1\,x-\cfrac{c_1 c_2\,x^2}{1-(c_2+c_3)\,x-\cfrac{c_3 c_4\,x^2}{\cdots}}}.$$

PROOF. — The two formulas are respectively obtained from Lemma 1 by iteration, starting from the first and the second row. □

LEMMA 3. — *Let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$ be three sequences of complex numbers. If*

$$1+bx+\theta\,x^2+\cfrac{b_1\,x^2}{1+\cfrac{b_2\,x^2}{1+b\,x+\theta\,x^2+\cfrac{b_3\,x^2}{1+\cfrac{b_4\,x^2}{\cdots}}}} \tag{3.1}$$

$$= 1+a_1\,x+\cfrac{c_1\,x^2}{1+a_2\,x+\cfrac{c_2\,x^2}{1+a_3\,x+\cfrac{c_3\,x^2}{1+a_4\,x+\cfrac{c_4\,x^2}{\cdots}}}}.$$

then, for $n\geq 1$, we have $a_{2n-1}=b$, $a_{2n}=0$ and

$$b_1 = c_1 - \theta, \qquad b_1 b_2 = c_1 c_2,$$

$$b_{2n}+b_{2n+1} = c_{2n}+c_{2n+1}-\theta, \qquad b_{2n+1}b_{2n+2} = c_{2n+1}c_{2n+2}.$$

PROOF. — Denote by $A_0(x)$ and $B_0(x)$ the left and right side of (3.1) and set

$$A_0(x) = 1 + b\,x + \theta\,x^2 + \cfrac{b_1\,x^2}{1 + \cfrac{b_2\,x^2}{A_2(x)}},$$

$$B_0(x) = 1 + a_1\,x + \cfrac{c_1\,x^2}{1 + a_2\,x + \cfrac{c_2\,x^2}{B_2(x)}}.$$

From the equality $A_0(x) = B_0(x)$, we derive immediately the following :

$$a_1 = b, \quad a_2 = 0, \quad c_1 = b_1 + \theta, \quad c_1c_2 = b_1b_2,$$

and

$$b_2x^2 + A_2(x) = c_2x^2 + B_2(x).$$

The proof can then be readily completed by induction. □

By Theorem 1, $\phi(x)$ is even, so we may write the following continued expansion :

$$\phi(x) = \cfrac{x^2}{1 + \cfrac{a_1\,x^2}{1 + \cfrac{a_2\,x^2}{1 + \cfrac{a_3\,x^2}{\cdots}}}}.$$

It follows that

$$\phi\Big(\frac{x}{x+1}\Big) = \cfrac{x^2}{(1+x)^2 + \cfrac{a_1\,x^2}{1 + \cfrac{a_2\,x^2}{(1+x)^2 + \cdots}}}. \tag{3.2}$$

On the other hand, since $\phi(x/(1+x)) - x^2$ is odd, we also have the following expansion :

$$\phi\Big(\frac{x}{x+1}\Big) = x^2 - \cfrac{b_1\,x^3}{1 + \cfrac{b_1b_2\,x^2}{1 + \cfrac{b_2b_3\,x^2}{1 + \cfrac{b_3b_4\,x^2}{\cdots}}}}. \tag{3.3}$$

By Lemma 2 with $c_0 = 1$ and $c_{2k-1} = -c_{2k} = -b_k$, $k \geq 1$, we get

$$\phi\Big(\frac{x}{x+1}\Big) = \cfrac{x^2}{1 + b_1\, x + \cfrac{b_1^2\, x^2}{1 + (b_2 - b_1)\, x + \cfrac{b_2^2\, x^2}{1 + (b_3 - b_2)\, x + \cdots}}}. \tag{3.4}$$

By Lemma 3, we get succesively from (3.2) and (3.4)

$$b_1 = b_2 = 2,\ b_3 = b_4 = 4,\ \ldots,\ b_{2n-1} = b_{2n} = 2n \qquad (n \geq 1). \tag{3.5}$$

Substituting these values in (3.4) yields

$$\phi\Big(\frac{x}{x+1}\Big) = \cfrac{x^2}{1 + 2x + \cfrac{2^2\, x^2}{1 + \cfrac{2^2\, x^2}{1 + 2x + \cfrac{4^2\, x^2}{1 + \cfrac{4^2\, x^2}{1 + 2x + \cdots}}}}}. \tag{3.6}$$

By writing $x/(1-x)$ for x in (3.6), we get

$$\begin{aligned} \phi(x) &= \cfrac{x^2}{1 - x^2 + \cfrac{2^2\, x^2}{1 + \cfrac{2^2\, x^2}{1 - x^2 + \cfrac{4^2\, x^2}{1 + \cdots}}}} \\ &= \cfrac{x^2/(1-x^2)}{1 + \cfrac{2^2\, x^2/(1-x^2)}{1 + \cfrac{2^2\, x^2/(1-x^2)}{1 + \cfrac{4^2\, x^2/(1-x^2)}{1 + \cfrac{4^2\, x^2/(1-x^2)}{\cdots}}}}}. \end{aligned} \tag{3.7}$$

Substituting now $4x^2/(1-x^2)$ by y and applying (2.8), we get

$$h(y) = \sum_{n \geq 1} (-1)^{n-1} H_{2n-1} y^n = \cfrac{y}{1 + \cfrac{1^2\, y}{1 + \cfrac{1^2\, y}{1 + \cfrac{2^2\, y}{1 + \cfrac{2^2\, y}{\cdots}}}}}, \tag{3.8}$$

which implies immediately (1.5). Finally, on account of (2.9), (3.3) and (3.5), we obtain

$$\sum_{m=1}^{\infty}(-1)^m 2^{2m-1} G_{2m} x^{2m-1} = -\cfrac{2\,x}{1+\cfrac{2\cdot 2\,x^2}{1+\cfrac{2\cdot 4\,x^2}{1+\cfrac{4\cdot 4\,x^2}{\cdots}}}}, \tag{3.9}$$

which is clearly equivalent to (1.4).

Remark 1. — Some modified forms of our continued fractions in Section 3 may be obtained by applying the lemmas 2 and 3. For example, we have

$$g(x) = x^2 - \cfrac{x^3}{1+\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1-\cfrac{x}{1+\cfrac{2x}{1-\cfrac{2x}{\cdots}}}}}}}. \tag{3.10}$$

Remark 2. — Values of Hankel determinants of a sequence are known to be related to coefficients of continued fraction expansions of its ordinary generating function. For instance, from (1.4) and (1.5) we derive

$$\begin{vmatrix} H_1 & H_2 & \cdots & H_{n+1} \\ H_2 & H_3 & \cdots & H_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+1} & H_{n+2} & \cdots & H_{2n+1} \end{vmatrix} = \prod_{i=1}^{n} \left\lceil \frac{i}{2} \right\rceil^{2(n-i+1)}, \tag{3.11}$$

and

$$\begin{vmatrix} G_2 & G_3 & \cdots & G_{n+2} \\ G_3 & G_4 & \cdots & G_{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n+2} & G_{n+3} & \cdots & G_{2n+2} \end{vmatrix} = \prod_{i=1}^{n} \left(\left\lceil \frac{i}{2} \right\rceil \left\lceil \frac{i+1}{2} \right\rceil \right)^{n-i+1}, \tag{3.12}$$

where $\lceil x \rceil$ means the least integer $n \geq x$ and $H_{2i} = G_{2i+1} = 0$.

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