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PROPERTIES OF INTEGERS AND FINITENESS CONDITIONS FOR SEMIGROUPS

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Abstract. Let $\mathbb{h}$ and $\mathbb{k}$ be integers greater than 1; we prove that the following statements are equivalent: 1) the direct product of $\mathbb{h}$ copies of the additive semigroup of non-negative integers is not $\mathbb{k}$-repetitive; 2) if the direct product of $\mathbb{h}$ finitely generated semigroups is $\mathbb{k}$-repetitive, then one of them is finite. Using this and some results of Dekking and Pleasants on infinite words, we prove that certain repetitivity properties are finiteness conditions for finitely generated semigroups.

1. INTRODUCTION.

The notion of repetitive semigroup has been introduced by Justin who studied it in [3-10] and, in collaboration with the author of this paper, in [11-12].

We recall that a semigroup $\mathbb{S}$ is *repetitive* if, for each integer $\mathbb{k}$ greater than 1, it is $\mathbb{k}$-*repetitive*; this means that, given any finite alphabet $\mathbb{A}$ and any morphism μ from $\mathbb{A}^+$ (the free semigroup on $\mathbb{A}$) into $\mathbb{S}$, the infinite words on $\mathbb{A}$ contain $\mathbb{k}$ consecutive factors with the same image under μ.

One can prove, using the Theorem of Ramsey or in a more direct way, that *finite semigroups are repetitive* and, using the Theorem of van der Waerden, that *the additive semigroup* $\mathbb{N}$ *of non-negative integers is repetitive* [8].

Now, to better explain the idea of this paper let us make three remarks.

1) According to the terminology of [12], *finite semigroups are "strongly repetitive"* [8] (see also [1]). In [12] it is proved first that this property is not satisfied by $\mathbb{N}$ and then that it is a finiteness condition for finitely generated semigroups.

2) *Finite semigroup are "uniformly repetitive"* [14]. It is proved in [16] that $\mathbb{N}$ is not uniformly repetitive and in [13] that a finitely generated uniformly repetitive semigroup is finite.

3) Similarly, in [9] it is proved that $\mathbb{N}x\mathbb{N}$, the direct product of two copies of $\mathbb{N}$, *is not repetitive* and in [17] it is proved that if S is a finitely generated semigroup such that the direct product SxS is repetitive, then S is finite.

So, in a sense, in the domain of repetitivity, the "negative properties" of $\mathbb{N}$ reflect finiteness conditions for finitely generated semigroups.

Now let h and k be integers greater than 1; the aim of this paper is to prove that the following statements are equivalent:

1) *the direct product of* h *copies of* $\mathbb{N}$ *is not* k*-repetitive*;

2) *if the direct product of* h *finitely generated semigroups is* k*-repetitive, then one of them is finite.*

From this and from some results of Pleasants [15] and Dekking [2], we obtain that *a finitely generated semigroup* S *is finite if the direct product of two copies of* S *is* 4*-repetitive or the direct product of three copies of* S *is* 3*-repetitive or the direct product of five copies of* S *is* 2*-repetitive.*

2. DEFINITIONS AND PRELIMINARY RESULTS.

2.1. Words and factors.

We refer to [14] for the terminology concerning the *free monoid* A^* and the *free semigroup* $A^+=A^*-\{1\}$ generated by the *alphabet* A. We call the elements of A^* (finite) *words* and those of A *letters* .

We denote by $\mathbb{N}$ the set as well as the additive semigroup (monoid) of non-negative integers.

We extend the notion of a word to infinite words: a (right) *infinite word* on A is a map t of $\mathbb{N}$ into A. We write

$$t=t(0)t(1)...t(i)...$$

By word and factor we will always mean a finite nonempty word, except where otherwise stated.

If t is a word (either finite or infinite), let $t(i)$ be the letter of A *occurring* at "rank i" in t and let $t(i,j)$, $i\leq j$, be the *factor* $t(i)...t(j)$ of t. If a is a letter of the alphabet A and w is a word of A^+, we denote by $|w|_a$ the number of occurrences of a in w. The *length* $|w|$ of w is the sum of the $|w|_a$.

Now, let w, u, w_1, w_2, ..., w_k be elements of A^+. If $w=u^k$ we say that w is a k-*power* (*square* for k=2 and *cube* for k=3). If $w=w_1w_2...w_k$ and for each i, j ∈ {1, 2, ..., k} and for each a ∈ A, one has

$$|w_i|_a = |w_j|_a$$

then we say that w is an *abelian* k-*power* (*abelian square* for k=2 and *abelian cube* for k=3). A word (finite or infinite) is k-*power free* (resp. *abelian* k-*power free*) if it does not contain a factor which is a k-power (resp. an abelian k-power). For k=2 (resp. k=3) we have the notions of *square free* and *abelian square free* (resp. *cube free* and *abelian cube free*) *words*.

2.2. Infinite words generated by morphisms.

This is an important class of infinite words [14]; the cube free Thue infinite word on a two-letter alphabet, for example, belongs to this class [14]. All the infinite words we use in this paper belong to it.

Theorem 1. (Dekking, [2]). *There exists an infinite word on a two-letter alphabet without abelian 4-powers.*

This improves an analogous result due to Justin [9].

Theorem 2. (Dekking, [2]). *There exists an infinite word on a three-letter alphabet without abelian cubes .*

Theorem 3. (Pleasant, [15]). *There exists an infinite word on a five-letter alphabet without abelian squares.*

2.3. Lemma of Koenig.

The following lemma is an easy consequence of the well known Lemma of Koenig.

Lemma 1. *If* A *is a finite alphabet and* E *is an infinite subset of* A^+, *then there exists an infinite word* s *such that each factor of* s *is a factor of at least one word of* E.

Now, let G be a *set of generators* of a semigroup S and let μ be the *morphism* from G^+ into S defined by

$$\mu(g)=g$$

for each g ∈ G.

We say that the word u of G^+ is *irreducible* if for each word w ∈ G^+ such that μ(u)=μ(w) one has |u|≤|w|.

The following lemma is an easy consequence of Lemma 1 and of the fact that *a factor of an irreducible word is also irreducible.*

Lemma 2. *Let* S *be an infinite semigroup and* G *be a finite set of generators of* S. *There exists an infinite word* s *on the alphabet* G *such that each factor of* s *is irreducible.*

2.4. **Repetitivity.**

Definition 1. Let E be a set, π a map from A^+ into E and k an integer greater than 1. Furthermore, let w, w_1, w_2, ..., w_k be elements of A^+. If

$$w=w_1w_2...w_k$$

and, for $1\le i\le j\le k$,

$$\pi(w_i)=\pi(w_j)$$

then we say that w is a k-*power modulo* π.

Definition 2. A map $\pi : A^+ \to E$ is k-*repetitive* if each infinite word on A contains a factor which is a k-power modulo π.

Definition 3. A semigroup S is k-*repetitive* if each morphism from a finitely generated free semigroup into S is k-repetitive. A semigroup S is *repetitive* if it is k-repetitive for each integer k greater than 1.

3. RESULTS AND PROOFS.

If S is a semigroup and h an integer greater than 1, in this section we will denote by S^h the direct product of h copies of S. Our main result is the following theorem.

Theorem 4. *Let* h *and* k *be integers greater than* 1. *The following conditions are equivalent:*

1) N^h *is not* k-*repetitive;*

2) *if the direct product of* h *finitely generated semigroups is* k-*repetitive, then one of them is finite.*

Proof. 2) $\to$ 1) is trivial.

1) $\to$ 2). By hypothesis there exist a finite alphabet A, morphisms μ_1,

$\mu_2, ..., \mu_h$ from A^+ into N and an infinite word t on A such that t does not contain any k-power modulo the morphism

$$\mu=(\mu_1, \mu_2, ..., \mu_h)$$

from A^+ into N^h.

Now, let

$$t=t(0)t(1)...t(i)...$$

and let $G_1, G_2, ..., G_h$ be, respectively, finite sets of generators of the semigroups $S_1, S_2, ..., S_h$ whose direct product $S_1 x S_2 x ... x S_h$ is repetitive.

Suppose, by way of contradiction, that for each $i \in \{1, ..., h\}$ the semigroup S_i is infinite. By Lemma 2, for each $i \in \{1, ..., h\}$ there exists an infinite word s_i on the alphabet G_i with all its factors irreducible.

Now, consider the following factorization of s_i:

$$s_i=s_{i_0} s_{i_1} ... s_{i_j} ...$$

where $|s_{i_j}|=\mu_i(t(j))$.

The set $\{(s_{1_j}, s_{2_j}, ..., s_{h_j}); j \in N\}$ is finite.

As $S_1 x S_2 x ... x S_h$ is k-repetitive, there exists integers

$$0 \leq j(0)<j(1)<...<j(k)$$

such that, for all $i \in \{1, 2, ..., h\}$, one has the following equality *in the semigroup* S_i :

$$s_{i_{j(0)}} s_{i_{j(0)+1}} ... s_{i_{j(1)-1}} =$$

$$s_{i_{j(1)}} s_{i_{j(1)+1}} ... s_{i_{j(2)-1}} =$$

$$.............................=$$

$$s_{i_{j(k-1)}} s_{i_{j(k-1)+1}} ... s_{i_{j(k)-1}} .$$

By definitions of the factors s_{i_j} and by irreducibility of the infinite words s_i, we have that

$$|s_{i_{j(0)}} s_{i_{j(0)+1}} \dots s_{i_{j(1)-1}}| =$$

$$|s_{i_{j(1)}} s_{i_{j(1)+1}} \dots s_{i_{j(2)-1}}| =$$

$$\dots\dots\dots\dots\dots\dots\dots\dots =$$

$$|s_{i_{j(k-1)}} s_{i_{j(k-1)+1}} \dots s_{i_{j(k)-1}}|$$

i.e.,

$$\mu_i(t(j(0)))+\mu_i(t(j(0)+1))+\dots+\mu_i(t(j(1)-1))=$$

$$\mu_i(t(j(1)))+\mu_i(t(j(1)+1))+\dots+\mu_i(t(j(2)-1))=$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots =$$

$$\mu_i(t(j(k-1)))+\mu_i(t(j(k-1)+1))+\dots+\mu_i(t(j(k)-1)).$$

But this means that

$$t(j(0),j(1)-1)t(j(1),j(2)-1)\dots t(j(k-1),j(k)-1)$$

is a k-power modulo μ. Contradiction.•

The following lemma will be useful to prove our finiteness conditions for finitely generated semigroups.

Lemma 3. *Let* h, k *be integers greater than* 1. *If there exists an abelian* k*-power free infinite word on an* h*-letter alphabet, then* N^h *is not* k*-repetitive.*

Proof. Let A be an h-letter alphabet and t be an abelian k-power free infinite word on A and suppose, by way of contradiction, that N^h is k-repetitive.

Consider, for each $a \in A$, the morphisms μ_a from A^+ into N defined as follows:

$$\mu_a(x)=\begin{cases} 1 \text{ if } x=a \\ 0 \text{ if } x\in A-\{a\} \end{cases}$$

Let $\mu_1, \mu_2, ..., \mu_h$ be a permutation of the μ_a and

$$\mu=(\mu_1, \mu_2, ..., \mu_h)$$

be the corresponding morphism from A^+ into N^h. By definition of k-repetitivity, the infinite word t must contain a k-power modulo μ. But, by definition of the morphism μ, this implies that t contains an abelian k-power. Contradiction.•

Using Theorems 1, 2 and 3 we have:
Theorem 5. *The following statements are true*:

1) N^2 *is not 4-repetitive*;
2) N^3 *is not 3-repetitive*;
3) N^5 *is not 2-repetitive.*

From this and from Theorem 4 we have:
Theorem 6. *Let S be a finitely generated semigroup.Then :*

1) *if* S^2 *is 4-repetitive, then S is finite.*
2) *if* S^3 *is 3-repetitive, then S is finite.*
3) *if* S^5 *is 2-repetitive, then S is finite.*

As an immediate consequence of the definition we have that for all integers k, k' and for each semigroup S, if $2 \leq k' \leq k$ and if S is k-repetitive, then each homomorphic image of S is k'-repetitive.

So, considering Theorem 5, only some problems remain open:

1) *is* N^2 *2-repetitive* (resp. *3-repetitive*)?
2) *is* N^3 *2-repetitive ?*
3) *is* N^4 *2-repetitive ?*

A well-known problem of combinatorics on words is the following: *does there exist an infinite abelian square free word on a four-letter alphabet* ? If such an infinite word exists, then the answer to the last question is no.

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