Ribbon Schur functions with full support and Schur positivity

O. Azenhas, A. Conflitti, R. Mamede

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Outline

Schur functions and support

- The Schur functions are considered to be the most important basis for the ring of symmetric functions.
- Given partitions $\mu \subseteq \lambda$, $A := \lambda/\mu$

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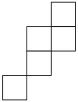
- $A = 3321/211 =$ 

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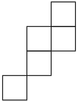
- $A = 3321/211 =$ 

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- $r(A)$ is the partition consisting of the row lengths of A, and c(A) is defined similarly. The support of A, considered as a subposet of the *dominance lattice*, has a top element $r(A)'$ and a bottom element $c(A)$,

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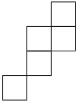
- $A = 3321/211 =$ 

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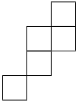
- $A = 3321/211 =$ 

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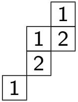
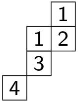
$[221; 41]$   .

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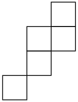
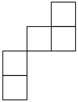
- $A = 3321/211 =$  , $B = 3311/21 =$ 

  $$s_A = s_{32} + s_{211} + 2s_{221} + s_{311}, \qquad s_B = s_{32} + s_{211} + 1s_{221} + s_{311}$$

  $\mathrm{supp} A = \mathrm{supp} B$,

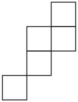
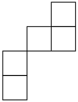
  $s_A - s_B = s_{221}$ is Schur positive but $s_B - s_A = -1s_{221}$ is not.

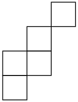
Skew shape equivalences

- Skew shapes yielding the same Schur function

  A an B are said to be *Schur equivalent* if $s_A = s_B$

  $[A] = \{B : s_A = s_B\}$

  $A =$  and $B =$  are not Schur equivalent but

  A and its antipodal rotation $A^\pi =$  are.

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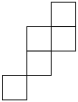
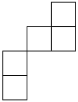
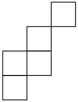
  $A =$  and $B =$  are not Schur equivalent but

  A and its antipodal rotation $A^\pi =$  are.

- Skew shapes yielding the same support

  A an B are said to be *support equivalent* if $\mathrm{supp} A = \mathrm{supp} B$

  $\lfloor A \rfloor = \{B : \mathrm{supp} B = \mathrm{supp} A\}$

  A, B and A^π are support equivalent.

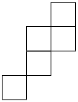
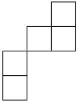
Partial orders on skew shape classes

- P_N is the poset of all Schur equivalence classes $[A]$ such that A has N boxes.

  $[A] \geq_s [B]$ if $s_A - s_B$ is Schur positive

- $Supp_N$ is the poset of all support equivalence classes $\lfloor A \rfloor$ such that A has N boxes.

  $\lfloor A \rfloor \geq_{supp} \lfloor B \rfloor$ if the support of B is contained in that of A

$A =$  and $B =$ 

$$[B] <_s [A] \quad in \quad P_5$$

$$\lfloor B \rfloor = \lfloor A \rfloor \quad in \quad Supp_5$$

Maximal supports among connected skew shapes

In *Maximal supports and Schur-positivity among connected skew shapes* arXiv:1107.4373 P. R. W. MacNamara, S. van Willigenburg classify the maximal connected skew shapes of $Supp_N$.

Theorem

An element $\lfloor R \rfloor$ in $Supp_N$ is a maximal connected element iff R is a ribbon in which the lengths of any two empty rows differ by at most one and the lengths of any two nonempty columns differ by at most one. The $\mathrm{supp} R$ is the full interval.

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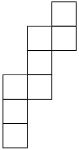
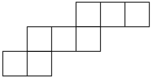
$R = (2,3,3) =$  $\lfloor R \rfloor = \lfloor R^{\pi} \rfloor$

$R' =$  $\lfloor R \rfloor' = \lfloor R' \rfloor$

Ribbon shapes with full support

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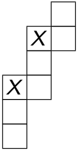
- What are the ribbon shapes $R = (r_1, \ldots, r_s)$ with $\mathrm{supp} R = [(r_{k_1}, \ldots, r_{k_s}); (\sum_{j \geq 1} r_j - s + 1, s - 1)]$?

  $s - 1$ the number of rows with length two in R

$R = (2, 3, 3) =$ 

$\mathrm{supp} R = [3^2\,2; 6\,2]$

$R = (32522271)$
$(7\,5\,3\,2\,2\,2\,2\,1) \preceq \xi = (8\,8\,8) \preceq (24-7, 7).$

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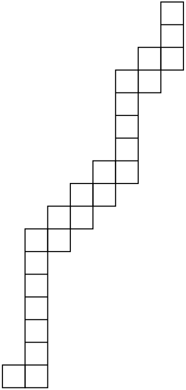

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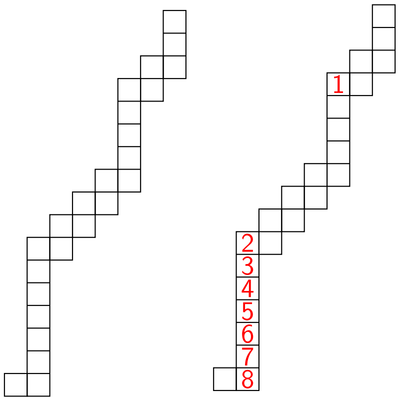

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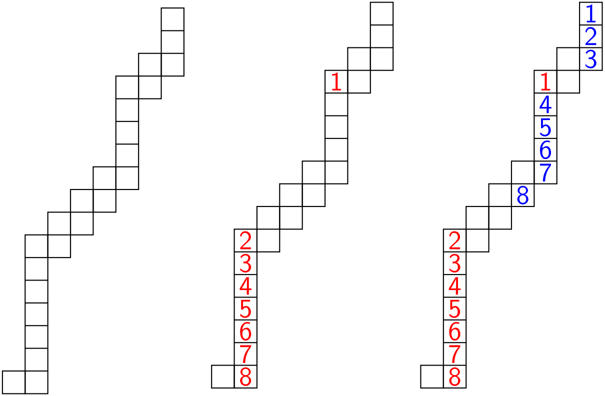

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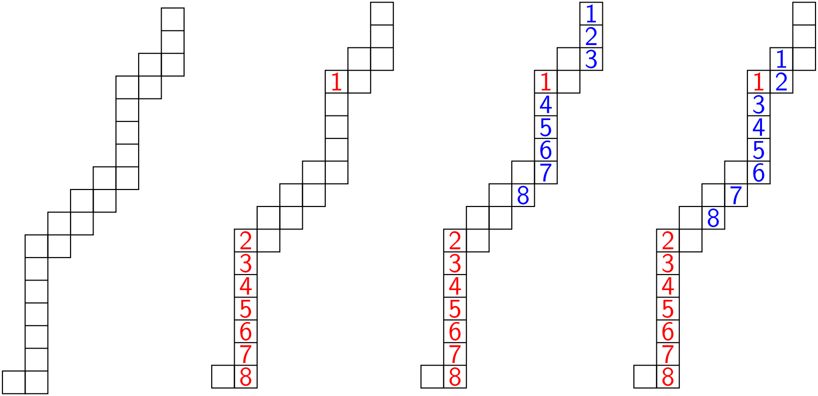

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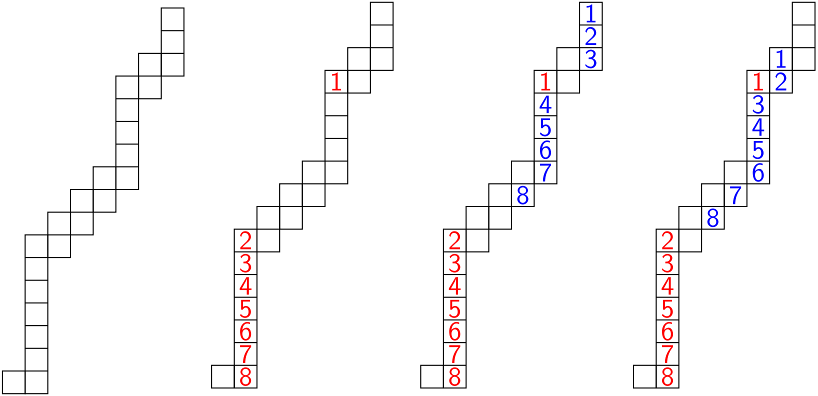
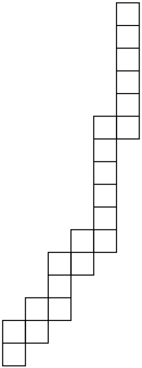


There is vertical space to put the last string of length 8.

$\sum_{i=1}^{2} \xi_i - \sum_{i=1}^{2} r_i = (8+8) - (7+5) = 4 > p = 3,$
$\xi_3 = 8 < \sum_{i \geq 3} r_i - p = 3+2+2+2+2+1-3, \quad (8\,8\,8) \in \mathrm{supp}(R)$
$\xi_3 = 8 = (\sum_{i \geq 3} r_i - p) - 1$

- $R = (662322) \quad (6^2\,3\,2^3) \preceq (7\,7\,7) \preceq (21, 21-5)$



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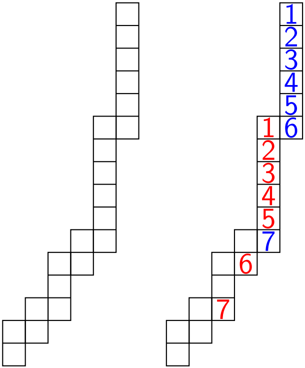

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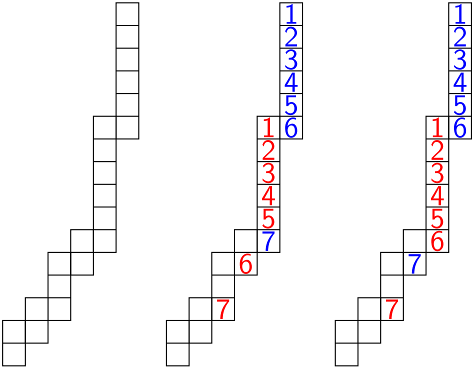

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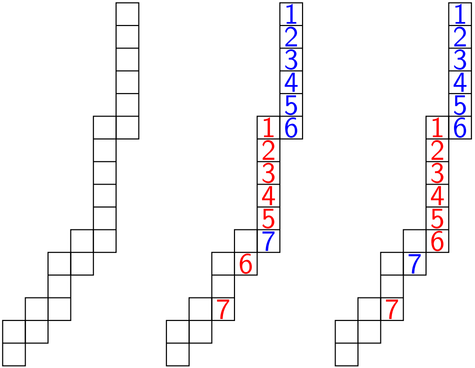

There are enough boxes to put the last string of length 7 but not enough vertical space: a row of length two remains.

$\xi_1 - r_1 + \xi_2 - r_2 = 7-6+7-6 \leq 3-1 \quad p = 3 \quad \xi_3 = 7 \geq 2+3+2+2-2$

$\xi = (777) \notin \mathrm{supp} R$

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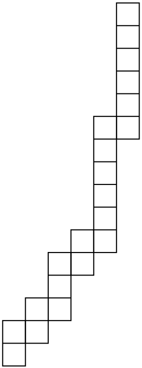

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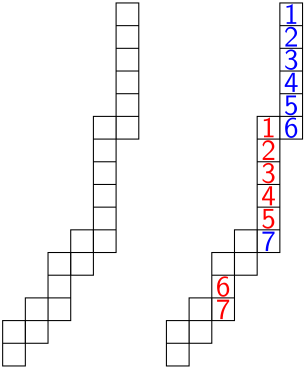

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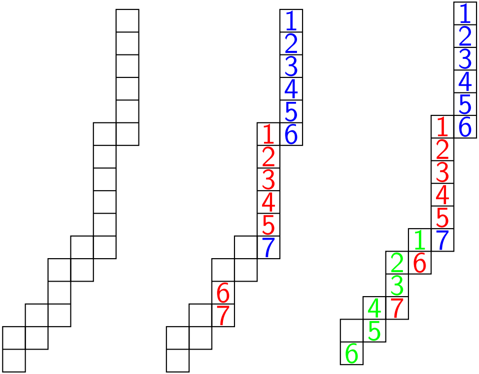

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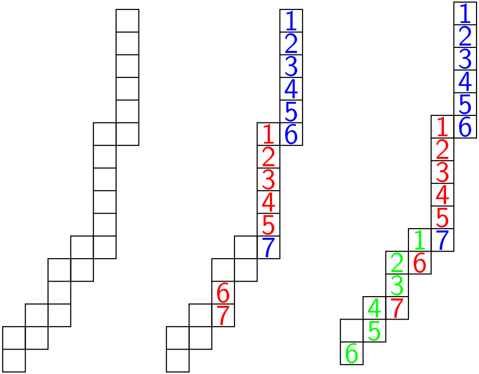

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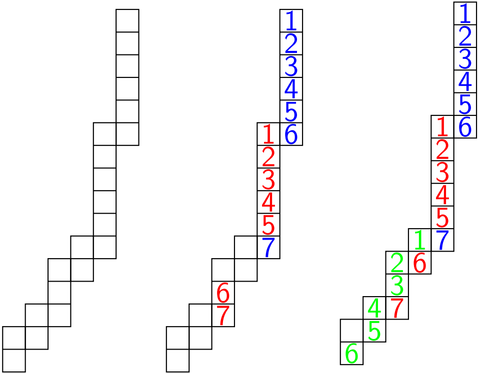

There is not enough vertical space to put a third string of length 7 but there is enough vertical space to put two more strings: one of length 6 and another of length 1.

$\xi_1 - r_1 + \xi_2 - r_2 = 7-6+7-6 = 2 \quad p = 3 \quad \xi_3 = 6 = 2+3+2+2-3$

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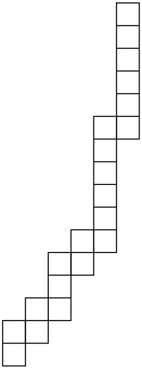

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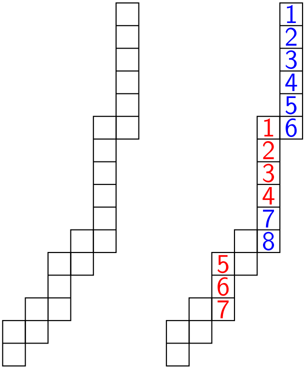

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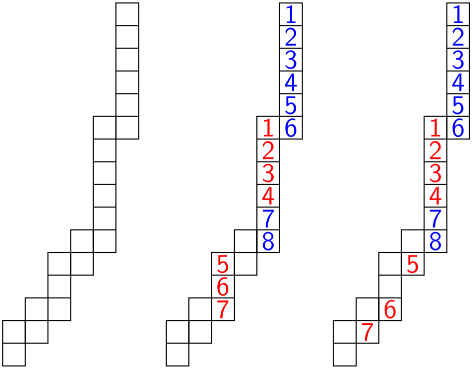

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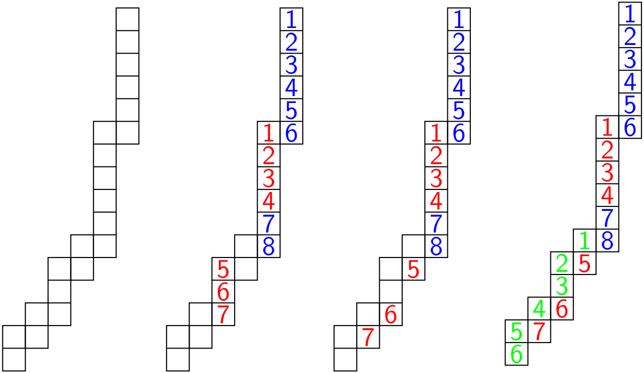

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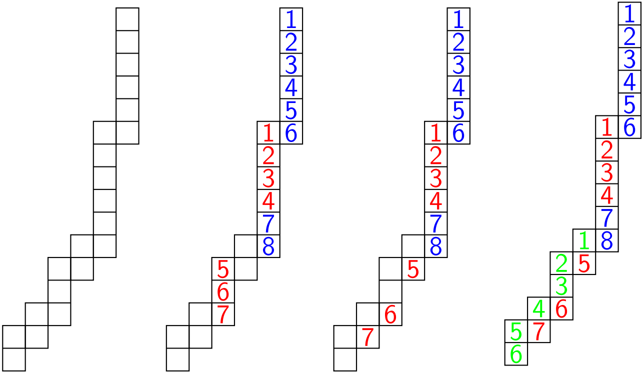

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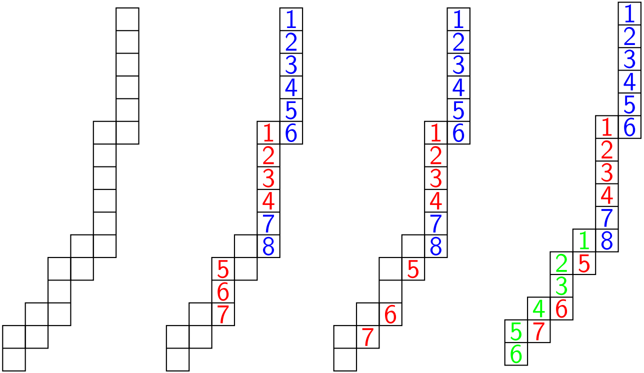

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Ribbon shape LR fillings

Lemma

Let $\xi = (\xi_1, \ldots, \xi_t)$ be a partition in the Schur interval $[(r_{k_1}, \ldots, r_{k_s}); (\sum_{j\geq 1} r_j - s + 1, s - 1)]$ but not in the support of R. Then there exists an $1 \leq i \leq t-1$ such that if $p \geq 1$ is the number of rows with length two among the columns indexed by $S = \{k_{i+1}, \ldots, k_s\}$, one has

$$\xi_{i+1} \geq \sum_{q\in S} r_q - p + 1 \quad \left(\Rightarrow \sum_{j=1}^{i} (\xi_j - r_{k_j}) \leq p - 1 \right). \tag{1}$$

This implies that the number p of rows of length two, among the adjacent columns indexed by S in R, can not be shortened by what remains $\sum_{j=1}^{i} (\xi_j - r_{k_j})$.

Theorem

Let $R = (r_1, \ldots, r_s)$, $s \geq 2$, be a ribbon. Then $\mathrm{supp} R \subsetneq [(r_{k_1}, \ldots, r_{k_s}); (\sum_{j\geq 1} r_j - s + 1, s - 1)]$ if and only if for some $1 \leq i \leq s-2$ with $p > 0$ rows of length two among the columns indexed by $\{k_{i+1}, \ldots, k_s\}$, there exist $g_1, \ldots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j = p - 1$, such that

$$r_{k_1} + g_1 \geq \sum_{j=i+1}^{s} r_{k_j} - p + 1$$

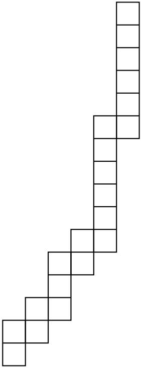
$$\vdots$$

$$r_{k_{i-1}} + g_{i-1} \geq \sum_{j=i+1}^{s} r_{k_j} - p + 1$$

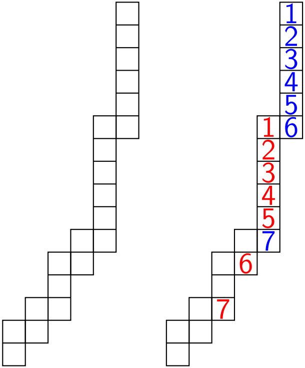
$$r_{k_i} + g_i \geq \sum_{j=i+1}^{s} r_{k_j} - p + 1$$

Moreover $(r_{k_1} + g_1, \ldots, r_{k_i} + g_i, \sum_{j=i+1}^{s} r_{k_j} - p + 1)_{\geq} \notin \mathrm{supp} R$.

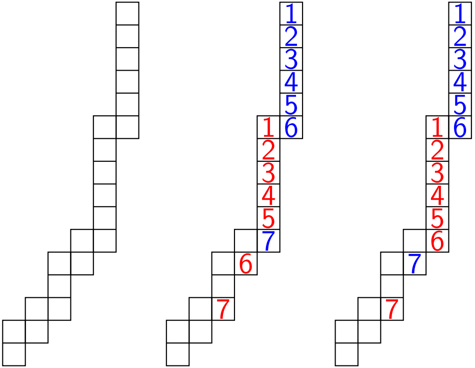
- $R = (662322) \quad (6^2\,3\,2^3) \preceq (7\,7\,7) \preceq (21, 21-5)$



- $R = (662322) \quad (6^2\,3\,2^3) \preceq (7\,7\,7) \preceq (21, 21-5)$



- $R = (662322) \quad (6^2\,3\,2^3) \preceq (7\,7\,7) \preceq (21, 21-5)$



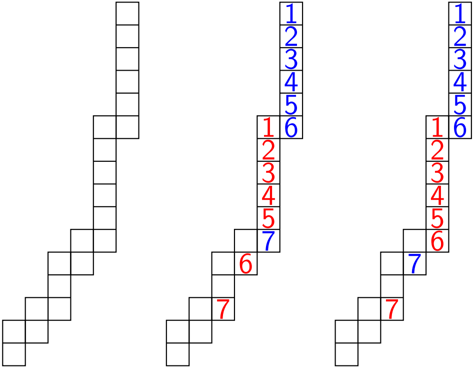
$\xi = (777) \notin \mathrm{supp} R$

$\xi_1 - r_1 + \xi_2 - r_2 = 7-6+7-6 \leq 3-1 \quad \xi_3 = 7 \geq 2+3+2+2-2$

$$r_1 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2$$

$$r_2 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2$$

- $R = (662322) \quad (6^2\,3\,2^3) \preceq (7\,7\,7) \preceq (21, 21-5)$



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$$r_1 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2$$

$$r_2 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2$$

$(876) \in \mathrm{supp} R$

Examples

- Ribbons whose column and row lengths differ in one unity have full support $[(t^m, (t-1)^n); (mt + n(t-1) - m - n + 1, m + n - 1)]$.
- The support of a ribbon $R = (r_1, r_2, r_3)$ has full interval except when $r = (r_1, r_2, r_3)$ or $r = (r_2, r_3, r_1)$ with $r_1 \geq r_2 + r_3$.
- $R = (6222276), \quad (7662222)$.

$$\begin{array}{l} 6+2 \geq 2+2+2+2-2 \\ 7,6 \geq 2+2+2+2-2 \end{array} .$$

Then $\xi = (6+2, 7, 6, 2+2+2+2-2) \notin supp(R)$.

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