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Endomorphisms of words in a quiver

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We present a purely combinatorial concept which has been useful in the representation theory of finite dimensional algebras. First we extend the classical concept of a word in an alphabet (as it is discussed for instance in the book [L] of M. Lothaire) to that of a word in a quiver. Then the endomorphisms of such a word are defined. They form a monoid which provides some information about recurrence and periodicity of the fixed word.

1. A *quiver* Q is an oriented graph, consisting of a set of vertices Q_0 and a set of arrows Q_1 such that to each arrow α in Q there is attached a starting vertex $s(\alpha)$ and a terminating vertex $t(\alpha)$. We add formal inverses α^{-1} for each arrow $\alpha \in Q_1$ with $(\alpha^{-1})^{-1} = \alpha$, $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. The set of formal inverses is denoted by Q_{-1}.

A sequence $w = w_1 w_2 \ldots w_n$ of arrows and formal inverses is called a *word* in Q of length $|w| = n$, if $w_{i+1} \neq {w_i}^{-1}$ and $s(w_{i+1}) = t(w_i)$ hold for each $i \in \{1, 2, \ldots, n-1\}$. The starting vertex and the terminating vertex of w are denoted by $s(w) = s(w_1)$ and $t(w) = t(w_n)$, respectively. Let $v = v_1 \ldots v_m$ be an additional word of length m in Q. The composite $vw = v_1 \ldots v_m w_1 \ldots w_n$ is defined by concatenating if this sequence is again a word in Q. In addition we need for each vertex x in Q the word e_x of length $|e_x| = 0$ with $s(e_x) = x = t(e_x)$. The composite $e_x w = w$ and $w e_x = w$, respectively, for a word w is defined if $s(w) = x$ and $t(w) = x$, respectively, are satisfied. We denote by Q^* the set of all words in Q.

Consider for some word $a = a_1 \ldots a_n$ of length $n > 0$

$$\sigma(a) = \begin{cases} 1 & a_1 \in Q_1, \\ -1 & a_1 \in Q_{-1}, \end{cases} \qquad \tau(a) = \begin{cases} 1 & a_n \in Q_1, \\ -1 & a_n \in Q_{-1} \end{cases}$$

and $a^{-1} = {a_n}^{-1} \ldots {a_1}^{-1}$. Extend this for e_x by $\sigma(e_x) = \tau(e_x) = 0$ and ${e_x}^{-1} = e_x$. We obtain *factors*, *quotients* and *divisors* of a word w as follows:

$$\mathrm{Fac}(w) \quad = \quad \{ (x, a, y) \in Q^* \times Q^* \times Q^* \mid w = xay \},$$

$$\begin{aligned} \mathrm{Quot}(w) &= \{ (x, a, y) \in \mathrm{Fac}(w) \mid \tau(x) \leq 0, \sigma(y) \geq 0 \} \text{ and} \\ \mathrm{Div}(w) &= \{ (x, a, y) \in \mathrm{Fac}(w) \mid \tau(x) \geq 0, \sigma(y) \leq 0 \}. \end{aligned}$$

We denote by $\pi(\alpha) = a$ the *projection* of a factor $\alpha = (x, a, y)$. Now we may introduce the set of *endomorphisms* of a word:

$$\mathrm{End}(w) = \{ (\varphi_s, \varphi_t) \in \mathrm{Quot}(w) \times \mathrm{Div}(w) \mid \pi(\varphi_s) = \pi(\varphi_t) \text{ or } \pi(\varphi_s) = \pi(\varphi_t)^{-1} \} \cup \{0\}.$$

Together with the composition which will be defined in section 3 the endomorphisms form a monoid.

2. Let M be an arbitrary monoid with *radical* $\mathrm{rad}\, M$ being the subset of non-invertible elements and $\mathrm{rad}^n M = (\mathrm{rad}\, M)^n$. We call M *local* if only the unit is invertible and if the set $\bigcap_{n \in \mathbb{N}} \mathrm{rad}^n M$ consists of precisely one element. We state the main result:

Theorem *Let w be a word in a quiver. Then the monoid* $\mathrm{End}(w)$ *of endomorphisms of w is local. For a factor monoid M of* $\mathrm{End}(w)$ *which is generated by two elements and a natural number n the following holds:*

(a) *The cardinality* $\mathrm{card}(M/\mathrm{rad}^n M)$ *is bounded by* $2n^2 - 2n + 2$.

(b) *If M and $M\langle x, y\rangle/\mathrm{rad}^n M\langle x, y\rangle$ are isomorphic, then $n \leq 3$. Here $M\langle x, y\rangle$ denotes the free monoid with two generators.*

Note that a polynomial bound in (a) has to be at least of degree 2 (cf. the example). Also the bound 3 in (b) is best possible (cf. Example 6.3 in [K]). For the proof we have to refer to [K], but we sketch here a somewhat weaker result to indicate the sort of techniques being used. First of all we complete the notation. In particular the composition in $\mathrm{End}(w)$ needs to be defined.

3. Let w be a word in Q and let $\alpha = (a_1, a, a_2)$ and $\beta = (b_1, b, b_2)$, respectively, be factors of w. The set $\mathrm{Fac}(w)$ is *partialy ordered* by

$$(a_1, a, a_2) \leq (b_1, b, b_2) \iff |a_i| \geq |b_i| \quad \text{for} \quad i \in \{1, 2\}.$$

The *union* of α and β is defined by

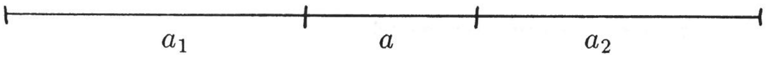
$$\alpha \cup \beta = \min\{ \gamma \in \mathrm{Fac}(w) \mid \alpha \leq \gamma, \beta \leq \gamma \}.$$

The factors α and β are *connected* if

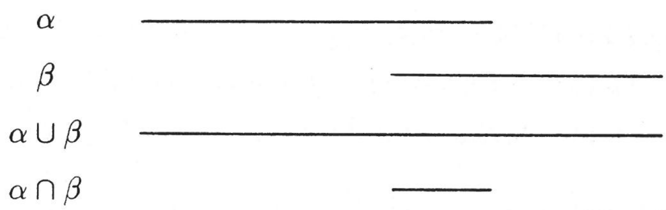
$$S = \{\gamma \in \mathrm{Fac}(w) \mid \gamma \leq \alpha, \gamma \leq \beta\} \neq \emptyset.$$

In the latter case $\alpha \cap \beta = \max S$ denotes the *intersection* of α and β.

A factor $\alpha = (a_1, a, a_2)$ of w may be visualised by the following diagram:



The line corresponds to w and the partition into three parts reflects the length of the words a_1, a and a_2 in Q^*. To compare different factors it usually suffices to present the projections according to their relative position:



Let v be an additional word in Q and suppose $\gamma = (c_1, c, c_2) \in \mathrm{Fac}(v)$. If $v = \pi(\beta)$, then the *composition* of γ and β is defined as follows

$$\gamma * \beta = (b_1 c_1, c, c_2 b_2).$$

It is obvious that $\alpha \leq \beta$ holds if and only if there exists a factor $\alpha_\beta \in \mathrm{Fac}(\pi(\beta))$ (uniquely determined by α and β) such that $\alpha = \alpha_\beta * \beta$.

For a factor α the *length* is defined by $|\pi(\alpha)|$ and we also use $\alpha^{-1} = ({a_2}^{-1}, a^{-1}, {a_1}^{-1}) \in \mathrm{Fac}(w^{-1})$.

We introduce the following notation for an endomorphism $\varphi = (\varphi_s, \varphi_t)$:

The *signum* of φ is $\mathrm{sgn}(\varphi) = \begin{cases} 1 & \pi(\varphi_s) = \pi(\varphi_t), \\ -1 & \text{else.} \end{cases}$

The *support* of φ is $\mathrm{supp}(\varphi) = \varphi_s \cup \varphi_t$.

The *shift* of φ is $\|\varphi\| = ||x'| - |x||$, if $\varphi_s = (x, a, y)$ and $\varphi_t = (x', a', y,)$.

The *image* of $\alpha \in \mathrm{Fac}(w)$ is $\alpha\varphi = (\alpha_{\varphi_s})^{\mathrm{sgn}(\varphi)} * \varphi_t$, if $\alpha \leq \varphi_s$.

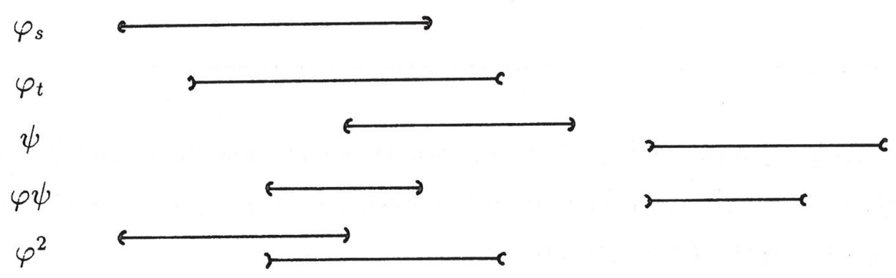
The *preimage* of $\alpha \in \mathrm{Fac}(w)$ is $\alpha\varphi^{-1} = (\alpha_{\varphi_t})^{\mathrm{sgn}(\varphi)} * \varphi_s$, if $\alpha \leq \varphi_t$.

The composition of two endomorphisms $\varphi, \psi \in \mathrm{End}(w)$ is defined as follows:

$$\varphi\psi = \begin{cases} (\alpha\varphi^{-1}, \alpha\psi) & \varphi = (\varphi_s, \varphi_t),\ \psi = (\psi_s, \psi_t) \text{ and } \alpha = \varphi_t \cap \psi_s \text{ exists,} \\ 0 & \text{else.} \end{cases}$$

The set End(w) is closed under the composition which is obviously associative.

The following diagrams illustrate the composition, assuming that sgn$(\varphi) = 1 =$ sgn(ψ). The marks at the ends of each line help to distinguish between quotients and divisors.



4. A word a in Q is called *p-periodic* if there exist $x_1, x_2 \in Q^*$ and $r \in \mathbf{N}$ such that $a = (x_1x_2)^r x_1$ and $|x_1x_2| = p$. A factor α of a word in Q is *p-periodic* if the projection $\pi(\alpha)$ is p-periodic. The following lemma plays a central role. It is easy to prove.

Lemma. *Let* $\alpha \in \mathrm{End}(w)$ *and suppose* $\alpha \neq 1$, $\alpha^2 \neq 0$. *Then* $\mathrm{sgn}(\alpha) = 1$ *and* $\mathrm{supp}(\alpha)$ *is* $\|\alpha\|$*-periodic.*

Proposition. *Let* α *and* β *be different endomorphisms in* $\mathrm{rad}\,\mathrm{End}(w) \setminus \mathrm{rad}^2\,\mathrm{End}(w)$. *Then* $\alpha\beta^n\alpha = 0$ *or* $\beta\alpha^n\beta = 0$ *holds for all* $n \geq 2$.

Corollary. *Let* $M\langle x, y\rangle$ *be the free monoid in two generators and let* M *be the following factor monoid:*

$$M = M\langle x,y\rangle/I \quad \textit{with} \quad I = (x^3, y^3, xyx, yxy) + (x,y)^5.$$

For a word w *in a quiver there is no factor monoid of* $\mathrm{End}(w)$ *isomorphic to* M.

Proof. To prove the Proposition we may assume that $\alpha^2, \beta^2 \neq 0$ and that $\|\beta\| \geq \|\alpha\|$. Now suppose $\alpha\beta^n\alpha = \gamma = (\gamma_s, \gamma_t) \neq 0$ for some $n \geq 2$. We conclude $\gamma_s\alpha \leq \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ and $\gamma_s\alpha\beta^n \leq \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$. Therefore

$$|\,\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)| \geq |\gamma_s\alpha \cup \gamma_s\alpha\beta^n| = \|\beta^n\| + |\gamma_s| \geq n\|\beta\| \geq 2\|\beta\| \geq \|\alpha\| + \|\beta\|.$$

Applying the Lemma this already implies that supp(α) and supp(β) are both $\|\alpha\|$- and $\|\beta\|$-periodic by Proposition 1.3.5 in [L] (cf. Lemma 2.4 in [K] for a simplification). Moreover, a careful analysis of the definition of endomorphisms and their composition shows that $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$ and that there exists an endomorphism δ such that $\alpha = \delta^r$, $\beta = \delta^s$ for some natural r, s. This contradicts $\alpha \neq \beta$ in $\operatorname{rad}\operatorname{End}(w) \setminus \operatorname{rad}^2 \operatorname{End}(w)$. Therefore the Proposition is proven and the Corollary immediately follows since $xy^2x, yx^2y \notin I$.

5. Example. The following example is based on the quiver Q with one vertex and two arrows, i.e. $Q_1 = \{x, y\}$. We fix some $n \in \mathbf{N}$. Consider $w_r = (x^{-1}y)^r x^{-1}$ for $r \in \mathbf{N}_0$ and $w = w_n$. Let

$$\alpha_s = (1, w_{n-1}, yx^{-1}), \quad \alpha_t = (x^{-1}y, w_{n-1}, 1),$$

$$\beta_s = (w, 1, 1) \quad \text{and} \quad \beta_t = (1, 1, w).$$

Then $\alpha = (\alpha_s, \alpha_t)$ and $\beta = (\beta_s, \beta_t)$ belong to $\operatorname{End}(w)$ with

$$\operatorname{End}(w) = \{\, \alpha^i \mid 0 \le i \le n \,\} \cup \{\, \alpha^i \beta \alpha^j \mid 0 \le i, j \le n \,\} \cup \{0\},$$

$$\operatorname{End}(w) \setminus \operatorname{rad}^n \operatorname{End}(w) = \{\, \alpha^i \mid 0 \le i < n \,\} \cup \{\, \alpha^i \beta \alpha^j \mid 0 \le i + j < n - 1 \,\},$$

$$\operatorname{card}(\operatorname{End}(w)/\operatorname{rad}^n \operatorname{End}(w)) = n^2/2 + n/2 + 1.$$

6. References

[K] H. KRAUSE, Endomorphisms of words in a quiver, Preprint, Universität Bielefeld (1991).

[L] M. LOTHAIRE, Combinatorics on words, Encyclopedia of mathematics and its applications, Vol. 17, Addison-Wesley, Reading (1983).

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