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Patterns of Negative Shifts and Signed Shifts

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Abstract. Given a function f from a linearly ordered set X to itself, we say that a permutation π is an *allowed pattern* of f if the relative order of the first n iterates of f beginning at some $x \in X$ is given by π. We give a characterization of the allowed patterns of signed shifts in terms of monotone runs of a certain transformation of π, which completes and simplifies the original characterization given by Amigó. Signed shifts, which are generalizations of the shift map where some slopes are allowed to be negative, are particularly well-suited to a combinatorial analysis. In the special case where all the slopes are negative, we give an exact formula for the number of allowed patterns. Finally, we obtain a combinatorial derivation of the topological entropy of signed shifts.

Keywords: pattern avoidance, signed shift, permutation, descent, dynamical system.

1 Introduction

Permutations realized by one-dimensional dynamical systems give insight into their short-term behavior and provide an important tool to distinguish random from deterministic time series [2]. Moreover, permutations allow us to give a combinatorial interpretation of topological entropy, an important measure of complexity of the dynamical system.

Given a linearly ordered set X, a map $f : X \to X$, and $x \in X$, consider the finite sequence $x, f(x), f(f(x)), \ldots, f^{n-1}(x)$. If these n values are different, then their relative order determines a permutation $\pi \in \mathcal{S}_n$, obtained by replacing the smallest value by a 1, the second smallest by a 2, and so on. We write $\mathrm{Pat}(x, f, n) = \pi$, and we say that π is an *allowed pattern* of f, or that π is *realized* by f, and also that x *induces* π. For example, if $f(x) = \{3x\}$, where $\{y\}$ denotes the fractional part of y (see the left of Figure 1 for a graph of this function), and $x = .12$, we obtain $(x, f(x), f^2(x), f^3(x)) = (.12, .36, .08, .24)$, and so $\mathrm{Pat}(f, x, 4) = 2413$. If there are repeated values in the first n iterations of f starting with x, then $\mathrm{Pat}(x, f, n)$ is not defined. Denote the set of allowed patterns by $\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n$ and $\mathrm{Allow}(f) = \bigcup_{n\geq 1} \mathrm{Allow}_n(f)$.

It was shown in [5] that if f is a piecewise monotone map on the unit interval, then the number of allowed patterns of length n grows at most exponentially in n, implying

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the existence of forbidden patterns, that is, permutations that are not realized by f. Additionally, the growth rate of the number of allowed patterns equals the topological entropy of f.

It is a difficult problem to characterize and enumerate the set of allowed patterns of a given function f. This problem was solved in [7] for the case when f is a positive shift, that is, $f(x) = \{Nx\}$ for some integer $N \geq 2$. Some progress when f is a symmetric tent map has been made in [9], and more recently in [4]. A characterization of allowed patterns when $f(x) = \{\beta x\}$ for a real number $\beta > 1$ was given in [8]. The case of negative β was recently studied in [6] and [10].

An important class of dynamical systems are the so-called signed shifts, which generalize positive and negative shifts, as well as the tent map. A first approach to characterizing the allowed patterns of signed shifts appears in [1], although it is cumbersome and incomplete; as discussed in [3]. The goal of this extended abstract is to provide a simple and precise characterization of the permutations realized by arbitrary signed shifts, which is given in Theorem 4. As a consequence of our characterization, we obtain an exact formula for the number of permutations realized by the negative shift in Section 6. Finally, in Section 7 we compute the topological entropy of an arbitrary signed shift using combinatorial tools. Parts of this extended abstract are based on and expand results from two recent preprints by the authors [3, 10], which also contain some of the proofs omitted here due to space constraints.

2 Signed Shifts

We consider signed shifts, a generalization of the shift map that allows negative slopes. For $k \geq 2$, we denote the signature of a signed shift by $\sigma = \sigma_0\sigma_1 \ldots \sigma_{k-1} \in \{+,-\}^k$. Let $T_\sigma^+ = \{t : \sigma_t = +\}$ and $T_\sigma^- = \{t : \sigma_t = -\}$. Define the *signed sawtooth map* $M_\sigma : [0,1] \to [0,1]$, for each $0 \leq t \leq k-1$ and $x \in [\frac{t}{k}, \frac{t+1}{k})$ (where the right endpoint of the interval is included when $t = k-1$), by letting

$$M_\sigma(x) = \begin{cases} kx - t & \text{if } t \in T_\sigma^+, \\ t + 1 - kx & \text{if } t \in T_\sigma^-. \end{cases}$$

Some examples of the corresponding graphs appear in Figure 1.

Let $\mathcal{W}_k$ be the set of infinite words on the alphabet $\{0,1,\ldots,k-1\}$. In order to interpret M_σ as a shift on words, we define a linear order $<_\sigma$ on $\mathcal{W}_k$ that depends on the signature of σ, by letting $v_1v_2v_3 \ldots <_\sigma w_1w_2w_3 \ldots$ if one of the following holds:

- $v_1 < w_1$,
- $v_1 = w_1 \in T_\sigma^+$ and $v_2v_3 \ldots <_\sigma w_2w_3 \ldots$, or

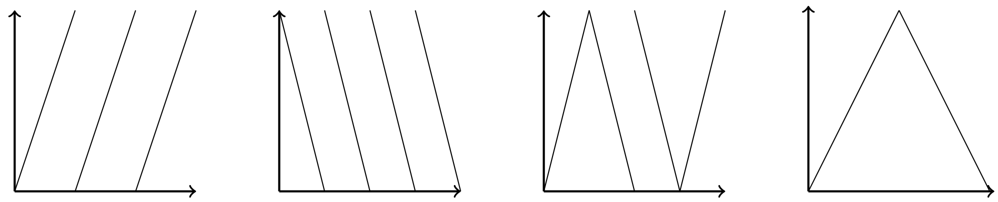

Figure 1: The graphs of M_σ for $\sigma = +^3$, $\sigma = -^4$, $\sigma = +--+$, and $\sigma = +-$, respectively.

- $v_1 = w_1 \in T_\sigma^-$ and $v_2v_3\ldots >_\sigma w_2w_3\ldots$.

We now define the *signed shift* $\Sigma_\sigma : (\mathcal{W}_k, <_\sigma) \mapsto (\mathcal{W}_k, <_\sigma)$ as the map $\Sigma_\sigma(w_1w_2w_3\ldots) = w_2w_3\ldots$, where $\mathcal{W}_k$ is ordered by $<_\sigma$.

The case when $\sigma = +^k$ (we use this notation to denote k copies of the $+$ sign) is called the *k-shift* or positive shift, and the order $<_\sigma$ is the lexicographic order. The signed shift with signature $\sigma = -^k$ is called the $-k$*-shift* or negative shift. The shift with signature $\sigma = +-$ is the well-known tent map.

Since M_σ and Σ_σ are order-isomorphic except at the points of discontinuity of M_σ, and these points do not influence the realized permutations, we have $\text{Allow}(M_\sigma) = \text{Allow}(\Sigma_\sigma)$. For our combinatorial analysis, it will be more suitable to work with the map Σ_σ.

Throughout this extended abstract, we write $w = w_1w_2\ldots$ and use the notation $w_{[i,j]} = w_iw_{i+1}\ldots w_j$ and $w_{[i,\infty)} = w_iw_{i+1}\ldots$. If d is a finite word, then d^m denotes concatenation of d with itself m times, and d^∞ denotes the corresponding infinite periodic word. We say that a finite word d is primitive if it cannot be written as a power of any proper subword, i.e. it is not of the form $d = a^m$ for any $m > 1$ and finite word a.

3 Characterization for Patterns of Signed Shifts

Our first aim is to give a characterization of the permutations realized by signed shifts. Let $\mathcal{C}_n^\star$ be the set of cyclic permutations of $[n]$ with a distinguished entry. We use the symbol $\star$ in place of the distinguished entry since its value can be recovered from the other entries. We will use both one-line notation and cycle notation while describing elements of $\mathcal{C}_n^\star$. For example, the cycle $(2,5,1,4,3) = 45231$ with the entry 2 marked is denoted by $(\star,5,1,4,3) = 45\star31 \in \mathcal{C}_5^\star$.

We use a bijection from $\mathcal{S}_n$ to $\mathcal{C}_n^\star$ introduced in [7], defined by $\pi \mapsto \hat{\pi}$ where, if $\pi = \pi_1\pi_2\ldots\pi_n$ in one-line notation, then $\hat{\pi} = (\star, \pi_2, \ldots, \pi_n)$ in cycle notation. Note that $\hat{\pi}$ satisfies $\hat{\pi}_{\pi_i} = \pi_{i+1}$ for $1 \le i \le n-1$, and $\hat{\pi}_{\pi_n} = \pi_1$, which is the entry marked with a $\star$.

For $1 \leq j \leq n-1$, we say that j is a *descent* of $\hat{\pi}$ if either $\hat{\pi}_j > \hat{\pi}_{j+1}$, or $\hat{\pi}_{j+1} = \star$ and $\hat{\pi}_j > \hat{\pi}_{j+2}$. Similarly, we say that a sequence $\hat{\pi}_i\hat{\pi}_{i+1}\dots\hat{\pi}_j$ is *decreasing* if the sequence obtained after deleting the $\star$, if applicable, is decreasing. Ascents and increasing sequences are defined in the same fashion.

Definition 1. A *σ-segmentation* of $\hat{\pi}$ is a set of indices $0 = e_0 \leq e_1 \leq \cdots \leq e_k = n$ such that

a) the sequence $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2}\dots\hat{\pi}_{e_{t+1}}$ is increasing if $\sigma_t = +$ and decreasing if $\sigma_t = -$;

b) if $\sigma_0 = +$ and $\hat{\pi}_1\hat{\pi}_2 = \star 1$ (equivalently, $\pi_{n-1}\pi_n = 21$), then $e_1 = 0$;

c) if $\sigma_{k-1} = +$ and $\hat{\pi}_{n-1}\hat{\pi}_n = n\star$ (equivalently, $\pi_{n-1}\pi_n = (n-1)n$), then $e_{k-1} = n-1$;

d) if $\sigma_0 = \sigma_{k-1} = -$ and both $\hat{\pi}_1 = n$ and $\hat{\pi}_{n-1}\hat{\pi}_n = 1\star$ (equivalently, $\pi_{n-2}\pi_{n-1}\pi_n = (n-1)1n$), then either $e_1 = 0$ or $e_{k-1} = n-1$;

e) if $\sigma_0 = \sigma_{k-1} = -$ and both $\hat{\pi}_1\hat{\pi}_2 = \star n$ and $\hat{\pi}_n = 1$ (equivalently, $\pi_{n-2}\pi_{n-1}\pi_n = 2n1$), then either $e_1 = 0$ or $e_{k-1} = n$;

f) and $e_t \neq \pi_n$ for all $1 \leq t \leq k-1$.

To each σ-segmentation of $\hat{\pi}$ we associate the finite word $\zeta = z_1z_2\dots z_{n-1}$, defined by $z_i = j$ whenever $e_j < \pi_i \leq e_{j+1}$, for $1 \leq i \leq n-1$. We say that the σ-segmentation *defines* ζ.

It is important to note that, because of condition f), each σ-segmentation of $\hat{\pi}$ defines a distinct associated word ζ.

Example 2. Consider $\sigma = ++$ and the permutation $\pi = 52413$. Then $\hat{\pi} = 34\star 12$ has a σ-segmentation given by $(e_0, e_1, e_2) = (0, 2, 5)$, which defines $\zeta = 1010$. Since $\pi_n = 3$, condition f) in Definition 1 prevents us from choosing $(e_0, e_1, e_2) = (0, 3, 5)$, which would have also defined the word $\zeta = 1010$.

Given a σ-segmentation of $\hat{\pi}$ and its associated word $\zeta = z_{[1,n-1]}$, we define the following indices and subwords of ζ. If $\pi_n \neq n$, let x be the index such that $\pi_x = \pi_n + 1$, and let $p = z_{[x,n-1]}$. Similarly, if $\pi_n \neq 1$, let y be such that $\pi_y = \pi_n - 1$, and let $q = z_{[y,n-1]}$. Moreover, for a finite word d on the alphabet $\{0, 1, \dots, k-1\}$, define $\|d\| = |\{i : \sigma_{d_i} = -\}|$; the parity of $\|d\|$ will play a role. For the k-shift, $\|d\|$ is always zero, and for the $-k$-shift, we have $\|d\| = |d|$. These two cases are considered in more detail in Section 5.

We will show that any word w inducing π has a certain form that may be described by σ-segmentations. In particular, we show in Lemma 7 that if w induces π, there is a σ-segmentation of $\hat{\pi}$ whose associated word is $\zeta = w_{[1,n-1]}$. For this reason, we will refer to ζ as a *prefix*.

Definition 3. A σ-segmentation of $\hat{\pi}$ is *invalid* if $\pi_n \notin \{1, n\}$ and the associated prefix ζ satisfies $p = q^2$ or $q = p^2$. Otherwise the segmentation is *valid*.

The rest of the section will be devoted to sketching the proof of the following theorem. This characterization is considerably simpler than the one given in [1], which also had some missing cases. Additionally, it allows us to obtain enumeration results in Section 6.

Theorem 4. *Given a permutation π, we have $\pi \in \mathrm{Allow}(\Sigma_\sigma)$ if and only if there exists a valid σ-segmentation of $\hat{\pi}$.*

The following example, with diagrams included in Figure 2, illustrates how Theorem 4 can be used to determine whether a permutation is an allowed pattern of a given signed shift.

Example 5. (a) Let $\sigma = ++$, and $\pi = 749862351$. Then $\hat{\pi} = \star 35912468$ has a σ-segmentation $(e_0, e_1, e_2) = (0, 4, 9)$ that defines the prefix $\zeta = 10111001$. Since $\pi_n = 1$, this σ-segmentation is valid. By Theorem 4, π is an allowed pattern of the 2-shift.

(b) Let $\sigma = +-$ and $\pi = 356124$. Then $\hat{\pi} = 245\star 61$ has a σ-segmentation $(e_0, e_1, e_2) = (0, 3, 6)$. This segmentation defines the prefix $\zeta = 01100$, which is valid because $p = 1100$ and $q = 01100$. By Theorem 4, π is an allowed pattern of the tent map.

(c) Let $\sigma = --$, and $\pi = 615423$. We see that $\hat{\pi} = 53\star 241$ has a unique σ-segmentation given by $(e_0, e_1, e_2) = (0, 4, 6)$, which defines the prefix $\zeta = 10100$. Since $p = 00$ and $q = 0$, this σ-segmentation of $\hat{\pi}$ is invalid. To get a glimpse of the ideas behind the proof of Theorem 4, let us see why there is no word $w = \zeta w_{[n,\infty)} \in \mathcal{W}_2$ inducing π. If w were to induce π, then $w_{[y,\infty)} <_\sigma w_{[n,\infty)} <_\sigma w_{[x,\infty)}$, that is

$$0w_{[n,\infty)} <_\sigma w_{[n,\infty)} <_\sigma 00w_{[n,\infty)} \tag{3.1}$$

which implies that $w_n = 0$. By the definition of $<_\sigma$, canceling the letter $0 \in T_\sigma^-$ implies $0w_{[n+1,\infty)} >_\sigma w_{[n+1,\infty)} >_\sigma 00w_{[n+1,\infty)}$, and so $w_{n+1} = 0$. It follows from this argument that the only possibility is $w_{[n,\infty)} = 0^\infty$, which doesn't satisfy (3.1). Since the only σ-segmentation of $\hat{\pi}$ is invalid, Theorem 4 implies that π is not an allowed pattern of the -2-shift.

The theorem follows from two main pieces. We first show in Lemmas 6 and 7 that if there is a word $w \in \mathcal{W}_k$ such that $\mathrm{Pat}(w, \Sigma_\sigma, n) = \pi$, then $\hat{\pi}$ has a valid σ-segmentation such that $\zeta = w_{[1,n-1]}$. Then, given a prefix ζ obtained from a valid σ-segmentation of $\hat{\pi}$, we define words of the form $w = \zeta w_{[n,\infty)}$ and in Lemma 9 show that they induce π. In the rest of the paper, we use k to denote the length of σ, that is, $\sigma \in \{+,-\}^k$. Using an argument similar to the one in Example 5(c), we obtain the following lemma.

Lemma 6. *If the prefix ζ defined by a σ-segmentation of $\hat{\pi}$ can be completed to a word $w = \zeta w_{[n,\infty)} \in \mathcal{W}_k$ with $\mathrm{Pat}(w, \Sigma_\sigma, n) = \pi$, then the σ-segmentation is valid.*

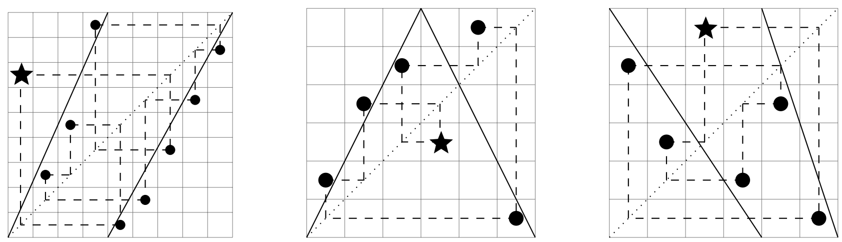

Figure 2: Plots of $\hat{\pi}$ for $\pi = 749862351$, $\pi = 356124$ and $\pi = 615423$, from left to right, as in Example 5. The line segments illustrate the σ-segmentation in each case.

Lemma 7. *If $w \in \mathcal{W}_k$ and $\mathrm{Pat}(w, \Sigma_\sigma, n) = \pi$, then there exists a valid σ-segmentation of $\hat{\pi}$ whose associated prefix is $\zeta = w_{[1,n-1]}$.*

Proof sketch. Let $w \in \mathcal{W}_k$ be such that $\mathrm{Pat}(w, \Sigma_\sigma, n) = \pi$. For $0 \le j \le k$, let $e_j = |\{1 \le r \le n : w_r < j\}|$, unless this definition makes $e_j = \pi_n$, in which case we take $e_j = \pi_n - 1$ instead. The sequence $0 = e_0 \le e_1 \le \cdots \le e_k = n$ is a valid σ-segmentation of $\hat{\pi}$ defining the prefix $\zeta = w_{[1,n-1]}$. By Lemma 6, this σ-segmentation is valid. □

In the next lemma, let ζ be the prefix defined by some σ-segmentation of $\hat{\pi}$.

Lemma 8. *Let p and q be defined as above, when applicable. Then*

(a) either p is primitive, or $p = d^2$, where d is primitive and $\|d\|$ is odd (likewise, either q is primitive or $q = d^2$, where d is primitive and $\|d\|$ is odd);

(b) if $\zeta = aqq$ for some a and $\|q\|$ is odd, then $p = q^2$ (likewise, if $\zeta = a'pp$ for some a' and $\|p\|$ is odd, then $q = p^2$).

In particular, if ζ is a prefix defined by an invalid σ-segmentation of $\hat{\pi}$, then either $p = q^2$, q is primitive and $\|q\|$ is odd; or $q = p^2$, p is primitive and $\|p\|$ is odd. It follows that, in the case when $\sigma = +^k$, all σ-segmentations are valid since $\|d\|$ is zero for any d.

We will next define a sequence of words $s^{(m)}$ and $t^{(m)}$ and show that, when $m \ge \frac{n}{2}$, they induce π. Denoting by Ω_σ and ω_σ the largest and the smallest words in $\mathcal{W}_k$ with respect to $<_\sigma$, respectively, we have

$$\Omega_\sigma = \begin{cases} (k-1)^\infty & \text{if } \sigma_{k-1} = +, \\ (k-1)0^\infty & \text{if } \sigma_{k-1} = -, \sigma_0 = +, \\ ((k-1)0)^\infty & \text{if } \sigma_{k-1} = 0, \sigma_0 = -; \end{cases} \quad \omega_\sigma = \begin{cases} 0^\infty & \text{if } \sigma_0 = +, \\ 0(k-1)^\infty & \text{if } \sigma_0 = -, \sigma_{k-1} = +, \\ (0(k-1))^\infty & \text{if } \sigma_0 = -, \sigma_{k-1} = -. \end{cases} \tag{3.2}$$

When $\pi_n \ne n$ (so that x and p are defined), let

$$s^{(m)} = \begin{cases} \zeta p^{2m} \omega_\sigma & \text{if } n \text{ is even or } \|p\| \text{ is even}, \\ \zeta p^{2m} \Omega_\sigma & \text{if } n \text{ is odd and } \|p\| \text{ is odd}. \end{cases}$$

Similarly, when $\pi_n \neq 1$ (so that y and q are defined), let

$$t^{(m)} = \begin{cases} \zeta q^{2m}\Omega_\sigma & \text{if } n \text{ is even or } \|q\| \text{ is even,} \\ \zeta q^{2m}\omega_\sigma & \text{if } n \text{ is odd and } \|q\| \text{ is odd.} \end{cases}$$

Lemma 9. *If $\pi_n \neq n$ and $m \geq \frac{n}{2}$, then $\mathrm{Pat}(s^{(m)}, \Sigma_\sigma, n) = \pi$. Likewise, if $\pi_n \neq 1$ and $m \geq \frac{n}{2}$, then $\mathrm{Pat}(t^{(m)}, \Sigma_\sigma, n) = \pi$.*

Combining the above lemmas above we obtain a proof of Theorem 4.

Corollary 10. If σ contains τ as a (not necessarily consecutive) subsequence, then

$$\mathrm{Allow}(\Sigma_\tau) \subseteq \mathrm{Allow}(\Sigma_\sigma).$$

Example 11. Let $\tau = ++$ and $\sigma = +-++$ be signed shifts. Take $\pi = 3741526 \in \mathrm{Allow}(\Sigma_\tau)$, and so $\hat{\pi} = 56712\star4$. The τ-segmentation given by $(e_0, e_1, e_2) = (0, 3, 7)$ defines $\zeta_\tau = 011010$. Removing σ_1 and σ_3 leaves τ, so we can take $(e'_0, e'_1, e'_2, e'_3, e'_4) = (0, 3, 3, 7, 7)$ as our σ-segmentation. The σ-segmentation is valid because we re-assigned the letters in the prefix in a way that respects the sign associated to each letter. This segmentation defines the prefix $\zeta_\sigma = 022020$, and we conclude that $\pi \in \mathrm{Allow}(\Sigma_\sigma)$.

4 Allowed Intervals

For a fixed signed shift, Σ_σ, this section provides a complete description of the set of words $w \in \mathcal{W}_k$ inducing π. This description is used in Theorem 13 to give an upper bound on the number of allowed patterns of Σ_σ, and later in Section 7 to calculate the topological entropy. Theorem 13 can also be used to improve the best known bounds on the number of allowed patterns of the tent map, as will be shown in an upcoming paper.

Theorem 12. *Let Σ_σ be a signed shift. Then w induces π if and only if there exists a valid σ-segmentation of $\hat{\pi}$ with associated prefix $\zeta = w_{[1,n-1]}$ such that the following conditions (depending on π_n) are satisfied:*

- *if $\pi_n \neq 1$ and $\pi_n \neq n$, then $q^\infty <_\sigma w_{[n,\infty)} <_\sigma p^\infty$;*
- *if $\pi_n = 1$, then $\omega_\sigma \leq_\sigma w_{[n,\infty)} <_\sigma p^\infty$;*
- *if $\pi_n = n$, then $q^\infty <_\sigma w_{[n,\infty)} \leq_\sigma \Omega_\sigma$.*

Given $\pi \in \mathcal{S}_n$ and a valid σ-segmentation of $\hat{\pi}$ with associated prefix ζ, we use Theorem 12 to associate an interval in $(\mathcal{W}_k, <_\sigma)$ of words of the form $w = \zeta w_{[n,\infty)}$ inducing π. For example, if $\pi_n \neq 1$ and $\pi_n \neq n$, a σ-segmentation of $\hat{\pi}$ with prefix ζ

corresponds to the open interval with endpoints ζq^∞ and ζp^∞. As we let π and the σ-segmentation vary, these intervals, which we call *allowed intervals* for Σ_σ, partition the set of words $w \in \mathcal{W}_k$ for which $\mathrm{Pat}(w, \Sigma_\sigma, n)$ is defined. The endpoints of allowed intervals are of the form ζq^∞, ζp^∞, $\zeta \omega_\sigma$ and $\zeta \Omega_\sigma$ for some $\zeta \in \{0, 1, \ldots, k-1\}^{n-1}$, and p and q are suffixes of ζ satisfying the conditions in Lemma 8. Let $I_n(\Sigma_\sigma)$ be the total number of allowed intervals. Since each allowed pattern has some valid σ-segmentation, it is clear that $|\mathrm{Allow}_n(\Sigma_\sigma)| \leq I_n(\Sigma_\sigma)$. In general, the inequality may be strict because π may correspond to multiple intervals arising from different σ-segmentations of $\hat{\pi}$.

Recall that $\psi_k(t) = \sum_{d|t} \mu(\frac{t}{d}) k^d$ is the number of primitive words of length t on k letters, where μ denotes the number-theoretical Möbius function. The number of words in $\mathcal{W}_k$ of the form $z_{[1,n-i-1]}(z_{[n-i,n-1]})^\infty$ for some i, where $z_{[n-i,n-1]}$ is primitive, is given by

$$a(n,k) := \sum_{i=1}^{n-1} k^{n-i-1} \psi_k(i). \tag{4.1}$$

Theorem 13. *For a fixed Σ_σ and n, we have $|\mathrm{Allow}_n(\Sigma_\sigma)| \leq I_n(\Sigma_\sigma)$. Additionally,*

- *if $\sigma_0 = \sigma_{k-1} = +$, then $I_n(\Sigma_\sigma) = a(n,k) + (k-2)k^{n-2}$;*
- *if $\sigma_0 \neq \sigma_{k-1}$, then $I_n(\Sigma_\sigma) = a(n,k) + (k-1)k^{n-2}$;*
- *if $\sigma_0 = \sigma_{k-1} = -$, then $I_n(\Sigma_\sigma) = a(n,k) + (k^2-2)k^{n-3}$.*

Proof sketch. To enumerate allowed intervals, we take one representative from each pair of endpoints, namely those of the form ζp^∞ and $\zeta \Omega_\sigma$, where ζ arises from a valid σ-segmentation. By Lemma 8, the endpoints of the form ζp^∞ are counted by $a(n,k)$. The second summand in each formula counts the words $\zeta \Omega_\sigma$ that have not already been counted by $a(n,k)$. This number depends on Ω_σ as given by Equation (3.2). □

5 The Negative Shift

Restricting to the positive and negative shifts, Theorem 4 allows us to derive simple formulas for the smallest positive integer k such that π is realized by the k-shift, and similarly for the $-k$-shift. In the rest of the paper, we will use the notation Σ_k and k-segmentation (respectively Σ_{-k} and $-k$-segmentation) to refer to Σ_σ and σ-segmentations where $\sigma = +^k$ (respectively $\sigma = -^k$). Let $\mathrm{des}(\hat{\pi})$ (respectively $\mathrm{asc}(\hat{\pi})$) denote the number of descents (respectively ascents) of $\hat{\pi}$ with $\star$ removed.

For the positive shift, let $N(\pi) = \min\{k : \pi \in \mathrm{Allow}(\Sigma_k)\}$. It was shown in [7] that

$$N(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where $\epsilon(\hat{\pi}) = 1$ if $\pi_{n-1}\pi_n = 21$ or $\pi_{n-1}\pi_n = (n-1)n$; and $\epsilon(\hat{\pi}) = 0$ otherwise. This formula can be deduced from Theorem 4 by noticing that each descent of $\hat{\pi}$ requires a new index in the segmentation, that an additional index is required when conditions b) or c) in Definition 1 hold, and finally using the fact that all k-segmentations are valid by Lemma 8. Notice that any permutation π has a unique $N(\pi)$-segmentation.

The analogous definition for the negative shift is

$$\overline{N}(\pi) = \min\{k : \pi \in \text{Allow}(\Sigma_{-k})\}.$$

Using Theorem 4, we try to construct a valid $-k$-segmentation for $\hat{\pi}$ with the smallest possible k. An index in the segmentation is needed for each ascent of $\hat{\pi}$, and, unless conditions d) or e) in Definition 1 apply, a $-k$-segmentation exists as long as $k \geq 1 + \text{asc}(\hat{\pi})$. In this case, we call the unique $-(1+\text{asc}(\hat{\pi}))$-segmentation the *minimal negative segmentation* of $\hat{\pi}$. However, there are cases in which we need a larger k, either because of conditions d) or e) or because the minimal negative segmentation is invalid.

Definition 14. We say that π is

- *cornered* if $\pi_{n-2}\pi_{n-1}\pi_n = 2n1$ or $\pi_{n-2}\pi_{n-1}\pi_n = (n-1)1n$ (equivalently, we invoke d) or e) in Definition 1);
- *collapsed* if the minimal negative segmentation of $\hat{\pi}$ is invalid;
- *regular* if π is neither cornered nor collapsed.

We point out that a permutation cannot be simultaneously cornered and collapsed. Indeed, a collapsed permutation requires the words p and q to be defined, which only happens if $\pi_n \notin \{1, n\}$. We obtain the following result as a corollary to Theorem 4.[2]

Theorem 15. *We have*

$$\overline{N}(\pi) = 1 + \text{asc}(\hat{\pi}) + \epsilon(\hat{\pi})$$

where $\epsilon(\hat{\pi}) = 1$ if π is cornered or collapsed; and $\epsilon(\hat{\pi}) = 0$ when π is regular. Additionally, the number of valid $-\overline{N}(\pi)$-segmentations of $\hat{\pi}$ is 1 if π is regular, 2 if π is cornered, and $\min\{|p|, |q|\}$ *if π is collapsed.*

Example 16. Let $\pi = 3651742$. Then $\hat{\pi} = 7{\star}62154$ has minimal negative segmentation $(e_0, e_1, e_2) = (0, 5, 7)$, defining the prefix $\zeta = 010010$, which yields $p = (010)^2 = q^2$. By Theorem 4, π is not realized by the -2-shift. By Theorem 15, $\overline{N}(\pi) = 3$. Indeed, we may obtain a valid -3-segmentation by placing an additional index to separate one of the three pairs of equal letters $z_i = z_{i+3}$ for $i = 1, 2, 3$. The distinct prefixes defined by -3-segmentations are $\zeta^{(1)} = 121021$, $\zeta^{(2)} = 021020$ and $\zeta^{(3)} = 010020$.

Corollary 17. The smallest forbidden patterns of the $-k$-shift have length $k+2$ and there are always exactly 4 of them.

[2] After we posted an earlier preprint including this result on `arxiv.org`, we were informed by Charlier and Steiner that they independently obtained Theorem 15 and Corollary 17 in unpublished work [6].

Proof sketch. The pattern $\pi = 12\dots(k{+}1)(k{+}2)$ corresponds to $\hat{\pi} = 23\dots(k{+}2)\star$, which has k ascents. Since π is neither cornered nor collapsed, we obtain $\overline{N}(\pi) = k+1$ by Theorem 15. By symmetry, the same holds for $\pi = (k{+}2)(k{+}1)k\dots21$.

The pattern $\pi = 12\dots k(k{+}2)(k{+}1)$ corresponds to $\hat{\pi} = 23\dots(k{+}2)\star(k{+}1)$, which has $k-1$ ascents. Since $\hat{\pi}_k > \hat{\pi}_{k+2}$, a minimal negative segmentation defines a prefix with $z_k = z_{k+1}$. However, this gives $q = z_{[k,k+1]} = (z_{k+1})^2 = p^2$, and so π is collapsed and $\overline{N}(\pi) = k+1$. By symmetry, the same holds for $\pi = (k{+}2)(k{+}1)k\dots312$.

A cornered permutation with $\overline{N}(\pi) = k+1$ would require that $\hat{\pi}$ has $k-1$ ascents, but one can see that this is not possible in either case. For example, having $\hat{\pi}_1 = k+2$ and $\hat{\pi}_{k+1}\hat{\pi}_{k+2} = 1\star$ leaves only $k-2$ remaining possible locations for an ascent. □

Compare Corollary 17 with the analogous result for the k-shift, proved in [7], stating that its smallest forbidden patterns have length $k+2$ and there are exactly 6 of them.

Example 18. The smallest forbidden patterns of the -4-shift are 123456, 654321, 123465, 654312. The smallest forbidden patterns of the 4-shift are 615243, 324156, 342516, 162534, 453621, 435261.

6 Enumeration for the Negative Shift

The exact counting of patterns of length n realized by the $-k$-shift is more complicated than in the positive case [7], since the same permutation π may correspond to multiple allowed intervals for the $-\overline{N}(\pi)$-shift, coming from different prefixes ζ, as described in Theorem 15. Among the potential distinct prefixes, we choose a canonical one by requiring it to be the smallest prefix with respect to $<_\sigma$ among those prefixes defined by valid $-\overline{N}(\pi)$-segmentations of $\hat{\pi}$. The segmentation defining the canonical prefix is called the *canonical $-\overline{N}(\pi)$-segmentation*. A $-j$-segmentation $(e_0, e_1, \dots, e_j)$ is called a *refinement* of a $-k$-segmentation $(e'_0, e'_1, \dots, e'_k)$ if $k \le j$ and $\{e'_0, e'_1, \dots, e'_k\} \subseteq \{e_0, e_1, \dots, e_j\}$ as multisets.

Lemma 19. *For $n \ge 3$ and $k \ge 2$, let $p(n,k)$ be the number of allowed intervals for the $-k$-shift that correspond to $-k$-segmentations obtained as refinements of a canonical $-\overline{N}(\pi)$-segmentation. Then*

$$p(n,k) = a(n,k) + (k^2-2)k^{n-3} - 2\sum_{j=1}^{k-1} j^{n-3} - 2\sum_{\substack{c=1\\ odd}}^{\frac{n-1}{2}} \sum_{j=1}^{k-1} \frac{c-1}{c}\binom{c+k-j-1}{k-j} j^{n-2c-1}\psi_j(c).$$

Theorem 20. *For $n \ge 3$ and $k \ge 2$, let $b(n,k)$ be the number of permutations $\pi \in \mathcal{S}_n$ with $\overline{N}(\pi) = k$, that is, $b(n,k) = |\mathrm{Allow}_n(\Sigma_{-k}) \setminus \mathrm{Allow}_n(\Sigma_{-(k-1)})|$. We have*

$$p(n,k) = \sum_{j=0}^{k-2} \binom{n+j-1}{j} b(n,k-j).$$

n \ k	2	3	4	5	6	7
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

n \ $-k$	2	3	4	5	6	7
3	6					
4	20	4				
5	54	62	4			
6	140	408	168	4		
7	336	2084	2196	412	4	
8	800	9152	19556	9804	972	4

Table 1: $|\{\pi \in \mathcal{S}_n : N(\pi) = k\}|$ (left) and $b(n,k) = |\{\pi \in \mathcal{S}_n : \overline{N}(\pi) = k\}|$ (right).

Equivalently,

$$\sum_{k=2}^{n} b(n,k)x^k = (1-x)^n \sum_{k\geq 2} p(n,k)x^k.$$

Proof sketch. Let $\pi \in \text{Allow}_n(\Sigma_{-k})$, and so $\overline{N}(\pi) = k - j$ for some $0 \leq j \leq k-2$. The locations of the first $k-j+1$ indices in a $-k$-segmentation of $\hat{\pi}$ are those of the canonical segmentation. We may choose the locations for the remaining j indices in $\binom{n+j-1}{j}$ ways. □

Theorem 20 and Lemma 19 provide a formula for $|\text{Allow}_n(\Sigma_{-k})| = \sum_{j=2}^{k} b(n,j)$. The values of $b(n,k)$ for small n and k are given in Table 1, where for comparison we have also included the analogous values for the k-shift, obtained in [7].

We remark that obtaining a formula for $|\text{Allow}_n(\Sigma_\sigma)|$ for arbitrary σ would be more complicated, because there is no obvious way to generalize Theorem 15. Even for the tent map Σ_{+-}, since we may choose e_1 to be on either side of the peak of $\hat{\pi}$, most allowed patterns have two $+-$-segmentations, defining two allowed intervals for π. However, it is possible for one or both of these segmentations to be invalid depending on the position of n with respect to π_x and π_y.

7 Topological Entropy of Signed Shifts

It is shown in [5] that the permutation topological entropy of a piecewise monotone map f on a real interval I equals the topological entropy of f, and is given by

$$\lim_{n\to\infty} \frac{\log(|\text{Allow}_n(f)|)}{n-1}. \tag{7.1}$$

The following consequence of Theorem 12 provides a combinatorial way to recover the topological entropy of the signed shift, which was computed in [11] using different tools.

Corollary 21. For any $\sigma \in \{+,-\}^k$, the topological entropy of M_σ is $\log(k)$.

Proof sketch. Recall that $|\text{Allow}_n(M_\sigma)| = |\text{Allow}_n(\Sigma_\sigma)|$. For $\pi \in \text{Allow}_n(\Sigma_\sigma)$, the number of distinct prefixes defined by a σ-segmentation of $\hat{\pi}$ is at most $\binom{n+k-2}{k-1}$. It follows that

$$\frac{I_n(\Sigma_\sigma)}{\binom{n+k-2}{k-1}} \leq |\text{Allow}_n(\Sigma_\sigma)| \leq I_n(\Sigma_\sigma).$$

Since $I_n(\Sigma_\sigma) \sim nk^{n-1}$ by Theorem 13, it suffices to take limits and use Equation (7.1). □

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