On the top coefficients of Kazhdan-Lusztig polynomials

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Plan of the work

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- Notations and preliminaries
    - The symmetric group and Bruhat order
    - Kazhdan-Lusztig polynomials
    - Special matchings

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  - Motivations
  - Main Conjecture
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Given an element $v \in S_n$ we write v in disjoint cycle form or in the line notation.

Example:

- $v = (1, 2)(3, 4)$ is the disjoint cycle form.
- $v = 2143$ is the line notation, meaning that

$$v(1) = 2, v(2) = 1, v(3) = 4, v(4) = 3$$

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Observation: S_n with the set of generators

$$S := \{(i, i+1) : i \in [n-1]\}$$

is a Coxeter group.

With the set of generators, we can define the length function. For a generic Coxeter group W and an element $v \in W$ the length of v is the minimum numbers of generators necessary to express v.

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In the symmetric group the length function is:

$$l(v) := |\{(i,j) \in [n]^2 : i < j, v(i) > v(j)\}|$$

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Observation: Given $u, v \in S_n$ for brevity we denote:

$$l(u,v) := l(v) - l(u)$$

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- given $v \in S_n$ the right descent set

$$D_R(v) := \{i \in [n] : v(i) > v(i+1)\}$$

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Definition

Given $u, v \in S_n$ we say that $u \leq v$ if $\exists t_1, \ldots, t_r \in T$ $(r \geq 0)$ such that:

$$ut_1 \cdots t_r = v$$

and

$$l(u) < l(ut_1) < \ldots < l(ut_1 \cdots t_r) = l(v)$$

this order is called Bruhat order.

Given $u, v \in S_n$ we say that u is covered by v, and denote this by $u \triangleleft v$, if $u \leq v$ and $l(u, v) = 1$.

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Given an interval $[u, v]$ its Hasse diagram is the graph $G = (V, E)$ where:

- $V := [u, v]$
- $E := \{\{x, y\} \in V^2 : x \lhd y\}$

Example In the figure we show the Hasse diagram of S_4

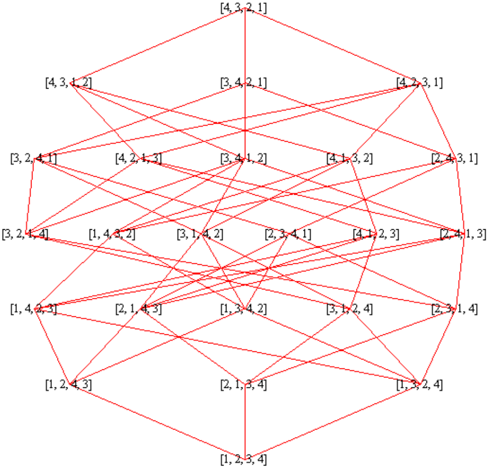

Figure: $[1234, 4321]$

In my work I consider the coatom and atom sets. Given an interval $[u, v]$ we define:

$$c(u, v) := \{z \in [u, v] : z \lhd v\}$$

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$$c(u, v) := \{z \in [u, v] : z \lhd v\}$$

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Finally I use also the following rank generating function:

$$r_{u,v}(q) := \sum_{i=0}^{l(u,v)} r_i q^i$$

where $r_i := |\{z \in [u, v] : l(u, z) = i\}|$

Kazhdan-Lusztig polynomials

In their fundamental paper **[Representations of Coxeter groups and Hecke algebras]**, Kazhdan and Lusztig defined, for every Coxeter group W a family of polynomials indexed by a pair of elements of W.

These polynomials are intimately related to the Bruhat order of W and depend on the descendent set of an elements.

There are several way to introduce these polynomials, here we use the best for our purpose. So by Definition-Theorem we define first the R-polynomials and then we use these we define the Kazhdan-Lusztig polynomials.

Theorem (Kazhdan-Lusztig)

There is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *such that:*

- $R_{u,v}(q) = 0$ *if* $u \not\leq v$.
- $R_{u,v}(q) = 1$ *if* $u = v$.
- *If* $s \in D_R(v)$ *then:*

$$R_{u,v}(q) = \begin{cases} R_{us,vs} & \text{if } s \in D_R(u) \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } s \notin D_R(u) \end{cases}$$

Theorem (Kazhdan-Lusztig)

There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *(that we call Kazhdan-Lusztig polynomial) such that:*

- $P_{u,v}(q) = 0$ *if* $u \leq v$.
- $P_{u,v}(q) = 1$ *if* $u = v$.
- $deg(P_{u,v}(q)) \leq \frac{l(u,v)-1}{2}$ *if* $u < v$
- *Se* $u \leq v$ *then:*

$$q^{l(v)-l(u)} P_{u,v}(\frac{1}{q}) = \sum_{a \in [u,v]} R_{u,a}(q) P_{a,v}(q)$$

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Definition (Top coefficient function)

Given W a Coxeter group, $u, v \in W$ with $u \leq v$:

$$\overline{\mu}(u,v) := \begin{cases} [q^{\frac{l(v)-l(u)-1}{2}}]P_{u,v} & \text{if } l(u,v) \equiv 1 mod 2 \\ 0 & \text{otherwise} \end{cases}$$

where with $[q^i]P_{u,v}$ we denote the coefficient of q^i in $P_{u,v}$.

Special matchings

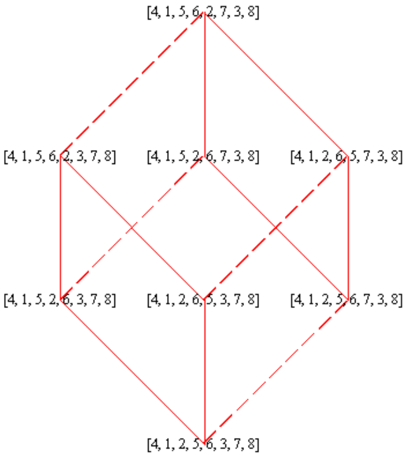
Given P a poset and $G := (V, E)$ its Hasse diagram, then we say that the function

$$M : V \to V$$

is a special matching if:

- M is an involution such that $\{v, M(v)\} \in E$ for all $v \in V$.
- $x \lhd y \Rightarrow M(x) \leq M(y)$ for all $x, y \in V$ such that $M(x) \neq y$

Example: the dot line in the following Figure is a special matching of $[41256378, 41562738]$.



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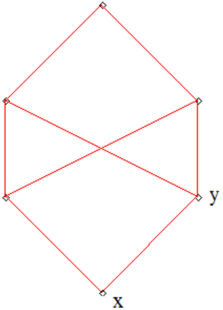
<u>**Observation:**</u> if $x \lhd y$



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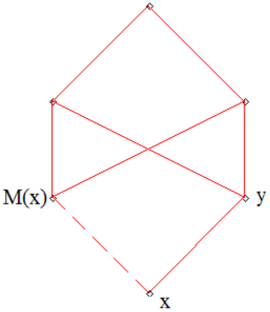
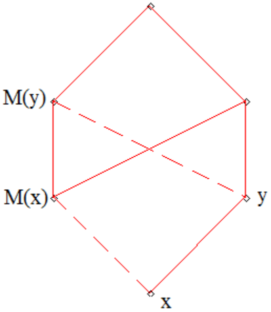
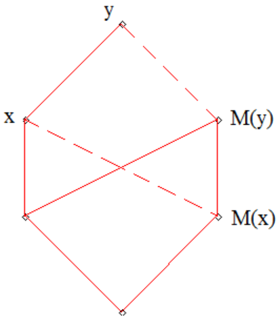


the last special matching condition ($x \lhd y \Rightarrow M(x) \leq M(y)$) imply in particular that:

Observation: if $x \lhd y$ and $M(x) \rhd x$ then $M(y) \rhd y$ and $M(y) \rhd M(x)$.



Observation: Dually if $x \lhd y$ and $M(y) \lhd y$ imply $M(x) \lhd x$ and $M(x) \lhd M(y)$.



There is a Proposition very important in my work

Proposition (Coatom's condition)

Given $u, v \in S_n$ with $u \leq v$ then:

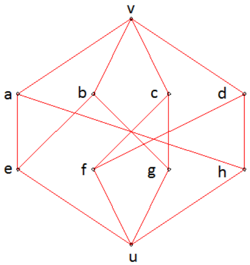
$$|c(u,v)| - 1 > |c(u,v')| \ \forall v' \lhd v$$

$$\Downarrow$$

$[u, v]$ *doesn't have a special matching*

We show this Proposition by an example

Example:



In this example the previous proposition is true.

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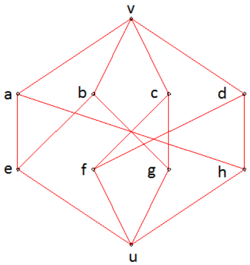

In this example the previous proposition is true.
If we choose $M(v) = a$ then we must have that:

$$M(b) \lhd a, M(c) \lhd a, M(d) \lhd a$$

Note that is also true

Proposition (Atoms condition)

Given $u, v \in S_n$ with $u \leq v$ then:

$$|a(u,v)| - 1 > |a(u',v)| \ \forall u \lhd u'$$

$$\Downarrow$$

$[u, v]$ *doesn't have a special matching*

Main motivations

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- There is some connection between special matching and Kazhdan-Lusztig polynomials
- Can we use the connection between special matching and Kazhdan-Lusztig polynomials to prove the combinatorial invariance?

Conjecture (Lusztig 1980, Dyer 1987)

Given $u, v \in W$ *and* $x, y \in W^{'}$ *then:*

$$[u, v] \cong [x, y] \Rightarrow P_{u,v} = P_{x,y}$$

Recalling that an interval $[u, v]$ (with $u, v \in S_n$) is irreducible if doesn't exixts $x, y \in S_m$ and $z, t \in S_p$ (with $m, p \leq n$) such that

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then we can show the Conjecture:

Conjecture (Brenti)

Given $u, v \in S_n$ with $[u, v]$ irreducible, $l(u, v) > 1$ and $l(u, v)$ odd then:

$$[u, v] \text{ has a special matching} \Leftrightarrow \overline{\mu}(u, v) = 0$$

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This Conjecture is due to Brenti in **[Kazhdan-Lusztig polynomials: history, problems, and combinatorial invariance]** and is verified for $1 \leq l(u, v) \leq 5$.

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- The poset $[u,v]$ is irreducible. In fact using the following result due to Stanley in **[Enumerative Combinatorics]**:

  if doesn't exixts $x, y \in S_m$ and $z, t \in S_p$ (with $m, p \leq n$) such that $r_{u,v}(q) = r_{x,y}(q) r_{z,t}(q)$ then $[u,v]$ is irreducible

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  I compute the rank generating function:

$$r_{u,v}(q)=(1+q)(1+5q+13q^2+20q^3+19q^4+8q^5+q^6)$$

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  I prove that doesn't exixts a pair of permutation in $z,t\in S_m$ (with $m\le 7$) such that :

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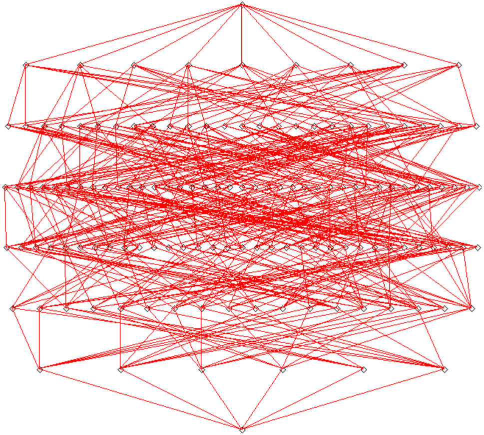

Figure: $[231564, 562341]$, $|c(u,v)| - 1 > |c(u,v')|$ for all $v' \lhd v$

So we study this new Conjecture

Conjecture (Bosca)

Given $u, v \in W$ *with* $l(u, v) > 1$ *then:*

$$[u, v] \text{ has a special matching} \Rightarrow \overline{\mu}(u, v) = 0$$

Main results

In my work I study the previous Conjecture for some classes of coxeter group and elements. The step of the prove are the following:

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- Take $u, v \in S_n$ such that $\overline{\mu}(u, v) \neq 0$.

Main results

In my work I study the previous Conjecture for some classes of coxeter group and elements. The step of the prove are the following:

- Take $u, v \in S_n$ such that $\overline{\mu}(u, v) \neq 0$.
- show that $[u, v]$ doesn't have a special matching using the fact that $|c(u, v)| - 1 > |c(u, v')|$ for all $v' \lhd v$.
- show that $[u, v]$ doesn't have a special matching using the fact that $|a(u, v)| - 1 > |a(u', v)|$ for all $u \lhd u'$.

We consider now the permutations $u, v \in S_n$ such that $u \leq v$ and $D_R(v) \subseteq \{1, n-1\}$. By the following theorem:

Theorem (B. Shapiro, M. Shapiro, A. Vainshtein)

Given $u, v \in S_n$ be such that $u \leq v$ and $D_R(v) \subseteq \{1, n-1\}$. Then

$$P_{u,v}(q) = (1+q)^r$$

where $r := |\{j \in [v(n)+1, v(1)-2] : \sum_{i=1}^{j} u(i) = \binom{j+1}{2}\}|$

by an isomorprhism between poset and other combinatorial constructions I have prove that:

Proposition (Bosca)

Given $u, v \in S_n$ with $u \leq v$ and $D_R(v) \subseteq \{1, n-1\}$. All pair such that $\overline{\mu}(u, v) \neq 0$ and $[u, v] \ncong [e, w]$ (for some $w \in S_n$) up to isomorphism are of the type:

$$v = n, 2, \ldots, n-1, 1$$

$$u = i, 1, \ldots, \widehat{i}, \ldots \widehat{j}, \ldots, n, j$$

and $j - i = n - 3$.

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and $j - i = n - 3$.

By this theorem and the Coatoms condition I can conclude that:

Theorem (Bosca)

Given $u, v \in S_n$, $u \leq v$ and $D_R(v) \subseteq \{1, n-1\}$ be such that $\overline{\mu}(u, v) \neq 0$. Then $[u, v]$ doesn't have a special matching.

Example:

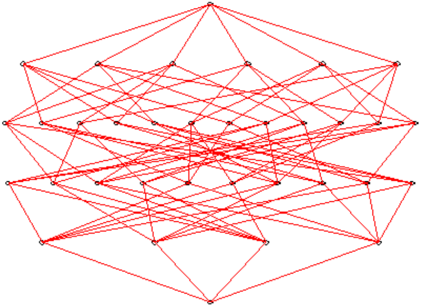


Figure: $[4123576, 7124563] \cong [21354, 52341]$

Grasmannian permutation

I study the conjecture also for permutation in the following set:

$$S_n^{S\setminus\{(i,i+1)\}} = \{x \in S_n : x(1) < \ldots < x(i) \text{ and } x(i+1) < \ldots < x(n)\}$$

and for this permutations we consider the following partition

$$\Lambda_v := (v(i) - i, \ldots, v(1) - 1)$$

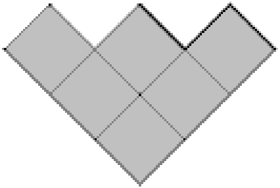
and its diagram

$$\{(i,j) \in \mathbb{N} : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$$

Example: Given $v = 2461357 \in S_n^{S\setminus\{(3,4)\}}$ then:

$$\Lambda_v = (v(3) - 3, v(2) - 2, v(1) - 1) = (3, 2, 1)$$

and its diagram (Russian notation):



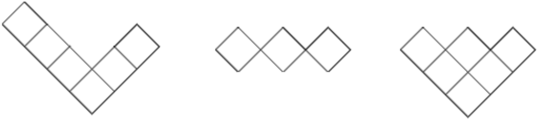
In my work I consider pair of permutations $u, v \in S_n^{S\setminus\{(i,i+1)\}}$ such that the diagram of the following partition:

$$\Lambda := \Lambda_v - \Lambda_u = (v_i - u_i, \ldots, v_1 - u_1)$$

is a Dyck cbs. A diagram is a Dyck cbs if:

- is connected .
- no contains 2×2 square.
- no cells in the diagram have the level strictly less than the rightmost and leftmost cells.

Example: the following are three example of no Dyck cbs



Using the following Corollary:

Corollary (Lascoux)

Given $u, v \in S_n^{S\setminus\{(i,i+1)\}}$ *then:*

$$\Lambda = \Lambda_v - \Lambda_u \text{ is a Dyck cbs} \Leftrightarrow \overline{\mu}(u,v) \neq 0$$

and other combinatorial constructions and isomorphism between poset I can state that:

Proposition (Bosca)

Given $u, v \in S_n^{S\setminus\{(i,i+1)\}}$ *with* $u \leq v$. *Then (up to isomorphism)* $\overline{\mu}(u,v) \neq 0$ *if and only if:*

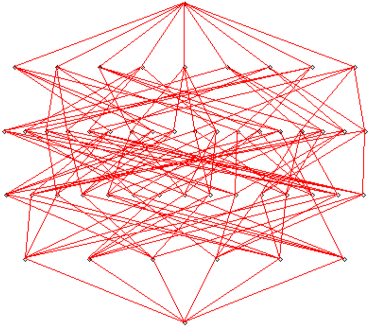
$$v = v(1), v(2), \ldots v(i-1), n, 1, \ldots \widehat{v}(1), \ldots, \widehat{v}(2), \ldots, \widehat{v}(i-1) \ldots, n-1$$

$$u = 1, v(1), v(2), \ldots n-1, 2, \ldots \widehat{v}(1), \ldots, \widehat{v}(2), \ldots, \widehat{v}(i-1), \ldots, n$$

Theorem (Bosca)

Given $u, v \in S_n^{S\setminus\{(i,i+1)\}}$ be such that $\overline{\mu}(u,v) \neq 0$ then $[u,v]$ doesn't have a special matching

Example: Given $u = 145236$ and $v = 456123$ in $S_6^{(S\setminus(3,4))}$ we have that the poset $[u,v]$ doesn't have special matching.



Boolean elements

In this part of my work I extend my Conjecture for linear Coxeter group

Definition

A Coxeter system $(W, \{s_1, \ldots, s_n\})$ is called linear if:

- $(s_i s_j)^r = e$ for $r \geq 3$ if $|i - j| = 1$.
- $s_i s_j = s_j s_i$ if $1 < |i - j| < n - 1$.

W is called strictly linear if also $s_1 s_n = s_n s_1$.

Recalling that given $v \in W$:

$$v = s_1 \cdots s_k$$

is called reduced expression of v if $l(v) = k$.

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We use the work of Marietti in **[Parabolic Kazhdan-Lusztig and R-polynomials for Boolean elements in the symmetric group]** and we consider this pair of elements:

Recalling that given $v \in W$:

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We use the work of Marietti in **[Parabolic Kazhdan-Lusztig and R-polynomials for Boolean elements in the symmetric group]** and we consider this pair of elements:

- **Boolean reflection:** elements $t \in W$ such that there t admits a reduced expressions:

$$t = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$$

Recalling that given $v \in W$:

$$v = s_1 \cdots s_k$$

is called reduced expression of v if $l(v) = k$.
We use the work of Marietti in **[Parabolic Kazhdan-Lusztig and R-polynomials for Boolean elements in the symmetric group]** and we consider this pair of elements:

- **Boolean reflection:** elements $t \in W$ such that there t admits a reduced expressions:

$$t = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$$

- **Boolean elements:** $v \in W$ such that smaller than a Boolean reflection.

Given v Boolean element we define:

$V_h :=$ the number of occurrences of s_h in a reduced expression of v

Then we can use

Theorem (Marietti)

Given u, v Boolean elements in W with $u \leq v$. Then:

$$P_{u,v}(q) = (1+q)^b$$

where:

$$b = |\{k \in [n] : V_k = V_{k+1} = 2, U_{k+1} = 0\}|$$

Proposition (Bosca)

Given $(W, \{s_1, \ldots, s_n\})$ *be a linear Coxeter system and*

$$v = s_i \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_i$$

be a Boolean reflection and $u \leq v$*. Then* $\overline{\mu}(u, v) \neq 0$ *if and only if:*

$$u = s_i \cdots s_{k-1} \widehat{s}_k \widehat{s}_{k+1} \cdots \widehat{s}_{k+r} s_{k+r+1} \cdots s_{j-1} \cdots \widehat{s}_{k+r} \cdots \widehat{s}_{k+1} s_k \cdots s_i \quad (1)$$

for some $i \leq k \leq j-2$ *and* $0 \leq r \leq j-k-2$*.*

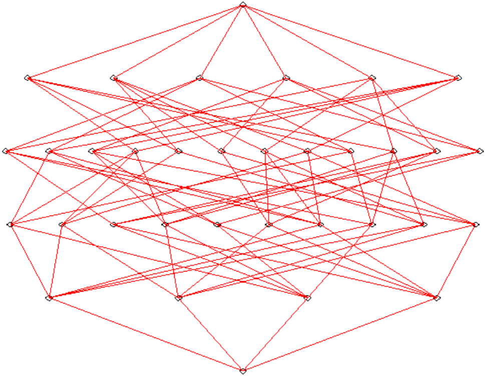
and so conlude that:

Theorem (Bosca)

Given $(W, \{s_1, \ldots, s_n\})$ *be a linear Coxeter system,* v *be a boolean reflection and* $u \leq v$ *be such that* $\overline{\mu}(u, v) \neq 0$*. Then* $[u, v]$ *doesn't have special matching.*

Example: Given the following poset:

$$[s_1 s_5 s_2 s_1, \; s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1]$$



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