Sorting monoids on Coxeter groups

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arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
arXiv:0912.2212v1 [math.CO]
+ research in progress

Bubble (anti) sort algorithm

1234

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1243

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1423

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4123

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4132

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Underlying combinatorics: right permutohedron

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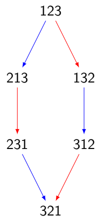

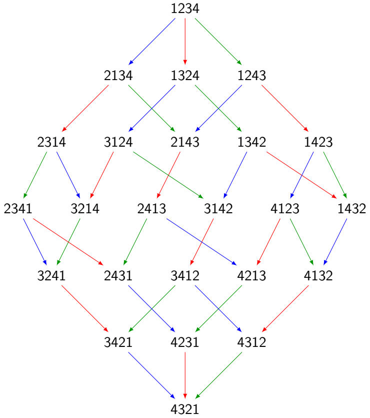

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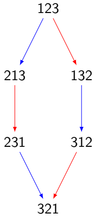
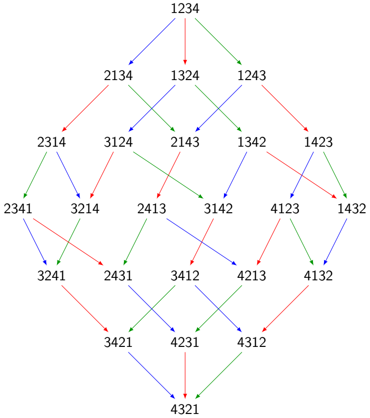
Underlying combinatorics: right permutohedron





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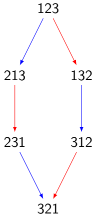

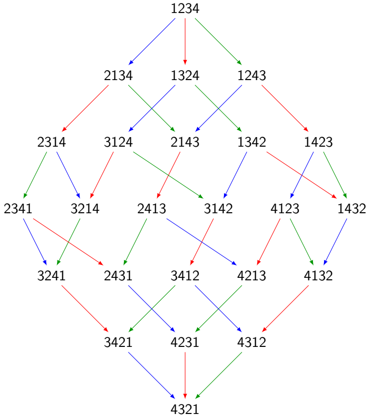

Elementary transpositions: $s_1, s_2, s_3, \ldots$

Relations: $s_i^2, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1$

Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

- Reduced word
- Length

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Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

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Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$

- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05]
- {blocks of w}: sub-lattice of the Boolean lattice

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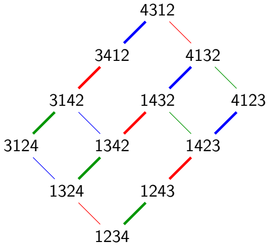

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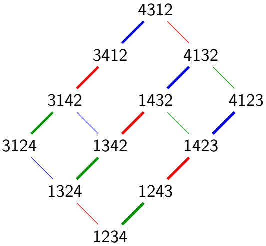
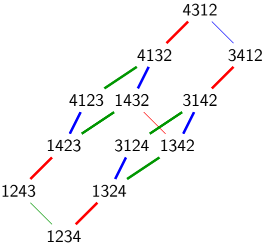
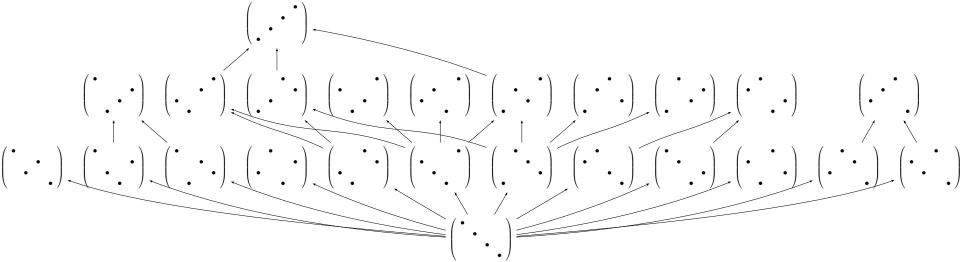




The cutting poset of a Coxeter group

Definition (HST09: Cutting poset $(W, \sqsubset)$)

$u \sqsubset w$ if $u = w^K$ with K block



- (almost) lattice
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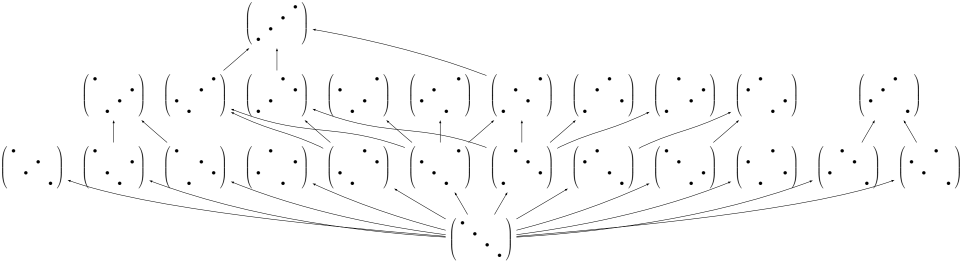

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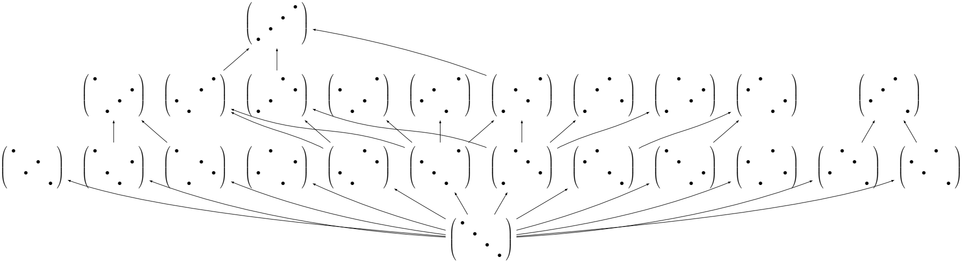

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Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \ldots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem (Norton 79, Carter 86, Krob-Thibon 97, Denton 09)

$|H_0(W)| = |W|$
$+$ *lots of nice properties*

Motivation: simple combinatorial model (bubble sort) appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

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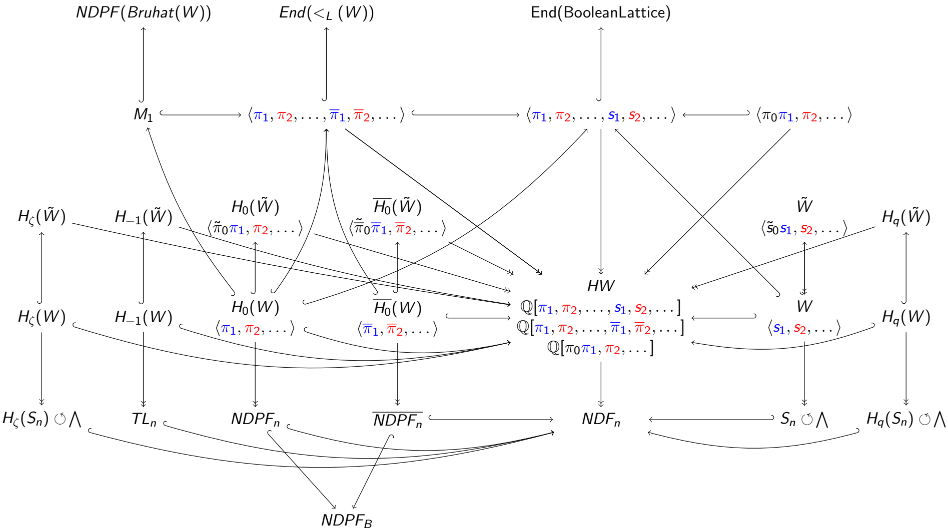
The Big Picture



The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

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Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Key role of idempotents:

- eV projective module: $V = eV \oplus (1-e)V$
- If $f = uev$ then fM is isomorphic to a submodule of eM:

$$\mathrm{fM} \xhookrightarrow{u\times} \mathrm{eM}$$
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Representation theory of finite monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes determine the simple modules.

Definition (Schüztenberger)

A monoid is aperiodic if it contains no subgroups

- A dream:
  Purely combinatorial representation theory for aperiodic monoids
- Work in progress: J-trivial monoids, R-trivial monoids

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  Purely combinatorial representation theory for aperiodic monoids
- Work in progress: J-trivial monoids, R-trivial monoids

Representation theory of finite monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes (essentially) determine the simple modules.

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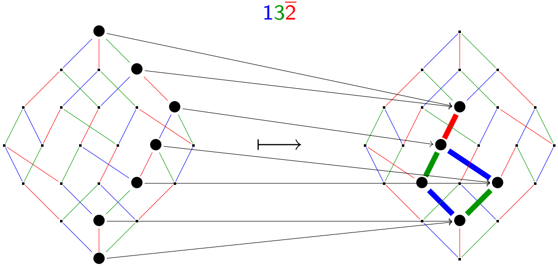

Lemma

For $f \in M(W)$ and $w \in W$: $\quad (s_i w).f = w.f \quad$ *or* $\quad s_i\,(w.f)$

Proof.

Exchange property / associativity □

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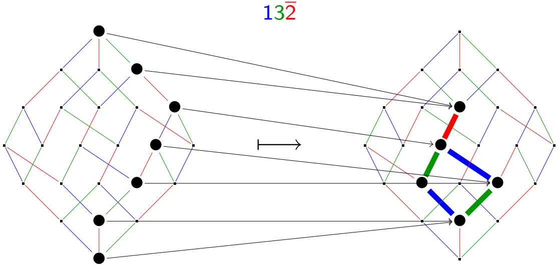

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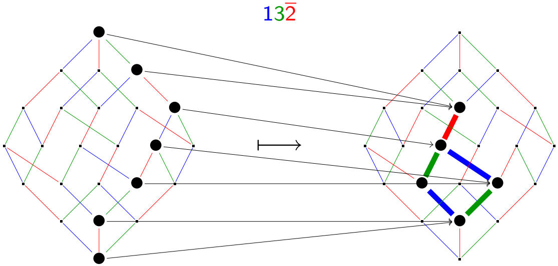

Corollary

- *If $w = uv$, then $(uv).f = u' \, (v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *Preservation of Bruhat order: $u \leq_B v \Longrightarrow u.f \leq_B v.f$*
- *f in $M(W)$ is determined by its fibers and $1.f$*

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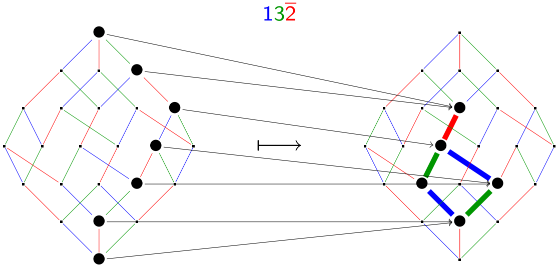

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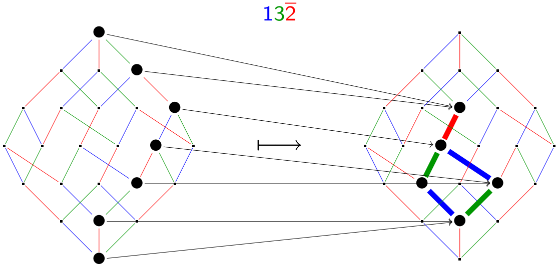

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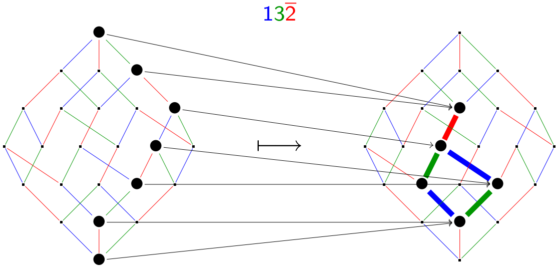

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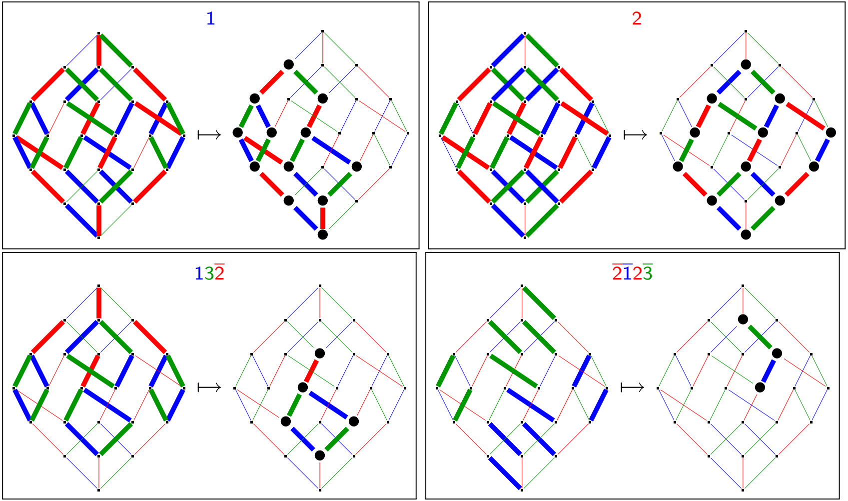
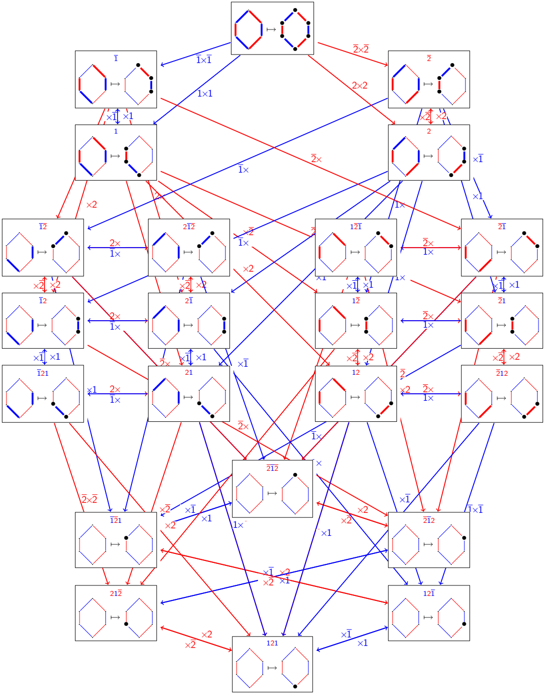
Some elements of the monoid





Representation theory of $M(W)$

Theorem (HST 08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im}(f)$ is a subinterval of $\mathrm{im}(e)$
- M is aperiodic

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Problem

Dimension of simple and projective modules?

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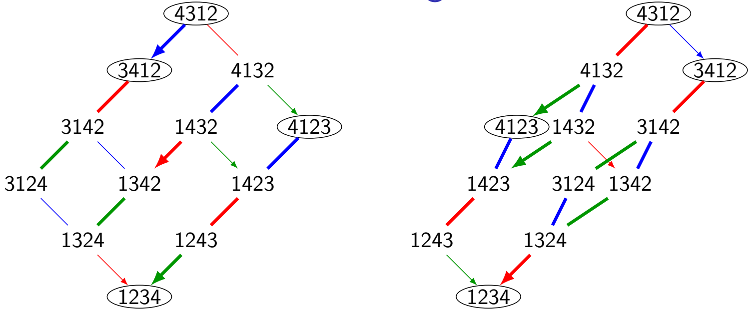
Dimension of simple and projective modules?

Translation algebras



Definition (Translation algebra)

$\mathcal{H}W^{(w)} := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\} \implies$ Submodules P_J
- **Theorem**[HST 09] $\mathcal{H}W^{(w)}$ is max. algebra stabilizing all P_J
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Proof: explicit triangular basis of $\mathcal{H}W^{(w)}$ w.r.t. Bruhat order

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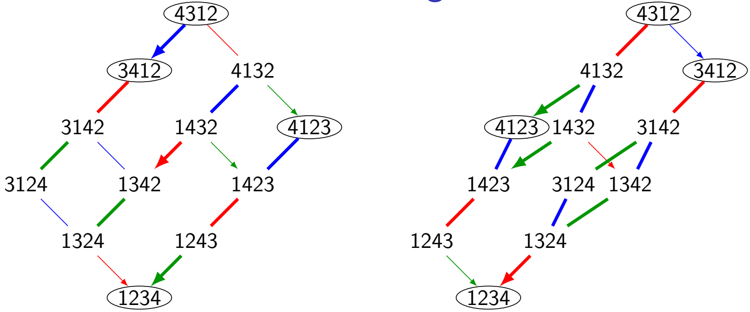

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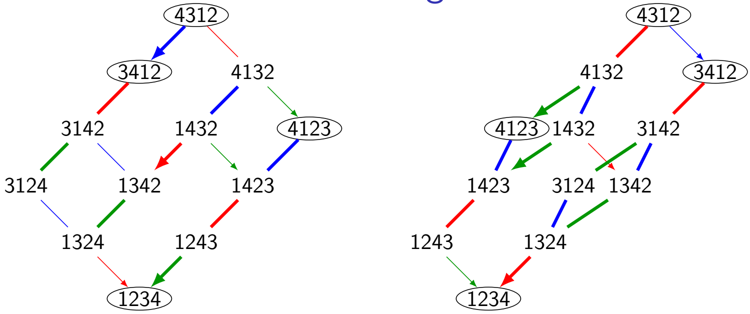

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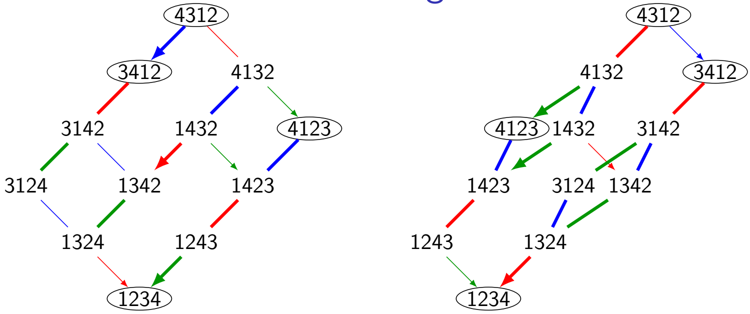

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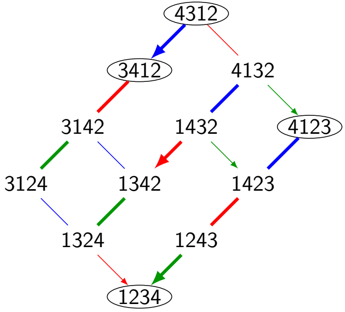

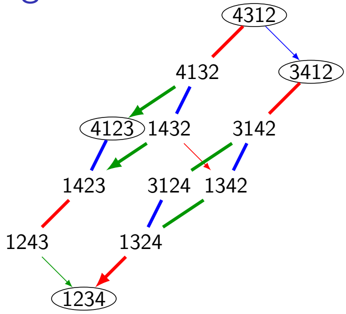

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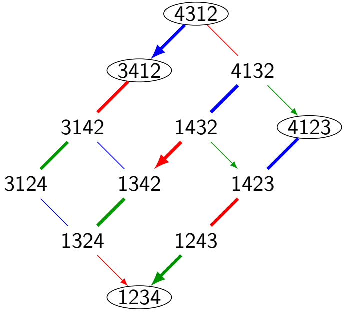
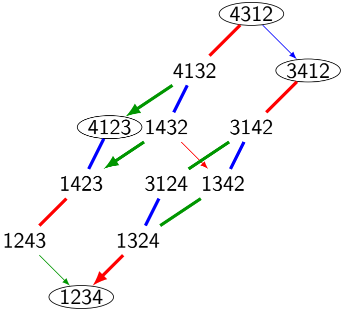


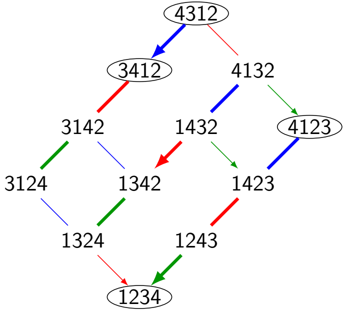


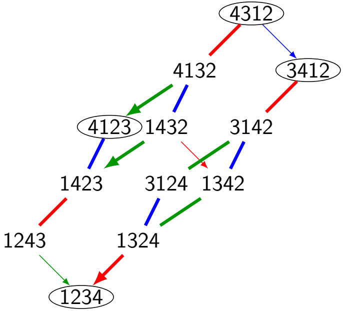
Theorem (HST 09)

$\mathcal{H}W^{(w)}$ quotient of $\mathbb{Q}[M(W)]$
Top: simple module S_w of M
Dimension: inclusion-exclusion along the cutting poset

Generating series calculation?

Translation algebras





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Work in progress

- Explicit construction of projective modules
- Study of restriction and induction functors
- Further links with the affine Hecke algebra?
- Rep. theory of J-trivial monoids
  (orthogonal idempotents, quiver, q-Cartan matrix, ...)
- Generalizations to R-trivial and aperiodic monoids
- Fast implementation in Sage (using `Semigroupe`)