PARTIAL PROOFS OF THOMASSEN'S CONJECTURE

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Abstract: THOMASSEN's Conjecture (1986) claims that every line-graph with vertex-connectivity number $\kappa \geq 4$ is hamiltonian. This conjecture is equivalent to several other conjectures. The partial proofs which will be summed up here do not start directly from a line-graph, but from a graph G such that its line-graph L(G) has the properties as above. It will be shown that both G and L(G) must fulfill a great number of restrictions if L(G) is to be a counterexample to THOMASSEN's Conjecture. These restrictions are both structural properties and inequalities related to several graph invariants. It is proved e.g. that a counterexample must have at least 23 vertices, it has a 2-cover, and it is not locally connected.

1. THOMASSENS's Conjecture and some related problems

All graphs considered here are simple finite undirected graphs (without loops or multiple edges). THOMASSEN's Conjecture (1986) claims:

C1: Every 4-connected line-graph is hamiltonian.

The importance of this conjecture is illustrated by the following conjectures, which are closely related to it. First of all, C2 and C3 are equivalent to C1 (FLEISCHNER and JACKSON 1989).

C2: Every cyclically 4-edge connected 3-regular graph G has a cycle C such that G - V(C) is an independent set of vertices.

C3: Every essentially 4-edge connected graph G has an eulerian subgraph H such that G - V(H) is an independent set of vertices (for a definition of "essentially k-edge connected graphs" see FLEISCHNER and JACKSON 1989).

C4: Every cyclically 4-edge connected 3-regular graph G has a cycle C such that G - V(C) is acyclic (JAEGER, quoted from FLEISCHNER and JACKSON 1989).

A stronger conjecture is due to MATTHEWS and SUMNER (1984):

C5: Every 4-connected claw-free graph is hamiltonian.

A partial proof both to C1 and C5 is supplied by ZHAN (1991): Every line-graph with $\kappa \geqslant 7$ is hamiltonian.

2. Notation

As usual, G is a graph with p vertices and q edges, and with vertex-set V(G) and edge-set E(G); L(G) is its line-graph. Other graph invariants are: minimum degree δ, maximum degree Δ, connectivity number κ, edge-connectivity number κ_1, greatest clique order ω, girth g, circumference c, independence number β_0, diameter diam(G), toughness tough(G), binding number bind(G); p_i denotes the number of vertices with degree i.

3. Outline of the present approach

In the sequel, two different graphs will be considered: firstly, a graph G such that L(G) is 4-connected, but not hamiltonian - such that L(G) is a counterexample to C1 - and secondly the line-graph L(G) itself. For both graphs some structural properties and several bounds for graph invariants will be derived.

Part of the results was found by the use of the computer program KBGRAPH, which is a knowledge-based system for the support of graph-theoretical proofs (GERNERT 1989). Nevertheless, this paper is written in the usual mathematical style and can be read independently from that program (an optional output of KBGRAPH is the derivation of each partial result).

4. Partial results related to the "root graph" G

Since terminology is not unique the following definition is proposed here:

> A cycle C within a graph H is called an edge-dominating cycle in H if every edge from E(H) is incident with at least one vertex in C.

Now a theorem by HARARY and NASH-WILLIAMS (1965) states that L(H) is hamiltonian if and only if H is either isomorphic to $K_{1,m}$ $(m \geq 3)$ or H has an edge-dominating cycle. This implies that G cannot have an edge-dominating cycle, and hence G is not hamiltonian. Because the line-graph of an eulerian graph would be hamiltonian, G cannot be eulerian. From the equivalence of C1 and C2 it follows that $c \leq p - 2$.

According to JAEGER (1979) every graph H with $\kappa_1(H) \geq 4$ is supereulerian (has an eulerian spanning subgraph), and hence we have $\kappa_1(G) \leq 3$. It is easy to show that $\kappa \geq 2$ can be assumed. LAI (1991) proved the following theorem: If every edge of a 2-connected graph H belongs to a cycle C_3 or C_4 then L(H) is hamiltonian. Therefore G must contain at least one edge which does not belong to a C_3 or C_4. A result by VELDMAN (1988) leads to $\mathrm{diam}(G) \geq 3$.

From $\kappa(L(G)) \geq 4$ it follows that $\Delta(L(G)) \geq 4$. This permits the case distinction:

Case 1: $\Delta(L(G)) = 4$
Case 2: $\Delta(L(G)) \geq 5$.

Case 1: Graphs which only contain vertices with degrees 2 and 4 must be excluded since they would be eulerian, and only cubic graphs remain. According to FLEISCHNER and JACKSON (1989) it is sufficient to consider cyclically 4-edge connected cubic graphs, and it is easy to see that such graphs are trianglefree. BAUER (1985) proved: If H is a 2-connected cubic graph with $p(H) \leq 13$ then L(H) is hamiltonian. This implies $p(G) \geq 14$.

Next, special theorems on cycles in cubic graphs (BAU and HOLTON 1991) together with the conditions that G has no edge-dominating cycle and that $G - V(C)$ is not an independent set (from the equivalence of C1 and C2) lead to $p \geq 16$ and $q \geq 24$.

Case 2: Now let be $\Delta(L(G)) \geq 5$. In the simplest case we have $\Delta = 4$. Vertices with degree 2 may occur, but by virtue of $\delta(L(G)) \geq 4$, every 2-vertex can be adjacent only to 4-vertices. Similarly two 4-vertices cannot be adjacent. The smallest graph to fulfill all these conditions is $S(K_5)$, the graph generated from K_5 by subdividing each edge exactly once. Of course, $S(K_5)$ is eulerian and must be modified. It would not be sufficient to replace some 2-vertices by 3-vertices because the resulting graph would be super-eulerian (such that $L(G)$ would be hamiltonian). Rather, at least two 3-vertices must be added. Hence we have graphs with $p_4 \geq 5$, $p_3 \geq 2$, $p_2 \geq 10$, and consequently $q \geq 23$. A similar proof is possible for $\Delta \geq 5$.

By combining Case 1 and Case 2 we find that anyway $q \geq 23$. Further properties of G can be listed as follows:

1. $4p/3 + 2 \leq q \leq (p - 4)(p - 5)/2 + 10$
2. $4 \leq \Delta \leq p - 3$
3. $c \geq 12$
4. $4 \leq \beta_0 \leq p/2 - 1$
5. G has at least one induced subgraph $K_{1,3}$.
6. If G is regular then $\delta \geq 3$ and δ is odd.
7. If G is a minimal counterexample then $\kappa = \kappa_1 = \delta$.

5. Partial results related to L(G)

It follows from $q \geq 23$ that $p(L(G)) \geq 23$. A line-graph has no induced subgraph $K_{1,3}$, and so $\text{tough}(L(G)) = \kappa(L(G))/2 \geq 2$ (MATTHEWS and SUMNER 1984, GODDARD and SWART 1990). According to ENOMOTO et al. (1985) every graph H with $\text{tough}(H) \geq k$ has a k-factor if $kp(H)$ is even and $p(H) \geq k + 1$. Therefore $L(G)$ has a 2-factor. OBERLY and SUMNER (1979) showed that every 4-connected locally connected graph without an induced $K_{1,3}$

is hamiltonian. Hence L(G) is not locally connected.

Further properties of L(G) are as follows:

1. $4 \leq \kappa(L(G)) \leq 6$
2. $4 \leq \Delta(L(G)) \leq p(L(G)) - 5$
3. $4 \leq \omega(L(G)) \leq p(L(G)) - 5$
4. $g(L(G)) = 3$
5. $\mathrm{diam}(L(G)) \geq 3$
6. $\mathrm{bind}(L(G)) < 3/2$ and $\mathrm{bind}(L(G)) \leq 1 + 12/(p(L(G)) - 6)$
7. $2 \leq \mathrm{tough}\ (L(G)) \leq 3$
8. L(G) is not planar.
9. L(G) contains a chordless cycle C_m with $m \geq 5$.

6. Concluding remarks

A great lot of bounds to graph invariants were not reported here for the sake of space. Further results can be expected from an extension of the program KBGRAPH in the near future. It seems quite plausible to assume that THOMASSEN's Conjecture is correct. In the light of ZHAN's theorem quoted above, a promising next step may be to prove the hamiltonian property e.g. for 6-connected line-graphs.

Acknowlegdments: The author wants to thank Robert C. BRIGHAM and Ronald D. DUTTON (Univ. of Central Florida, Orlando) for their permanent support. Essential contributions to library search and to details of the proofs were made by Mrs. Andrea ROTT.

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