Mahonians and parabolic quotients

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September 21, 2011

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Parabolic subgroups and quotients

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If $J = \{s_{n-k+1}, \ldots, s_{n-1}\}$ *then* $W_J \cong S_k$

$$\sum_{\sigma \in {}^J W} q^{\ell(\sigma)} = \frac{[2]_q[3]_q \cdots [n]_q}{[2]_q[3]_q \cdots [k]_q} = [k+1]_q[k+2]_q \cdots [n]_q.$$

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We have

$$\sum_{\sigma \in S_n} q^{\ell(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} \text{ BUT } \mathrm{maj}(\sigma) \neq \mathrm{maj}(\sigma_J) + \mathrm{maj}({}^J\sigma).$$

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Nevertheless,

Theorem (Panova, 2010)

If $W = S_n$ and $J = \{s_{n-k+1}, \ldots, s_{n-1}\}$ then

$$\sum_{\sigma \in {}^J W} q^{\mathrm{maj}(\sigma)} = [k+1]_q [k+2]_q \cdots [n]_q$$

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An idea of Adin-Gessel-Roichman

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$$^{J}W = \{\sigma = [\dots, n-k+1, \dots, n-k+2, \dots, n, \dots]\}.$$

Example

If $n = 5$ and $k = 3$ then

$$^{J}W = \{[12345], [13452], [21345], [23145], [23451], [31245], \dots\}$$

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Let

$$ s(\sigma) := \begin{cases} \sigma(n) - 1, & \text{if } \sigma(n) \in [n-k]; \\ n-k, & \text{otherwise.} \end{cases} $$

A recursion

We let

$$f_{n,k}(q,z) = \sum_{\sigma \in {}^J W} \epsilon^{\ell(\sigma)} q^{\mathrm{maj}(\sigma)} z^{s(\sigma)}$$

where $\epsilon = -1$.

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Theorem

For $k = 1, 2, \ldots, n-1$

$$\begin{aligned} f_{n,k}(q,z) &= \frac{1}{1+z}\Big(\big(\epsilon^k z^{n-k} + (-q)^{n-1}\big) f_{n-1,k}(q,1) + \\ &\quad + \epsilon^n z(1-q^{n-1}) f_{n-1,k}(q,-z)\Big) + z^{n-k} f_{n-1,k-1}(q,1). \end{aligned}$$

Does not restrict to a recursion for $f_{n,k}(q,1)$.

Explicit formulas

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Theorem (C, 2011)

If $k < n$ is odd we have

$$f_{n,k}(q,z) = [k+1]_{-q}[k+2]_q \cdots [n-1]_{\epsilon^n q} \cdot \Big(\sum_{i=0}^{n-k-1} \epsilon^{(n+1)(n-i-1)} z^i q^{n-i-1} + z^{n-k}[k]_{\epsilon^{n-1}q} \Big).$$

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The specialization

Corollary

For $J = \{s_{n-k+1}, s_{n-k+2}, \ldots, s_{n-1}\}$ *we have*

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Complex reflection groups

- The group of r-colored permutations:

$$G(r,n) = \{[\sigma_1^{z_1}, \ldots, \sigma_n^{z_n}] : \sigma \in S_n \text{ and } z_i \in \mathbb{Z}_r\}.$$

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- And other related groups: we let $C_p = \langle [1^{r/p}, \ldots, n^{r/p}] \rangle$ and

$$G^* := G(r,n)/C_p.$$

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$$\sum_{g \in G^*} q^{\mathrm{fmaj}(g)} = [d_1]_q [d_2]_q \cdots [d_n]_q,$$

where d_i are the fundamental degrees of G.

A bijection à la Garsia-Gessel

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Lemma

The map

$$\begin{array}{rcl} G^* \times \mathcal{P}_n \times \{0,1,\ldots,p-1\} & \longrightarrow & \mathbb{N}^n \\ (g,\lambda,h) & \mapsto & f=(f_1,\ldots,f_n), \end{array}$$

where $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} + h\frac{r}{p}$ *for all* $i \in [n]$, *is a bijection. And in this case we say that* f *is* g*-compatible.*

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For $k < n$ we let
$C_k = \{[\sigma_1^0, \sigma_2^0, \ldots, \sigma_k^0, g_{k+1}, \ldots, g_n] \in G^* : \sigma_1 < \cdots < \sigma_k\}$.

The result

Theorem (C. 2011)

Let $G = G(r, p, n)^*$. *Then*

$$\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})} = [p]_{q^{kr/p}} [r(k+1)]_q \cdots [r(n-1)]_q [rn/p]_q.$$

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Corollary

If $G = G(r,n)$, then C_k is a system of coset representatives for the (parabolic) subgroup $G(r,k)$ and

$$\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})} = [r(k+1)]_q[r(k+2)]_q \cdots [rn]_q.$$

Elements starting with a longest 0-colored increasing subsequence

$$\Pi_{r,n,k} := \{g = [\sigma_1^0, \ldots, \sigma_{n-k}^0, \sigma_{n-k+1}^{z_{n-k+1}}, \ldots, \sigma_n^{z_n}] \in G(r,n) : \sigma_1 < \cdots < \sigma_{n-k} \text{ and no increasing subsequence of length } n-k+1 \text{ colored with } 0\}.$$

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Theorem

If $n \geq 2k$ we have that

$$\sum_{g \in \Pi_{r,n,k}} q^{\mathrm{fmaj}(g^{-1})} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} [r(n-i+1)]_q [r(n-i+2)]_q \cdots [rn]_q.$$

Open problems

Problem

Let $J' = [k]$. Numerical evidence shows that

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This is known to have nice factorization if $k = 0$ (Biagioli-C.)