FIBONACCI NUMBERS AND WORDS

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Abstract. *Let Φ be the golden ratio $(\sqrt{5}+1)/2$, f_n the n-th Fibonacci finite word and f the Fibonacci infinite word. Let r be a rational number greater than $(2+\Phi)/2$ and u a non empty word. If u^r is a factor of f, then there exists $n \geq 1$ such that u is a conjugate of f_n and, moreover, each occurrence of u^r is contained in a maximal one of $(f_n)^s$ for some $s \in [2, 2+\Phi)$. Several known results on the Fibonacci infinite word follow from this.*

1. INTRODUCTION

In analogy with the definition of the Fibonacci numbers, one sets $f_0 = b$, $f_1 = a$, and, for $n \geq 2$, one defines the n-th Fibonacci finite word as the product $f_{n-1}f_{n-2}$ of the words f_{n-1} and f_{n-2} (see [4] and [7]). The two products $f_{n-1}f_{n-2}$ and $f_{n-2}f_{n-1}$ are almost the same, being different only on the last two letters. This is the amusing, very simple but very interesting "*near-commutative property*" used in [6] to study concrete algorithms. It plays an important role also in this paper.

The infinite Fibonacci word f is the *Sturmian word* associated with the golden ratio $\Phi = (\sqrt{5}+1)/2$ and can also be defined as the unique infinite word having, for each $n \geq 1$, f_n as a left factor.

If a power u^r has an occurrence in f, we try to extend it to the left and to the right as far as possible preserving periodicity. We call *maximal* the occurrences of the powers that cannot be locally extended and we prove that we always reach one of them. Denoted it by v^s, the main result of this paper says: *if $r > (2+\Phi)/2$ then $v = f_n$ for some $n \geq 1$, u is a conjugate of f_n and $s \in [2, 2+\Phi)$.* Several known results on f are consequences of this.

2. DEFINITIONS AND PRELIMINARY RESULTS

This paper is organized so as to be self-contained; terminology and notations are those currently used in theoretical computer science [4,7].

We consider only the *two-letter alphabet* $\{a, b\}$ and we call (finite) *words* the elements of the *free monoid* $\{a, b\}^*$; we denote by 1 the *empty word* and by $|u|$ the *length* of a word u. We consider a word u of length $k \geq 1$ as a map $u : \{0, 1, \ldots, k-1\} \to \{a, b\}$; we write $u = u(0) \ldots u(i) \ldots u(k-1)$ where $u(0)$, $u(i)$ and $u(k-1)$ are respectively the first, the i-th and the last letter of u.

A word u is a *factor* of a word v if there exist two words $u', u'' \in \{a, b\}^*$ such that

$v = u'uu''$. When $u' = 1$ (resp. $u'' = 1$) we say that u is a *left factor* (resp. *right factor*) of v. A *proper factor*, (resp. *proper left factor, proper right factor*) u of v is a factor (resp. left factor, right factor) u of v such that $|u| < |v|$.

A (right) *infinite word* on $\{a, b\}$ is a map q from the set of non-negative integers into $\{a, b\}$. We write $q = q(0)q(1)\dots q(i)\dots$. A word u is a *factor* of q if there exist a word u' and an infinite word q' such that $q = u'uq'$. If $u' = 1$ we say that u is a *left factor* of q.

A non empty word u may be a factor of another (finite or infinite) word w in several ways. So it is useful to speak about occurrences. For this reason, let i, j be integers such that $0 \leq i \leq j$ (and that $j < |w|$ if w is a finite word) and let us denote by $w(i, j)$ the word $w(i)\dots w(j)$. We say that the pair of integers (i, j) is an *occurrence* of the factor u in the word w if $u = w(i, j)$. We say that an occurrence (i_0, j_0) of u in w is *contained* in an occurrence (i_1, j_1) of v in w if $i_1 \leq i_0 \leq j_0 \leq j_1$. We only speak about occurrences of non empty words.

Now, let $\varphi : \{a, b\}^* \to \{a, b\}^*$ be the morphism whose restriction to $\{a, b\}$ is given by $\varphi(a) = ab$, $\varphi(b) = a$. Remark that φ is injective. Let us define the n–th Fibonacci finite words f_n in the following way: $f_0 = b$ and, for each $n \geq 0$,

$$f_{n+1} = \varphi(f_n).$$

In particular, we have: $f_1 = a$, $f_2 = ab$, $f_3 = aba$, $f_4 = abaab$, $f_5 = abaababa$, $f_6 = abaababaabaab$, $f_7 = abaababaabaababaababa\dots$. It is clear that, for each $n \geq 2$, f_n is the product (juxtaposition) $f_{n-1}f_{n-2}$ of f_{n-1} and f_{n-2}. Also, for each $n \geq 0$, $|f_n|$ is the n–th element F_n of the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8,13, 21, 34, 55, 89, 144, 233, 377 ....

Remark now that, for each $n \geq 1$, f_n is a left factor of f_{n+1}. So there exists an unique infinite word, namely the Fibonacci infinite word f, such that, for each $n \geq 1$, f_n is a left factor of f (see, [4] and [7]) and we have

$$f = abaababaabaababaababaabaababaabaababaababa\dots.$$

We denote by $F(f)$ the set of the **non empty** factors of f and by $LF(f)$ its subset containing the non empty left factors of f.

For each $n \geq 2$, we denote by g_n the product $f_{n-2}f_{n-1}$ and by h_n the common longest left factor of f_n and g_n. In particular, we have: $g_2 = ba$, $g_3 = aab$, $g_4 = ababa$, $g_5 = abaabaab$, ... and $h_2 = 1$, $h_3 = a$, $h_4 = aba$, $h_5 = abaaba$ ....

Remark that a non empty factor of an element of $F(f)$ is again in $F(f)$; for each $n \geq 2$, $g_n \in F(f)$, $\varphi(g_n) = g_{n+1}$ and $h_{n+1} \in LF(f)$; if $f(i) = b$ then $i > 0$ and $f(i-1) = f(i+1) = a$ and if $f(i, i+1) = aa$ then $i > 0$ and $f(i-1) = f(i+2) = b$ (i. e., $bb, aaa \notin F(f)$).

A factor v of f is *special* if $va, vb \in F(f)$. Let $k \geq 1$; we denote by $\tilde{v}$ the mirror image $u(k-1)u(k-2)\dots u(1)u(0)$ of the word $u = u(0)u(1)\dots u(k-2)u(k-1)$. We say that a non empty word v is a *palindrome* if $v = \tilde{v}$.

Lemma 1 belongs to the folklore (see for example [1], [2], [4] and [6]) and is very easy; so we can give just an hint of its proof. The point i) is the "*near-commutative property*" quoted in the introduction.

Lemma 1. *For each $n \geq 2$,*

i) $f_n = f_{n-1}f_{n-2} = f_{n-2}g_{n-1} = h_nxy$ and $g_n = f_{n-2}f_{n-1} = f_{n-1}g_{n-2} = h_nyx$, where $x, y \in \{a, b\}$, $x \neq y$ and if n is even then $xy = ab$ and if n is odd then $xy = ba$;

ii) $|h_n| = F_n - 2$;

iii) h_n *is a special factor;*

iv) $f_{n+3} = h_{n+1}xyh_nyxh_{n+1}xy$, *where* $x, y \in \{a, b\}$, $x \neq y$;

v) h_n *is a palindrome;*

vi) $h_{n+2} = f_nh_{n+1} = h_{n+1}\tilde{f}_n = h_n\tilde{f}_{n+1} = f_{n+1}h_n$;

vii) for each integer $m \geq 0$, h_n *is a left and a right factor of* h_{n+m}.

Moreover: viii) if $v \in F(f)$ *then* $\tilde{v} \in F(f)$.

Proof. One can prove i) easily by induction; ii), iii) and iv) are consequence of i); one can prove v) by induction using iv); vi) is a consequence of v); vii) is a corollary of vi) and finally one can prove viii) using v). ◇

Lemma 2. *For each* $n \geq 2$,

i) $f_n(0) = a(= f_1)$; $g_{n+1}(0) = a$;

ii) if $f(i,j) = f_{n+2}$ *then* $f(j+1) = a$; *if* $f(i,j) = g_{n+1}$ *then* $f(j+1) = a$;

iii) if $f(i,j) = h_{n+1}$ *then* $f(i, j+2) = f_{n+1}$ *or* $f(i, j+2) = g_{n+1}$.

Proof. i) is trivial; to prove ii) and iii) use the fact that aba is a right factor of each h_{n+2} (point vii) of Lemma 1) and the fact that $bb, aaa \notin F(f)$. ◇

Lemma 3. *Let* $u \in F(f)$. *Then* $\varphi^{-1}(u)$ *exists and belongs to* $F(f)$ *if and only if one of these two conditions holds: i)* $u(0) = a$ *and* $u(|u|-1) = b$; *ii)* $u(0) = a$, $u(|u|-1) = a$ *and* $ua \in F(f)$.

Proof. By induction on $|u|$. ◇

Remark. We have $\varphi^{-1}(aa) = bb \notin F(f)$. We often use Lemma 3 together with points i) and ii) of Lemma 2 in order to prove the existence of $\varphi^{-1}(u)$ in $F(f)$ for suitable $u \in F(f)$.

The following Lemma 4 says that no occurrence of g_n is too close to the left of f.

Lemma 4. *For each* $n \geq 2$, *if* $f(i,j) = g_n$ *then* $i \geq F_{n-1}$ *and* $f(i - F_{n-1}, i-1) = f_{n-1}$.

Proof. By induction. Let $n = 2$. If $f(i, i+1) = ba = g_2$ then $i \neq 0$ and $f(i-1) = a = f_1$. Now, let $n \geq 3$ and $f(i,j) = g_n$. As $f(0, F_n - 1) = f_n$, $i \neq 0$ and so $w = f(0, i-1)$ is non empty. By Lemmas 2 and 3, one has $\varphi^{-1}(wg_n) = w'g_{n-1} \in F(f)$ for some non empty $w' \in LF(f)$. By induction hypothesis f_{n-2} is a right factor of w'; so $i \geq F_{n-1}$ and $f(i - F_{n-1}, i-1) = f_{n-1}$. ◇

The following Lemma 5 belongs to the folklore. Point i) is proved, in [1] for example, using an auxiliary morphism which is not necessary here. Point ii) is a particular case of a more general result on sturmian words (see [4] and [5]). For each $k \geq 1$, let us denote by $s^{[k]}$ the mirror image of the left factor of f having length k.

Lemma 5. *For each* $k \geq 1$, *i) the unique special factor of length* k *is* $s^{[k]}$; *ii) in* $F(f)$ *there are exactly* $k+1$ *elements of length* k.

Proof. i). Remark that $s^{[k]}$ is a right factor of h_m for each m such that $F_m > k+1$ and so, by point iii) of Lemma 1, $s^{[k]}$ is special. Suppose now that for a given k there is a special factor v of length k which is different from $s^{[k]}$. The last letter of v is necessarily a, hence the greatest integer, such that h_n is a common right factor of $s^{[k]}$ and v, is greater than or equal to 3. Let also k' be the greatest integer such that $s^{[k]}(k') \neq v(k')$. We have $s^{[k]} = uxu'h_n$ and $v = u''yu'h_n$, for some $u, u', u'' \in \{a, b\}^*$ and for some $x, y \in \{a, b\}$, $x \neq y$. We have also $|u'| < F_{n-1}$ otherwise, by point vi) and vii) of Lemma 1, we have a contradiction with the maximality of n. Being a right factor of a special factor, $yu'h_n$

is special and so $yu'h_n a, yu'h_n b \in F(f)$. In both cases, n even or odd, $yu'g_n \in F(f)$ and, by Lemma 4, yu' is a right factor of f_{n-1}. Hence, xu' is not a right factor of f_{n-1}. Contradiction. ii) It is an easy consequence of i). ◇

Let us recall that an infinite word p is *periodic* (resp. *ultimately periodic*) if there exists $k \geq 1$ such that $p(j+k) = p(j)$ for each $j \geq 0$ (resp. for each $j \geq i$ for some $i \geq 0$).

Lemma 6. *The Fibonacci infinite word is not ultimately periodic.*

Proof. This is easy by point ii) of Lemma 5. ◇

Let $u, v, w, z, z' \in F(f)$. We say that (u, v, w) is a *non empty overlap* of z and z' if $uv = z$, $vw = z'$ and $uvw \in F(f)$. The possible non empty overlaps concerning f_n and g_n are considered in the following Lemma 7.

Lemma 7. *Let $n \geq 3$. Then*

i) $(f_{n-1}, f_{n-2}, g_{n-1})$ is the unique non empty overlap of f_n and f_n;

ii) $(f_{n-1}, f_{n-2}, f_{n-1})$ is the unique non empty overlap of f_n and g_n;

iii) $(f_{n-2}, f_{n-1}, f_{n-2})$ is the unique non empty overlap of g_n and f_n;

iv) there is no non empty overlap of g_n and g_n.

Proof. By induction. i) Let $n = 3$. As a is the unique word which is a proper non empty right and also left factor of aba, $(ab, a, ba) = (f_2, f_1, g_2)$ is the unique non empty overlap of f_3 and f_3; hence the statement is true for $n = 3$. Now, let $n \geq 4$. Clearly, $(f_{n-1}, f_{n-2}, g_{n-1})$ is a non empty overlap of f_n and f_n. Now, let (u, v, w) be a non empty overlap of f_n and g_n. By Lemmas 2 and 3 there exist $u' = \varphi^{-1}(u)$, $v' = \varphi^{-1}(v)$, $w' = \varphi^{-1}(w)$ in $F(f)$ such that (u', v', w') is a non empty overlap of f_{n-1} and f_{n-1}. By induction hypothesis, $u' = f_{n-2}$, $v' = f_{n-3}$, $w' = g_{n-2}$. Hence $u = \varphi(f_{n-2}) = f_{n-1}$, $v = \varphi(f_{n-3}) = f_{n-2}$, $w = \varphi(g_{n-2}) = g_{n-1}$; ii) the argument is analogous, but starting with the fact that a is the unique word which is a proper non empty right factor of aba and also a proper non empty left factor of aab; iii) ab is the unique word which is a proper non empty right factor of aab and also a proper non empty left factor of aba; iv) no word is a proper non empty right and also left factor of aab. ◇

Lemma 8. *Let $n \geq 5$. There are exactly two non empty overlaps of h_n and h_n, namely just*

$$(f_{n-1}, h_{n-2}, \tilde{f}_{n-1})$$

and

$$(f_{n-2}, h_{n-1}, \tilde{f}_{n-2}).$$

Proof. This is an easy consequence of point iii) of Lemma 2, Lemma 7 and point vi) of Lemma 1. ◇

Remark. In some sense, $(aba)(aba)$ can be considered as an "overlap" of h_4 and h_4 but we have chosen to consider only non empty overlaps. So there is a unique non empty overlap of h_4 and h_4 and this is is (ab, a, ba), in accordance with point i) of Lemma 7.

Lemma 9. *Let $v \in LF(f)$. Then the following two conditions are equivalent: i) v is a palindrome and ii) $v = h_n$ for some $n \geq 3$.*

Proof. ii) $\rightarrow$ i) is point v) of Lemma 1. We prove i) $\rightarrow$ ii) by induction. The palindromes of $LF(f)$, having length less than or equal to $F_5 - 2$, are $a = h_3$, $aba = h_4$ and $abaaba = h_5$. Let $n \geq 4$ and suppose, by induction hypothesis, that $h_3, \ldots, h_n$ and h_{n+1} are all the palindromes of $LF(f)$ having length less than or equal to $F_{n+1} - 2$. Suppose also, by way of contradiction, that there exists a proper left factor u of h_{n+2}

such that h_{n+1} is a proper left factor of u with u a palindrome. By points i) and vii) of Lemma 1, we can write $h_{n+2} = f_n h_{n-1} cdh'_n ba$, where h'_n is the left factor of h_n of length $F_n - 4$ and $c, d \in \{a, b\}$, $c \neq d$. We can see that $|u| \notin \{F_{n+1} - 1, F_{n+1}, F_{n+2} - 3\}$. So there exist a non empty left factor u' of h'_n, $x, y \in \{a, b\}$ and $u'' \in \{a, b\}^*$ such that $f_n = u'xyu''dc$ and $u = u'xyu''dch_{n-1}cdu'$. Being $u''dch_{n-1}cdu'$ a right factor of u, u a palindrome and a left factor of h_{n+2}, one has that $u''dch_{n-1}cdu'$ is a right factor of h_{n+2}. As $|u''dch_{n-1}cdu'| = F_{n+1} - 2$, one has that $u''dch_{n-1}cdu' = h_{n+1}$. By Lemma 8 and by $|u'xy| < F_n$, we have $u'xy = f_{n-1}$ and so $xy = cd$. Then $u = u'cdu''dch_{n-1}cdu'$ is not a palindrome. Contradiction. ◇

Lemma 10. *For each* $n \geq 2$,

i) $f_n g_n \notin F(f)$;

ii) if $f(i, j) = g_{n+1} h_{n+1}$ *then* $f(i - F_n, j) = f_{n+1} f_{n+1} h_n$;

iii) if $z, f_{n+1} z g_{n+1} \in F(f)$ *then* $|z| \geq F_{n-2}$;

iv) if $z, g_{n+1} z f_{n+1} \in F(f)$ *then* $|z| \geq F_{n-1}$.

Proof. i) follows from point i) of Lemma 1 and Lemma 4. ii) Let $f(i, j) = g_{n+1} h_{n+1}$. By Lemma 4, we have $f(i - F_n, i - 1) = f_n$. Hence, $f(i - F_n, j) = f_{n+1} f_{n+1} h_n$. The proofs of iii) and iv) are similar to that of Lemma 4. ◇

For each non empty finite word w, there exists a naturally associate periodic infinite word p_w, defined as follows: $p_w(0, |w| - 1) = w$ and, for each $i \geq 0$, $p_w(i + |w|) = p_w(i)$. We say that a word u is a *power* of the (finite!) word w if u is a left factor of p_w. We say that w is the *base* and $k = |u|/|w|$ is the *exponent* of this power and we write $u = w^k$. In general k is rational, but if k is an integer we obtain the usual notion of power. We consider only powers with exponent greater than or equal to 1. We say that two words u and v are *conjugate* if there exist two words u' and u'' such that $u = u'u''$ and $v = u''u'$. The following lemma is trivial.

Lemma 11. *Let* $r, s \geq 1$ *and* u,v *be non empty words of equal length. If* u^r *is a factor of* v^s *then* u *is a conjugate of* v.

4. MAXIMAL POWERS IN THE FIBONACCI INFINITE WORD

Let $u \in F(f)$, r rational and $u^r \in F(f)$. We say that the power u^r is *maximal* if for each word v such that $|u| = |v|$ and for each rational s, if u^r is a factor of v^s, then $u = v$ and $r = s$. We say that an occurrence (i_0, j_0) of u^r in f is *maximal* if for each $v \in F(f)$ such that $|u| = |v|$ and for each i_1, j_1 such that $i_1 \leq i_0 \leq j_0 \leq j_1$, if (i_1, j_1) is an occurrence of some power of v in f then $u = v$, $i_0 = i_1$ and $j_0 = j_1$. A power can have maximal occurrences even if it is not maximal. For example, a, $(ab)a$, $(aba)(aba)$, and $(abaab)(abaab)a$ are not maximal powers but the pairs (5,5), (8,10), (13,18) and (21,31) are respectively maximal occurrences of them in f.

Proposition 1. *Let* $u \in F(f)$ *and* $r \geq 1$. *If* (i_0, j_0) *is an occurrence of* u^r *in* f *then there exist a conjugate* v *of* u *and* $s \geq r$ *such that* (i_0, j_0) *is contained in a maximal occurrence of* v^s *in* f.

Proof. Let I be the set of all $i \geq 0$ such that (i, j_0) is an occurrence of some $(u')^{r'}$ such that $|u| = |u'|$. Since I contains at least i_0, it is non empty. Let i_1 be its minimum. Now, let J be the set of all $j \geq j_0$ such that (i_1, j) is an occurrence of some $(u'')^{r''}$ such that $|u| = |u''|$. Since J contains at least j_0, it is non empty. By Lemma 6, there is a maximum in J, say j_1. Clearly, (i_1, j_1) is a maximal occurrence of some v^s such that $|v| = |u|$ and $s \geq r$. By Lemma 11, u is a conjugate of v. $\diamondsuit$

Hereafter we denote by Φ the golden ratio $(\sqrt{5}+1)/2$. One of the arguments used proving the following Propositions 2–6 consists in reading some words in both directions, left–right and right–left.

Proposition 2. *Let $v \in F(f)$ and s be a rational number such that $(2+\Phi)/2 < s < 2$. Then no occurrence of v^s in f is a maximal one.*

Proof. By way of contradiction, suppose that (i_1, j_1) is a maximal occurrence of v^s. There exist a positive integer k, a proper non empty left factor v' of v and a proper non empty right factor v'' of v such that $|v'| = k$, $v = v'v''$ and $v^s = v'v''v'$. We pose $x = v''(0)$ and $x' = v''(|v''|-1)$ and we have $v'' = xv''' = v''''x'$ for some $v''', v'''' \in \{a,b\}^*$.

ia) $i_1 = 0$. Let $f(j_1+1) = y$. By maximality of v^s we have $x \neq y$, and so v' is special. By Lemma 5, $v' = s^{[k]}$ and, by definition of $s^{[k]}$, $v' = \tilde{v}'$. As $v' \in LF(f)$ and v' is a palindrome, we have, by Lemma 9, $v' = h_n$ for some $n \geq 3$.

ib) $i_1 \neq 0$. Let $f(i_1 - 1) = y'$, $f(j_1+1) = y$ and consider the words $v^s y = v'xv'''v'y$ and $y'v^s = y'v'v''''x'v'$. Remark that, by point viii) of Lemma 1, $\tilde{v}'x'\tilde{v}''''\tilde{v}'y' \in F(f)$. By maximality of v^s we have $x \neq y$ and $x' \neq y'$ and from this v' and $\tilde{v}'$ are both special factors. Then $v' = \tilde{v}' = s^{[k]}$ and so $v' \in LF(f)$ and v' is a palindrome. Hence, by Lemma 9, $v' = h_n$ for some $n \geq 3$.

Thus in *both* ia) and ib) we have $v' = h_n$ for some $n \geq 3$.

Moreover $n > 3$, otherwise we would have $v'v''v' = (av'')a = (av'')^s$ and $s = (2 + |v''|)/(1+|v''|) \leq 3/2 < (2+\Phi)/2$ which is a contradiction. So $v' = h_n$ for $n \geq 4$.

i1) $|v''| = 1$. By point vii) of Lemma 1, v' begins and ends with aba. As $aaa, abababa \notin F(f)$ we reach a contradiction in the case $v'' = a$ as well as in the case $v'' = b$.

i2) $|v''| = 2$. By maximality of v^s we have $v'xyv'yx \in F(f)$ where $x, y \in \{a,b\}$, $x \neq y$ and $v'' = xy$. By point iii) of Lemma 2 we have to consider two cases: $v'xy = f_n$ and $v'xy = g_n$.

i2a) If $v'xy = f_n$ then we have $v'xyv'yx = f_n g_n \in F(f)$ which is impossible by point i) of Lemma 10.

i2b) If $v'xy = g_n$ then we have $v'xyv' = g_n h_n$ which contradicts, by point ii) of Lemma 10, the maximality of v^s (more precisely, $g_n h_n$ is a fractional power of g_n with exponent $(2F_n - 2)/F_n < 2$ and each its occurrence is contained, as right factor, in an occurrence of $f_n f_n h_{n-1}$ which is a fractional power of f_n with exponent $(2F_n + F_{n-1} - 2)/F_n > 2$).

i3) $|v''| \geq 3$. We have $v'xyzv'yx \in F(f)$ where x, y are letters and z is a word. Again we have to consider two cases: $v'xy = f_n$ and $v'xy = g_n$.

i3a) If $v'xy = f_n$ then, by point iii) of Lemma 10, $|z| \geq F_{n-3}$.

i3b) If $v'xy = g_n$ then, by point iv) of Lemma 10, $|z| \geq F_{n-2}$.

So in *both* i3a) and i3b) we have $|z| \geq min\{F_{n-2}, F_{n-3}\} = F_{n-3}$. But then we have

$$s = (|v'| + |xyz| + |v'|)/(|v' + |xyz|) = 1 + (F_n - 2)/(F_n + |z|) <$$

$$< 1 + (F_n - 2)/(2F_{n-1}) < (2+\Phi)/2,$$

i. e., a contradiction. $\diamondsuit$

Proposition 3. *Let $v \in F(f)$. If v^2 has a maximal occurrence in f then $v = a$ or $v = aba$.*

Proof. As in the previous case we first prove that v is a palindrome and $v \in LF(f)$ and so that $v = h_n$ for some $n \geq 3$. By point vii) of Lemma 1, $abaaba$ is a right and a left factor of h_n for each $n \geq 5$. Since $abaabaabaaba \notin F(f)$, the only two possibilities are $v = h_3 = a = f_1$ or $v = h_4 = aba = f_3$. $\diamond$

Proposition 4. *Let $v \in F(f)$ and s be a rational such that $2 < s < 3$. If v^s has a maximal occurrence in f then either*

$$v = f_2 = ab \text{ and } s=5/2$$

or

$$v = f_n \text{ and } s = 2 + (F_{n-1} - 2)/F_n$$

for some $n \geq 4$.

Proof. In this case there exist a proper non empty left factor v' of v and a proper non empty right factor v'' of v such that $v = v'v''$ and $v^s = v'v''v'v''v'$. In analogy with the previous cases we first prove that $v'v''v' = h_m$ for some $m \geq 3$.

Clearly the case $m = 3$ is impossible.

If $m = 4$ then $v'v''v' = aba$, $v'v'' = ab$ and $v' = a$. Hence, $v = ab$ and $s = 5/2$.

Now, let $m \geq 5$. By Lemma 8, we have $v'v'' = f_{m-2}$ or $v'v'' = f_{m-1}$.

ia) If $v'v'' = f_{m-2}$ then, again by Lemma 8, we have $F_{m-1} - 2 = |v'| < |v'v''| = F_{m-2}$ and so $m = 4$. Contradiction.

ib) If $v'v'' = f_{m-1}$ then, again by Lemma 8, $v' = h_{m-2}$. Hence, for some $n = m - 1 \geq 4$, $v = f_n$ and $s = 2 + (F_{n-1} - 2)/F_n$. $\diamond$

Proposition 5. *Let $v \in F(f)$. If v^3 has a maximal occurrence in f then $v = aba$.*

Proof. In analogy with the proof of Proposition 3, the unique possibility is $v = h_4 = aba = f_3$. $\diamond$

Proposition 6. *Let $v \in F(f)$ and s be a rational such that $3 < s$. If v^s has a maximal occurrence in f then*

$$v = f_n \text{ and } s = 3 + (F_{n-1} - 2)/F_n$$

for some $n \geq 4$.

Proof. First, as in the proofs of Propositions 3 and 5, we have that the rational s is not an integer. So we can suppose that there exist a non empty left factor v' of v and a non empty right factor v'' of v such that $v = v'v''$ and $v^s = (v'v'')^k v'$ for some integer $k \geq 3$. In analogy with the previous cases we prove that $(v'v'')^{k-1}v' = h_m$ for some $m \geq 3$.

Clearly the cases $m = 3$ and $m = 4$ are impossible.

Let $m \geq 5$. By Lemma 8, we have $v'v'' = f_{m-1}$ or $v'v'' = f_{m-2}$.

ia) The case $v'v'' = f_{m-1}$ is impossible. In fact, again by Lemma 8, we would have $F_{m-1} = |v'v''| < |(v'v'')^{k-2}v'| = F_{m-2} - 2$, which is clearly a contradiction.

ib) If $v'v'' = f_{m-2}$ then, again by Lemma 8, we have that $(v'v'')^{k-2}v' = h_{m-1}$ and so $k = 3$ and $v' = h_{m-3}$. As $0 < |v'| = F_{m-3} - 2$ we must have $m \geq 6$. Hence, for some $n = m - 2 \geq 4$, $v = f_n$ and $s = 3 + (F_{n-1} - 2)/F_n$. $\diamond$

Remark. The word $abaabaa = (f_3)^2 a$ is not a maximal power, being always a factor of a (maximal) occurrence of $abaabaaba = (f_3)^3$ in f. The word $abaabaabaa = (f_3)^3 a$ does not belong to $F(f)$.

Remark. Consider the sequences $2 + (F_{n-1} - 2)/F_n$ and $3 + (F_{n-1} - 2)/F_n$; their elements are exponents of powers having a maximal occurrence in f and the numbers $1 + \Phi$ and $2 + \Phi$ are their respective limits for n going to infinity. By Propositions 2–6, no other value greater than $(2 + \Phi)/2$ is the limit of such a sequence. On the other hand, as one can easily see, in the interval $[1, (2 + \Phi)/2]$ infinitely many values have such a property.

Proposition 7. *Let s be a rational number greater than $(2 + \Phi)/2$ and $v \in F(f)$. If v^s has a maximal occurrence in f then there exists $n \geq 1$ such that*

$$v = f_n.$$

Proof. It follows by Propositions 2–6. ◊

Proposition 8. *Let $n \geq 1$ and s be a rational number greater than $(2 + \Phi)/2$. If $(f_n)^s$ has a maximal occurrence in f then:*

if $n = 1$ then $s = 2$;

if $n = 2$ then $s = 5/2$;

if $n = 3$ then $s = 2$ or $s = 3$;

if $n \geq 4$ then $s = 2 + (F_{n-1} - 2)/F_n$ or $s = 3 + (F_{n-1} - 2)/F_n$.

Proof. Again by Propositions 2–6. ◊

Remark. For $n \geq 4$ the two value are effectively realized.

Proposition 9. *Let r be a rational number greater than $(2 + \Phi)/2$ and $u \in F(f)$. If $u^r \in F(f)$, then there exists $n \geq 1$ such that u is a conjugate of f_n and, moreover, each occurrence of u^r is contained in a maximal one of $(f_n)^s$ for some $s \in [2, 2 + \Phi)$.*

Proof. By Proposition 1, 7 and 8 and by Lemma 11. ◊

Proposition 10. *For each $\epsilon > 0$ there exist a rational $t \in [(2 + \Phi)/2 - \epsilon, (2 + \Phi)/2)$ and a word w, such that w^t has a maximal occurrence in f and $|w| \neq F_n$ for each $n \geq 3$.*

Proof. Consider, for $n \geq 3$, the factorization

$$f_{n+4} = f_{n+1} f_n f_n f_n g_{n-1} f_{n-1} f_n.$$

Clearly, for $n \geq 3$, $(f_n f_n) f_n h_{n-1}$ has a maximal occurrence. As, for $n \geq 3$, $2F_n$ is not a Fibonacci number and $(2 + \Phi)/2$ is the limit of $(2F_n + F_{n+1} - 2)/2F_n$ for n going to infinity, the statement is proved. ◊

The following Propositions 11–13 are known results on the Fibonacci infinite word and are easy consequences of Proposition 9.

Proposition 11. (Séébold, [9]). *Let $u \in F(f)$. If $u^2 \in F(f)$ then u is a conjugate of f_n for some $n \geq 1$.*

Proof. This follows immediately from Proposition 9 and from $2 > (2 + \Phi)/2$. ◊

Proposition 12. (Karhumäki, [3]). *Let $u \in F(f)$. Then $u^4 \notin F(f)$.*

Proof. As already remarked by Séébold, Proposition 12 is a consequence of Proposition 11. ◊

Proposition 13. (Mignosi and Pirillo, [8]). *Let $u \in F(f)$ and r rational such that $u^r \in F(f)$. Then $r < 2 + \Phi$.*

Proof. By way of contradiction, suppose that for some $u \in F(f)$ and for some rational $r > 2 + \Phi$, $u^r \in F(f)$. By Proposition 9, each occurrence of u^r in f is contained in a maximal one of $(f_n)^s$ for some $s \in [2, 2 + \Phi)$ and some $n \geq 1$ such that $|u| = F_n$. So $r \leq s < 2 + \Phi$. Contradiction. ◊

Remark. The results of this paper have been announced at the International Conference on "Semigroups: Algebraic Theory and Applications to Formal Languages and Codes" (Luino, 22/27th June 1992), in a poster at the First European Congress of Mathematics (July 1992) and in a Colloquium at Oberwolfach (November 1992). A first version of this paper with the title "Maximal powers in Fibonacci infinite word" is in the Proceedings of the Luino Conference.

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