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Some contributions to the model theory of mono-unaries

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A mono-unary is a structure of the form (A,f), where A is a non-empty set, and f is a unary function from A to A. Many of the definitions and results apply also to (first-order) structures of a more general and richer type than mono-unaries. We describe mono-unaries in first-order logic with just one unary function symbol and equality (always interpreted as identity).

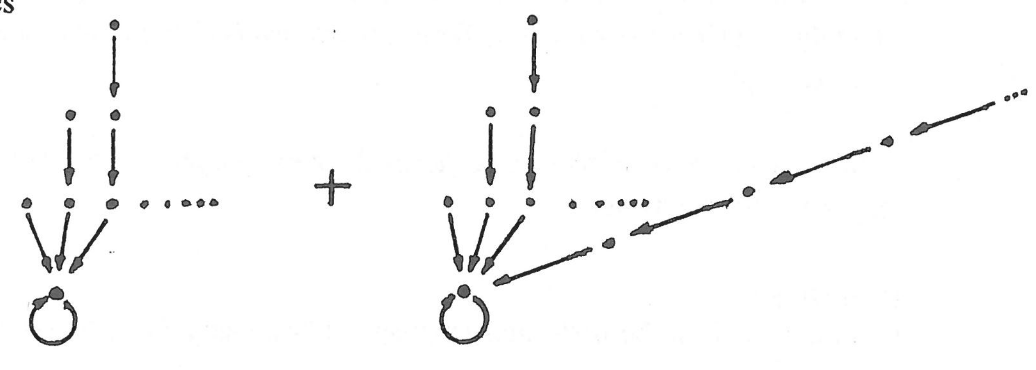
A structure is called *rigid*, if it has no nontrivial automorphisms. {Often, rigidity is defined by the absence of nontrivial endomorphisms. But if (A,f) is a mono-unary, then f is by necessity an endomorphism.} A structure is called *separative*, if for each two different elements a and b there is a formula in the first-order language of the structure, $\varphi(x)$, with exactly one free variable such that $\varphi(a)$, $\neg\varphi(b)$ hold in the structure. A structure is called *individuative*, if every element of it can be defined by a first-order formula; i.e. for every a from the universe of the structure there is a formula $\varphi(x)$ such that $\varphi(b)$ hold iff b = a for all b from the universe.

Trivially: individuative $\Rightarrow$ separative $\Rightarrow$ rigid.

Remark: The notions of *separativeness* and *individuativeness* can also be defined with respect to *higher-order* logics, or other logics extending first order. The arising problems seem to be very interesting for those who work in the field of 'model-theoretic' logics. E.g. the realm of model-theoretic logics should be of such an extension that for every rigid structure there exists a model-theoretic logic in which the structure is individuative, or at least separative.

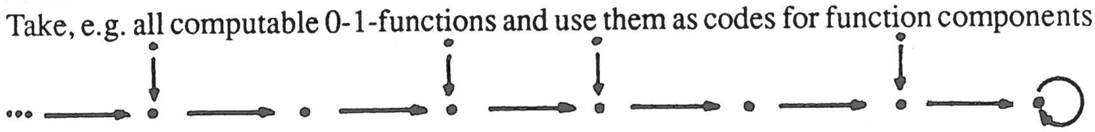
We show that in the *countable* case 1) rigidity $\not\Rightarrow$ separativeness and
2) separativeness $\not\Rightarrow$ individuativeness. We prove these non-implications by making use of appropriate mono-unaries

Ad 1



<u>Ad 2</u>

Take, e.g. all computable 0-1-functions and use them as codes for function components of the form:



Let (A,f) be a mono-unary. It is called *backward finite*, if for all $a \in A$ the preimage $f^{-1}\{a\}$ is finite.

<u>Theorem 1</u>

Let (A,f) be a backward finite rigid mono-unary. Then (A,f) is separative.

<u>Proof</u>: A König's -lemma-like argument.

<u>Theorem 2</u> (Comer and Le Tourneau)

ZFC $\vdash$ Let A be any nonempty set. Then there exists a rigid mono-unary (A,f)

ZFC $\nvdash$ There is an $f : \aleph_2 \to \aleph_2$ such that $(\aleph_2, f)$ is rigid and backward finite.

(otherwise ZFC $\vdash$ $\aleph_2 \leq 2^{\aleph_0}$)

<u>Problem</u>: ZF $\vdash$ There is a rigid mono-unary $(\aleph_1, f)$? {On the other hand: ZF $\vdash$ There is a rigid mono-unary $(\mathbb{R}, f)$. One cannot prove without AC that every infinite set has a rigid mono-unary on it. Moreover, we show

<u>Theorem 3</u>:

ZF $\vdash$ Let A be infinite and $f : A \to A$ such that (A,f) is rigid.

Then A can be surjected onto ω.

<u>Proof:</u> Let A be infinite and (A,f) be rigid. First, let f have infinitely many *finite* components. Since f is rigid, the heights of these finite components must be unbounded. This gives us a surjection of A onto ω. If f has only finitely many finite components then f must have infinite components. So let us consider one infinite component C. If C has a infinite forward path, then we are done. Thus we may suppose that C has as a sink a finite cycle. Then at least one 'tree' planted on this cycle must be infinite. Let T be an infinite tree planted on our finite cycle.

We claim that *all levels n* must be nonempty (and this will give us a surjective onto ω). For if for some n, all levels m, $m \geq n$ are empty, then we must have a point in our tree which branches infinitely. But since there are only finitely many levels above it, one cannot rigidify this part of the function graph.

Now we come to an interesting model-theoretic property of our notions of rigidity, separativeness and individuativeness.

<u>Theorem 4</u>

There is a set Γ in the first-order language of one unary function symbol without a rigid model, although each finite subset of Γ has an individuative model.

Remark: Thus the compactness theorem fails for *rigid, separative* and *individuative* models.

Proof: In order to construct the set Γ we need some notations.

The *expressions* $f^n x$ for $n \geq 0$ are defined recursively: $f^0 x := x$; $f^{n+1} x := ff^n x$.

$\exists^{\geq n}$ means: there are at least n

$\exists^{\leq n}$ means: there are at most n

$\exists^{=n}$ means: there are exactly n.

$H(0) := \exists x\, x \neq x$

$H(1) := H(0)$

for $n \geq 2$:

$$H(n) := \exists x_1 \ldots x_n (\wedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \forall y\, fy \neq x_1 \wedge fx_n = x_n \wedge \wedge_{1 \leq i < n} fx_i = x_{i+1})$$

{H(n) means that there are n individuals, x_1 being a point without preimage, x_n being a fixed point, while the rest lies on the f-thread between x_1 and x_n

$I := \{\exists^{=2} x\, fx = x\, ,\ \exists^{=2} x\, \forall y\, fy \neq x,$
$\forall x(fx = x \rightarrow \exists^{=2} y\, fy = x),$
$\forall x(fx \neq x \rightarrow \exists^{\leq 1} y\, fy = x)\}$

$II := \{\forall x(f^n x = x \rightarrow fx = x) : n \geq 1\}$

$III := \{H(n) \rightarrow H(n\text{-}2) \vee H(n\text{-}1) \vee H(n+1) \vee H(n+2) : n \geq 2\}$

$IV := \{\exists^{\geq n} : n \geq 1\}$

Now Γ is defined as the union $I \cup II \cup III \cup IV$.

And Δ is defined as $I \cup II \cup III$.

Lemma A:

Let k be a natural number ≥ 5. Then Δ has (up to isomorphism) exactly one individuative model of cardinality k.

Proof: If $k \geq 5$ is odd, a model of Δ with k points must look like:

If $k \geq 5$ is even, the model looks like:

Lemma B

Γ has no rigid model.

Proof: Obviously, Γ has no finite model. Let $M = (M,f)$ be an infinite model of Γ. Then the function f contains

1) two components of the form 

or

2) a component of the form 

In either case, M is not rigid.

Remark: If case 1) obtains, then f contains two components of the form
If f contains two components of the last form, the it contains two components as in 1). The Theorem follows now from the Lemmata A and B.

Our theory Γ has, incidentally, some interesting metamathematical properties besides being a counterexample to compactness: For $n \geq 2$ let [n,1] denote the conjunction $H(n) \wedge H(n+1)$, and let [n,2] denote the conjunction $H(n) \wedge H(n+2)$. Γ[n,1] denotes $\Gamma \cup \{[n,1]\}$ and Γ[n,2] denotes $\Gamma \cup \{[n,2]\}$. Finally, $\Gamma[\neg H]$denotes $\Gamma \cup \{\neg H(n) : n \geq 2\}$.

Proposition 1:
For $n \geq 2$, Γ[n,1] and Γ[n,2] are $\aleph_1$-categorical; $\Gamma[\neg H]$ is also $\aleph_1$-categorical. Hence, $\Gamma[\neg H]$, and for $n \geq 2$: Γ[n,1] and Γ[n,2] are complete and decidable (since they are all axiomatized).

Proposition 2:
Let Γ* be any complete (consistent) extension of Γ. Then Γ* is axiomatized by $\Gamma[\neg H]$ or by Γ[n,1] or Γ[n,2] for some $n \geq 2$.

Theorem 5: Γ is decidable.

Proof: The theorems of Γ are, of course, r.e. Therefore it suffices to show that the non-theorems are also r.e. Now, by the foregoing propositions 1 and 2 $\Gamma \nvdash \varphi$ is equivalent to: $\Gamma \vdash \neg \varphi$ or there are different labels A and B $\in$ {[n,1], [n,2], [¬ H]} such that $\Gamma[A] \vdash \varphi$ and $\Gamma[B] \vdash \neg \varphi$.

This is a nice example of an *incomplete* theory whose decidability can be shown by surveying *all* complete extensions of it. But the (known) decidability of the theory of one unary function (without) any non-logical axioms cannot be obtained in this way, because there is a recursively axiomatized theory T of a mono-unary such T is *essentially* undecidable; the mono-unary in question can even be taken as a permutation of ω.

We now come to a curious result concerning the relation between elementary equivalence and isomorphism in the realm of mono-unaries.

Theorem 6:
Let A = (A,f) and B = (B,g) be any two infinite mono-unary. If A ≡ B (elementarily equivalent), then there exist countably infinite substructures (A_0,f_0) and (B_0,g_0) of (A,f) and (B,g) respectively such that $A_0 \cong B_0$ (isomorphic).

Proof: We start from the assumption that A and B (i.e. the functions f and g) satisfy the same first-order sentences in the language of one unary function symbol. If f has a finite component, then this component can be completely described by a sentence, say σ. But then g satisfies, by elementary equivalence, this sentence σ too; and this means that g contains a component of the same shape. Now suppose that f contains infinitely many finite components. Then f and g contain countably infinite isomorphic substructures. If f contains only finitely many finite components, then f contains at least one infinite component. Now, lying not in one of the finitely many finite components can be expressed by a first order formula - since the finitely many finite components are fixed. So we may proceed as if we had no finite components at all. Consider now an infinite component C of f. First suppose that C contains a *finite* cycle c_0 (of length n). Suppose further that there is no other (infinite) component C' of f with a finite cycle of length n.

Now, at least one of the trees planted in c_0 must be infinite. If there is a backward infinite path in C, then we find, by elementary equivalence, the same situation in g. [Why?] Otherwise there must be a point which has infinitely many f-preimages. The same holds in g. It is easily seen, that the case that there are finitely many cycles of the length of c_0 reduces to the case of just one such cycle. If there are infinitely many cycles of the length of c_0 we are also done. In the case that f has no finite components and no finite cycles [this implies the first condition] our theorem holds trivially.

Let me conclude with the bare statement of the following two theorems:

Theorem 7: There are exactly countably many $\aleph_0$-categorical theories of one unary function; each of them is recursively axiomatizable.
(Some application of the Ryll-Nardzewski-Theorem.)

Theorem 8: A mono-unary is called totally inseparative, if it has exactly one 1-type. Then every totally inseparative mono-unary on ω has a decidable theory.

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