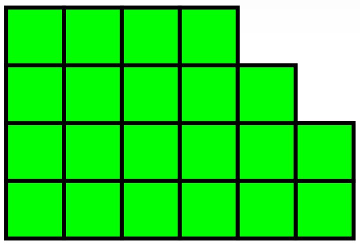
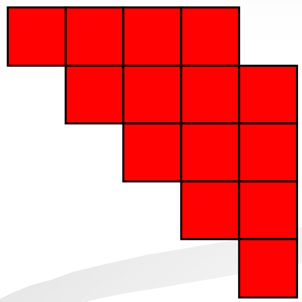
New Product Formulas for Tableaux





Ron Adin (Bar-Ilan U)

Ronald King (U Southampton)

Yuval Roichman (Bar-Ilan U)

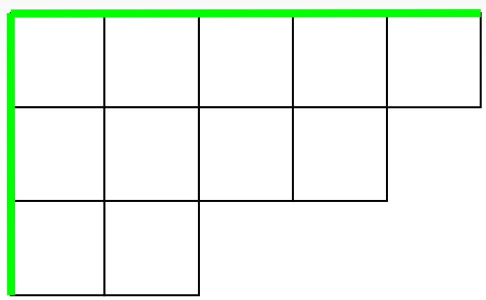
Product Formulas

Product formulas for the number of standard Young tableaux were known for two families of shapes – regular and shifted.

We present an unexpected addition to this list, consisting of certain **truncated** shapes.

Background

Regular Shapes



1	2	4	6	9
3	5	8	11	
7	10			

diagram

$\lambda = (5, 4, 2)$

$|\lambda| = 5 + 4 + 2 = 11$

standard Young tableau (SYT)

Regular Shapes

Theorem: [Frobenius-Young]

The number of SYT of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$

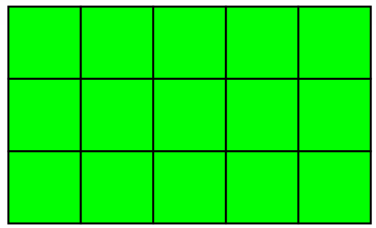
$(\lambda_1 \geq \ldots \geq \lambda_m \geq 0)$ is

$$f^{\lambda} = \frac{(|\lambda|)!}{\prod_i (\lambda_i + m - i)!} \cdot \prod_{i<j} (\lambda_i - \lambda_j - i + j)$$

There is an equivalent hook formula [FRT].

Regular Shapes

Example: For a rectangular shape $\lambda = (n^m) = (n, \ldots, n)$ (m parts),

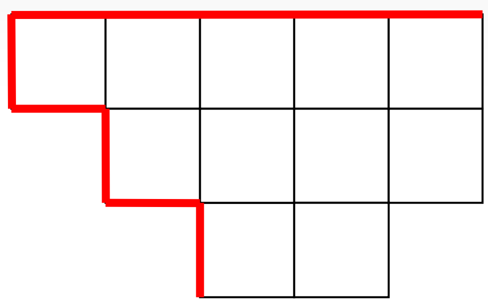


$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}}$$

where

$$F_m = \prod_{i=0}^{m-1} i!$$

Shifted Shapes



shifted diagram

$$\lambda = (5,4,2)$$

$$|\lambda| = 5+4+2 = 11$$

1	2	4	6	9
	3	5	8	11
		7	10	

standard Young tableau (SYT)

Shifted Shapes

Theorem: [Schur]

The number of SYT of shifted shape $\lambda = (\lambda_1, \ldots, \lambda_m)$ $(\lambda_1 > \ldots > \lambda_m > 0)$ is

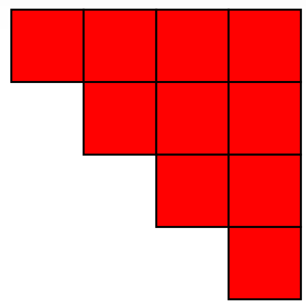
$$g^{\lambda} = \frac{(|\lambda|)!}{\prod_i \lambda_i!} \cdot \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

There is an equivalent hook formula.

Shifted Shapes

Example: For a shifted staircase shape

$\lambda = [m] := (m, m-1, \ldots, 1)$,



$$g^{[m]} = M! \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!}$$

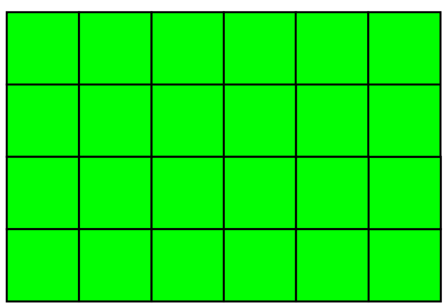
where

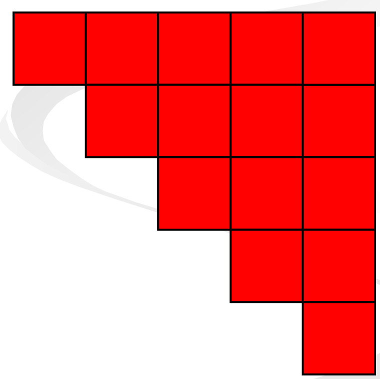
$$M = |[m]| = \binom{m+1}{2}.$$

Main Results

Truncation

Delete one or more cells from the NE (top right) corner of a regular or shifted shape.





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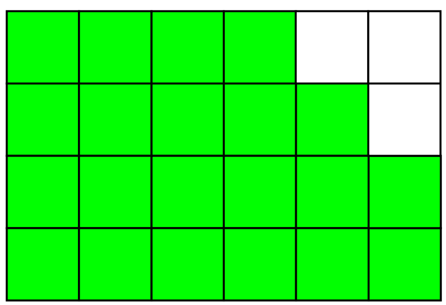

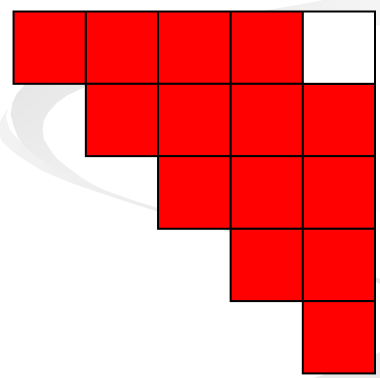

Truncated Shapes with Product Formulas

- Rectangle minus a square
- Rectangle minus a square, plus outer corner
- Shifted staircase minus a square
- Shifted staircase ninus a square, plus outer corner

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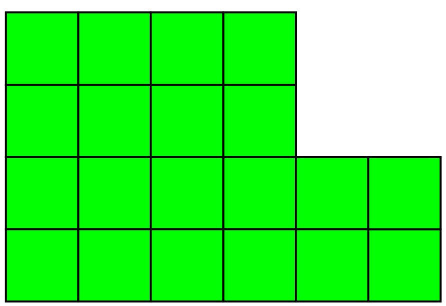

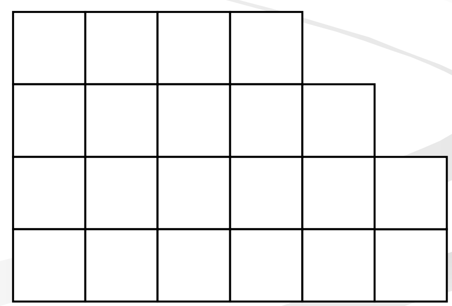

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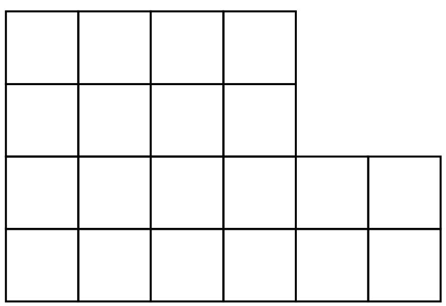

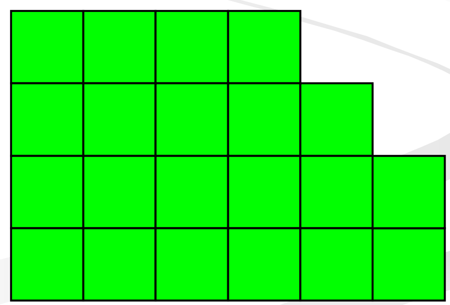

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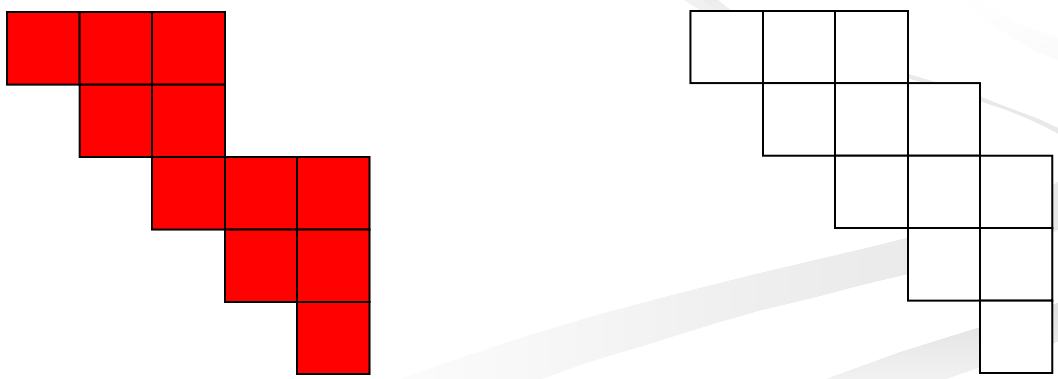

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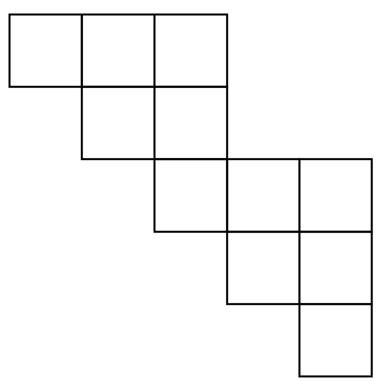
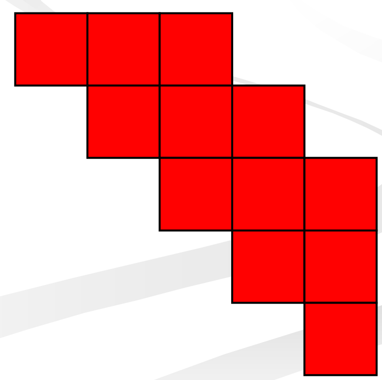
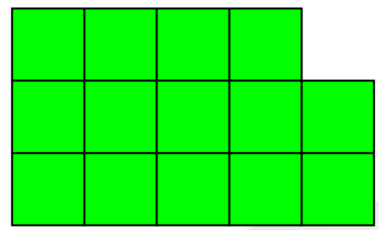




Sample Formulas

- Rectangle minus one cell:

$$\lambda = (n^m) \setminus (1)$$



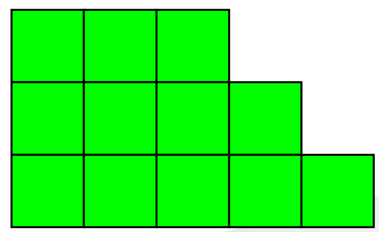
$$f^{\lambda} = N! \cdot \frac{2 \cdot (2m-3)!(2n-3)!}{(2m+2n-5)!(m+n-2)} \cdot \frac{F_{m-2} F_{n-2}}{F_{m+n-2}}$$

where $N = mn - 1$ is the size of the shape.

Sample Formulas

- Rectangle minus 2x2 square plus outer corner:

$$\lambda = (n^m) \setminus (2,1)$$



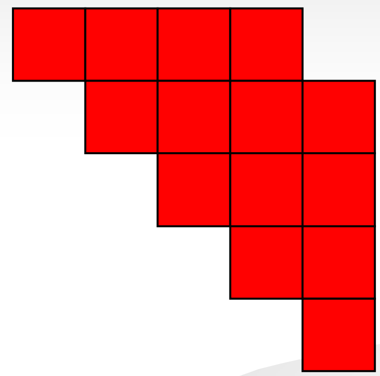
$$f^{\lambda} = N! \cdot \frac{(2m-4)!(2n-4)!}{(2m+2n-7)!} \cdot \frac{F_{m-2}F_{n-2}}{F_{m+n-2}}$$

where $N = mn - 3$ is the size of the shape.

Sample Formulas

- Shifted staircase minus one cell:

$$\lambda = [m] \setminus (1)$$

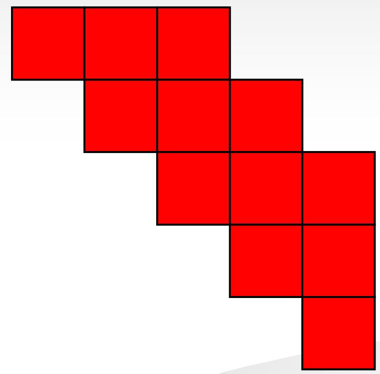


$$g^{\lambda} = N! \cdot \frac{4(2m-5)}{(4m-7)!(m-1)} \cdot \prod_{i=0}^{m-5} \frac{i!}{(2i+1)!}$$

where $N = \binom{m+1}{2} - 1$ is the size of the shape.

Sample Formulas

- Shifted staircase minus 2x2 square, plus outer corner:



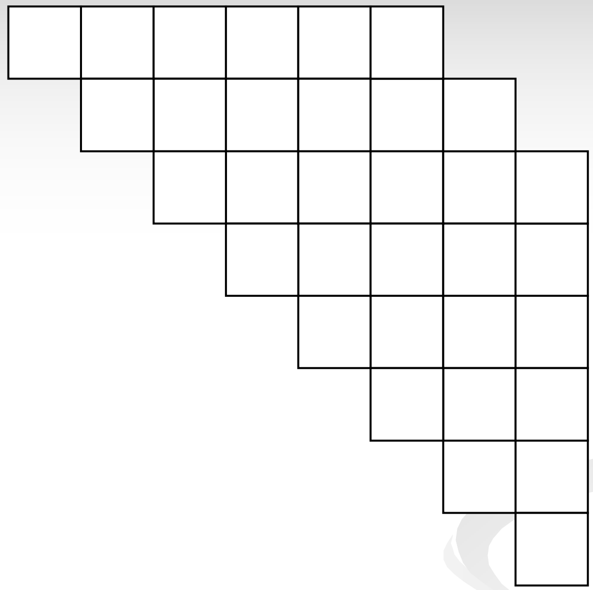
$$\lambda = [m] \setminus (2,1)$$

$$g^{\lambda} = N! \cdot \frac{2}{(4m-9)!(m-2)} \cdot \prod_{i=0}^{m-5} \frac{i!}{(2i+1)!}$$

where $N = \binom{m+1}{2} - 3$ is the size of the shape.

Ideas of Proof

Main Idea: Pivoting



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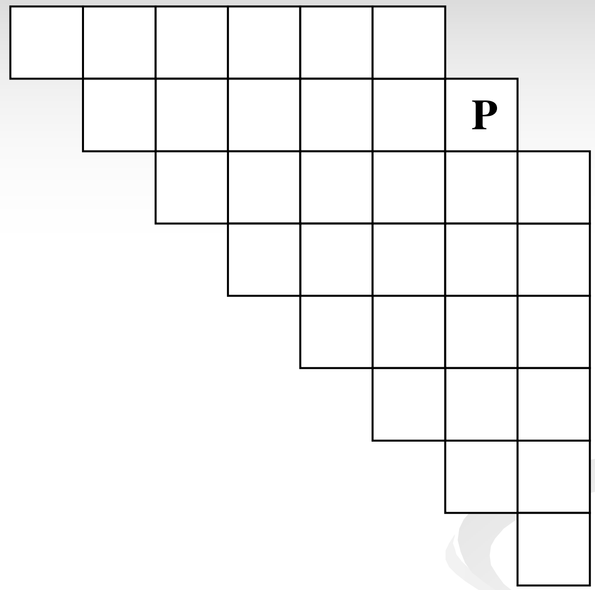
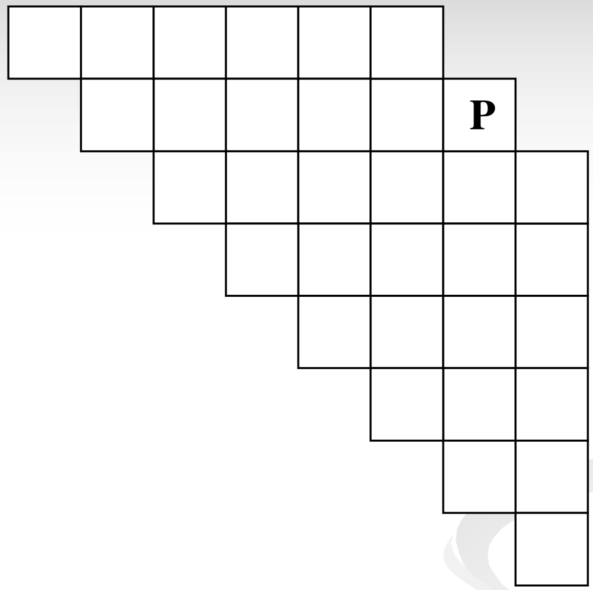


Choose a pivot cell **P** (on the NE boundary)

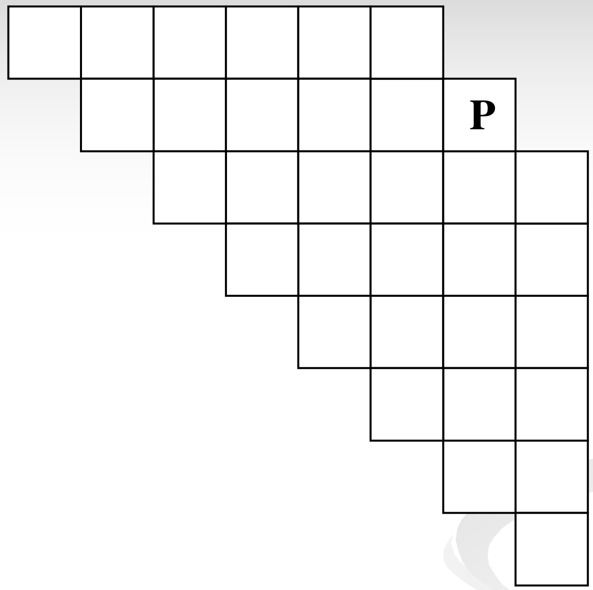
Main Idea: Pivoting



Choose a pivot cell **P** (on the NE boundary)

In an SYT, this cell contains some value k .

Main Idea: Pivoting

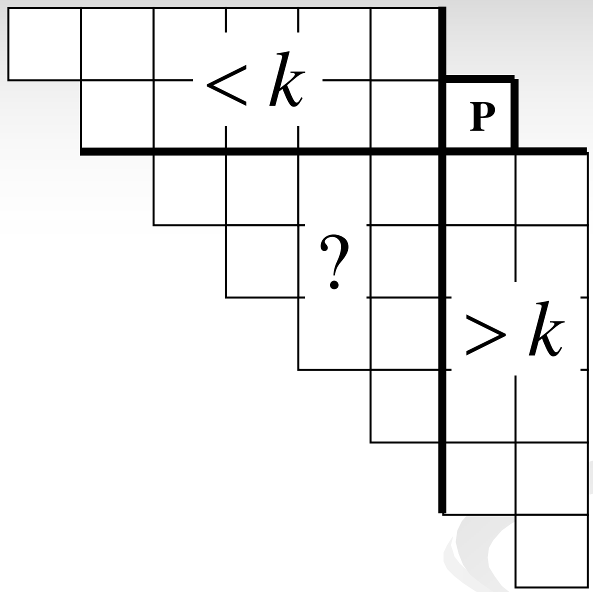


Choose a pivot cell **P** (on the NE boundary)

In an SYT, this cell contains some value k .

Where are the values $< k$? $> k$?

Main Idea: Pivoting



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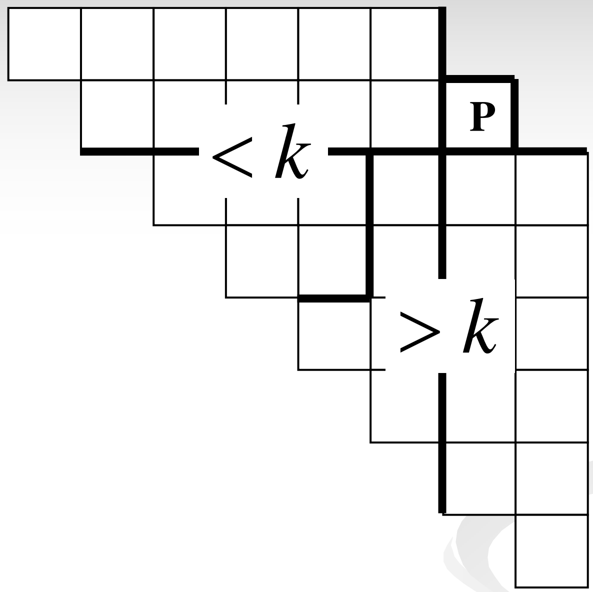

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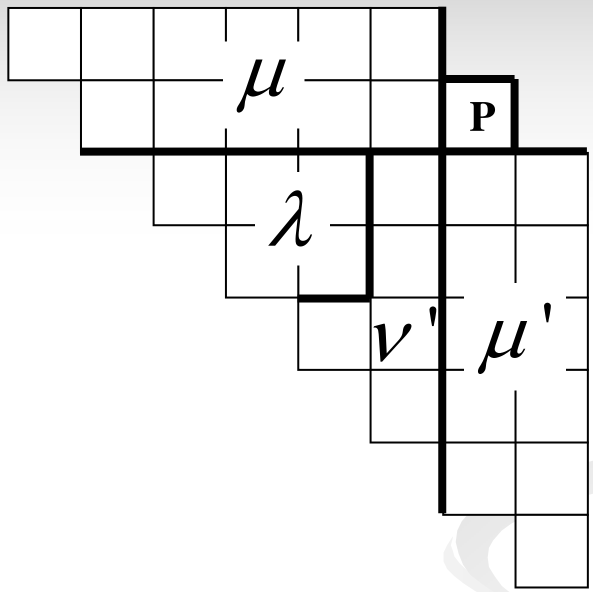
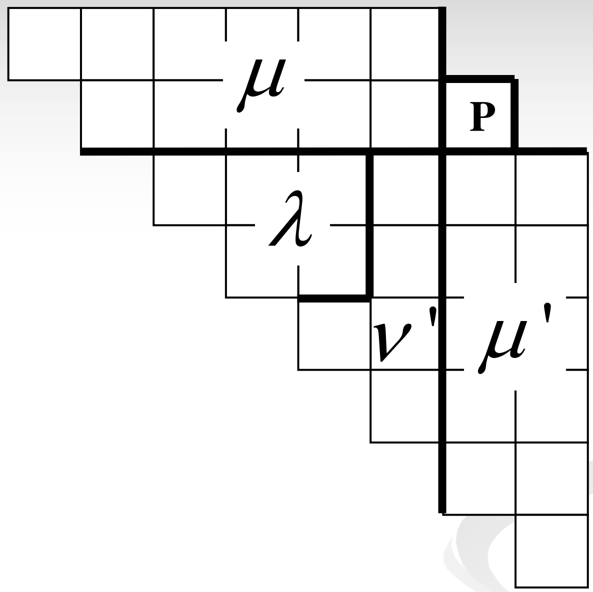


$$g^{\theta} = \sum_{\lambda \subseteq [m]} g^{\mu \cup \lambda} g^{\mu \cup \lambda^c} \qquad (\nu' = [m]/\lambda, \quad \nu = \lambda^c)$$

skew shape

Main Idea: Pivoting

θ



$\lambda = (3,2)$

$\nu = (4,1)$

$$g^{\theta} = \sum_{\lambda \subseteq [m]} g^{\mu \cup \lambda} g^{\mu \cup \lambda^c} \qquad (\nu' = [m]/\lambda, \quad \nu = \lambda^c)$$

skew shape　　set complement

Complementary Ideas

$$g^{\theta} = \sum_{\lambda \subseteq [m]} g^{\mu \cup \lambda} g^{\mu \cup \lambda^c}$$

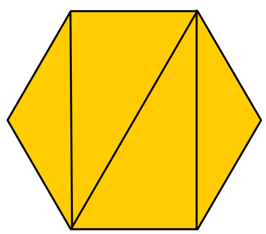
$$g^{[m]} = \sum_{\substack{\lambda \subseteq [m] \\ |\lambda| = t}} g^{\lambda} g^{\lambda^c} \quad (\forall t)$$

$$g^{\mu \cup \lambda} g^{\mu \cup \lambda^c} = c(\mu, |\lambda|, |\lambda^c|) \cdot g^{\lambda} g^{\lambda^c}$$

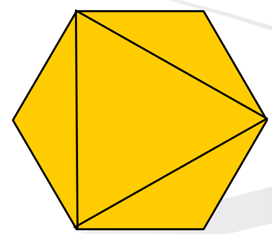
Further Comments

Motivation: Triangle-Free Triangulations

- **Definition:** A triangulation of a convex polygon is triangle-free (TFT) if it contains no "internal" triangle, i.e., a triangle whose 3 sides are diagonals of the polygon. The set of all TFT's of an n-gon is denoted $TFT(n)$.



TFT



non-TFT

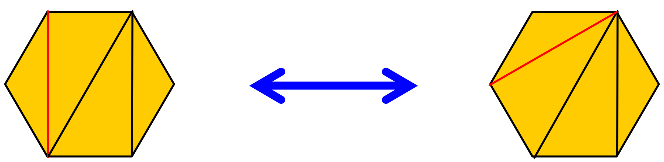
Motivation: Colored TFT

- **Note:** A triangulation is triangle-free iff the dual tree is a path.

- The triangles of a TFT can be linearly ordered (colored) in two "directions". Denote by $CTFT(n)$ the set of colored TFT's.

Motivation: Flip Graph

- Flip = replacing a diagonal by the other diagonal of the same quadrangle.



- The colored flip graph Γ_n has vertex set $CTFT(n)$ with edges corresponding to flips.

Motivation: Truncated Shifted Tableaux

- The standard Young tableaux of truncated shifted staircase shape $[4]\setminus(1)=(3,3,2,1)$:

$$
\begin{array}{cccc}
1 & 2 & 3 & \\
 & 4 & 5 & 6 \\
 & & 7 & 8 \\
 & & & 9
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 4 & \\
 & 3 & 5 & 6 \\
 & & 7 & 8 \\
 & & & 9
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 3 & \\
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 & & 6 & 8 \\
 & & & 9
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 4 & \\
 & 3 & 5 & 7 \\
 & & 6 & 8 \\
 & & & 9
\end{array}
$$

Motivation: Geodesics and Tableaux

- **Theorem:** [Adin-Roichman]

  The number of geodesics in Γ_n from a star TFT to its reverse is twice the number of standard Young tableaux of truncated shifted shape $(n-3, n-3, n-4, \ldots, 1) = [n-2] \setminus (1)$.

	3	4	5	6
1	1	2	4	
2		3	5	6
3			7	8
4				9

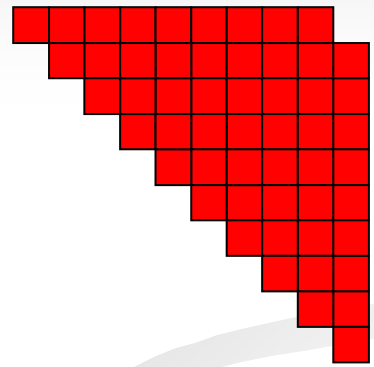
⟷ 13 14 24 15 25 … 36 46

sequence of flipped diagonals

Motivation: Numerical Evidence

- Shifted staircase minus one cell:

$$\lambda = [10] \setminus (1)$$
$$|\lambda| = 54$$



$$f^{\lambda} = 116528733315142075200$$
$$= 2^6 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$$

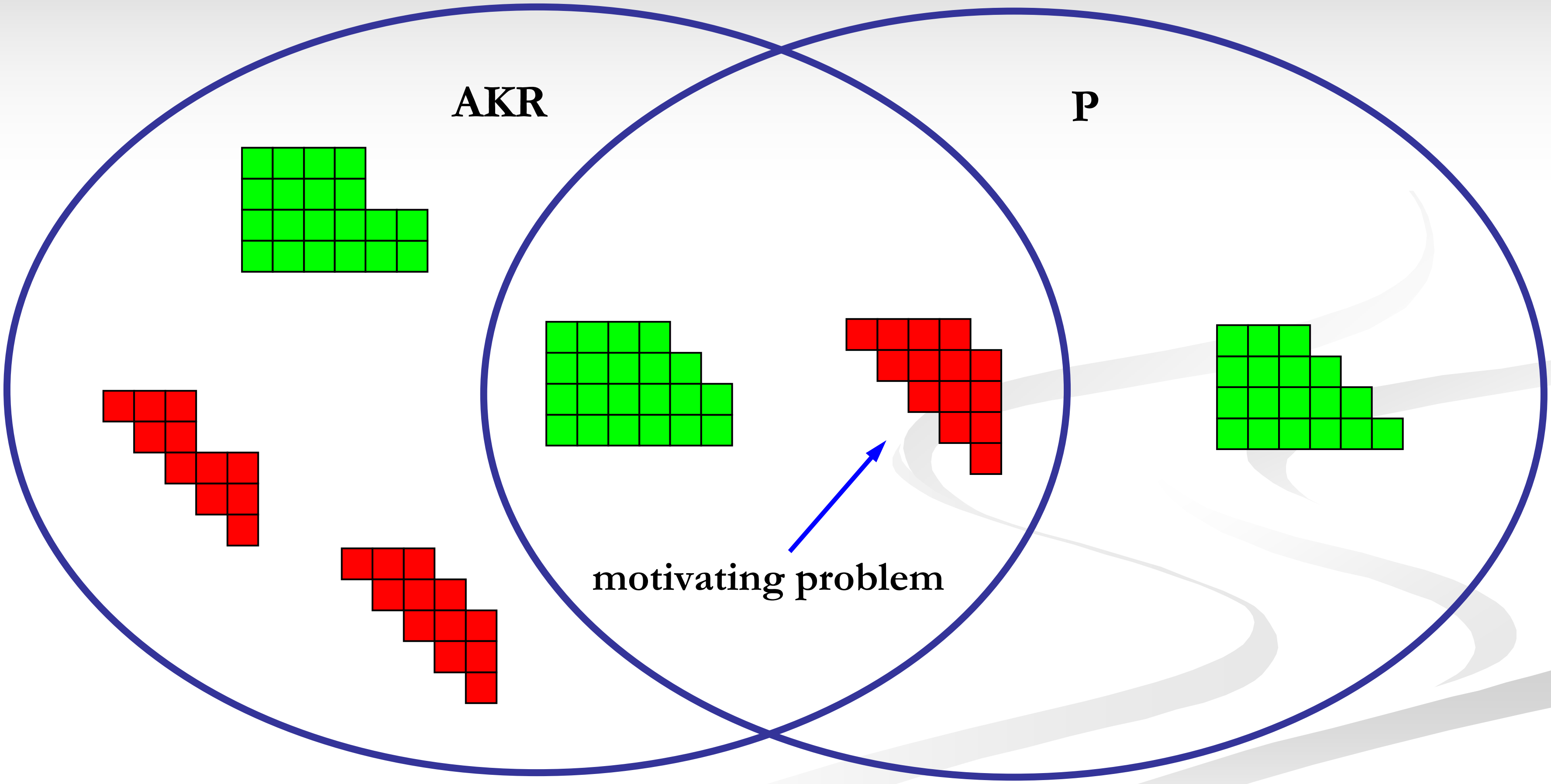
Largest prime factor is $\leq |\lambda|$!!!

Parallel Results

Greta Panova (Harvard U) has used quite different methods (including: bijections, Schur functions, polytope volume computation and contour integration) to prove product formulas in the following cases:

- Rectangle minus a staircase
- Rectangle minus a square, plus outer corner
- Shifted staircase minus one cell

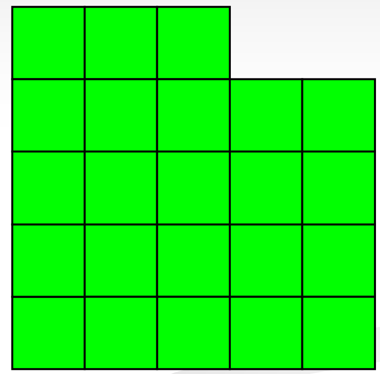
Parallel Results



Open Problems

- **Conjecture:** For a square minus two cells:

$$\lambda = (n^n) \setminus (2)$$



$$f^{\lambda} = (n^2 - 2)! \cdot \frac{6 \cdot (3n-4)!^2}{(6n-8)!(2n-2)!(n-2)!^2} \cdot \frac{F_{n-2}^{\;2}}{F_{2n-4}}$$

- Other shapes? Characterization?

Grazie
per l'attenzione!