Moment symbolic calculus in probability and in statistics

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Sept, 19th-21th 2011 – Bertinoro

67th Séminaire Lotharingien de Combinatoire
joint session with
XVII Incontro Italiano di Combinatoria Algebrica

- α : symbolic methods in probability and in statistics; classical umbral calculus;
- $n.\alpha$: U-statistics; moments of sampling distributions; Sheppard's corrections;
- $t.\alpha$: Lévy processes; time space harmonic polynomials; stochastic integration;
- $\gamma.\alpha$: cumulants; k statistics and polykays;
- $\gamma.\beta.\alpha$: Sheffer sequences; Lagrange inversion formula; Riordan arrays and connection constants; parametrization of cumulants; solving some linear recurrences;
- $\gamma.\beta.\boldsymbol{\mu}$: multivariate r.v.'s; multivariate Faà di Bruno Formula; multivariate Lévy processes; multivariate time space harmonic polynomials;
- $(n.\beta.\alpha)_\sigma$: work in progress: random matrices.

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Part I

Moment symbolic calculus in probability and in statistics (I)

1. **Why *symbolic methods* in probability and in statistics?**
    - Symbolic manipulation systems
2. **The classical umbral calculus**
    - The algebra of random variables
3. **Auxiliary umbrae**
    - Dot-operations
4. **U-statistics**
    - Symmetric polynomials
    - Computational issues
5. **Sheppard's corrections**
6. **Lévy processes**
    - Stochastic Finance
    - Symbolic Lévy processes
    - Time-space harmonic polynomials

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Typically, algebraic objects admit *exact computations* as opposed to the approximate character of *numerical computations*.

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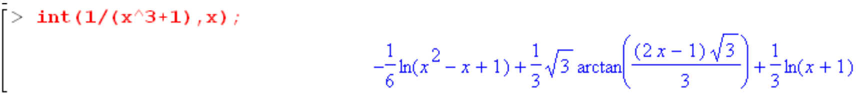

```
> int(1/(x^3+1),x);
```

$$-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{3}\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)+\frac{1}{3}\ln(x+1)$$

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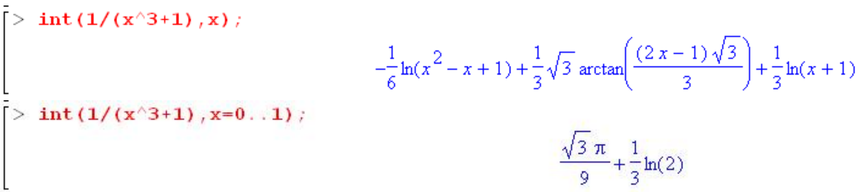

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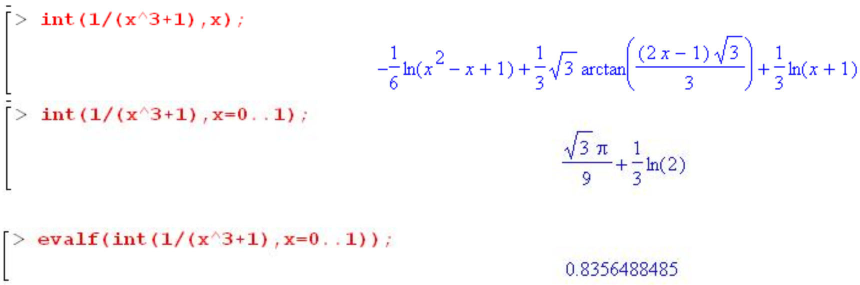

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> evalf(int(1/(x^3+1),x=0..1));
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```
0.8356488485
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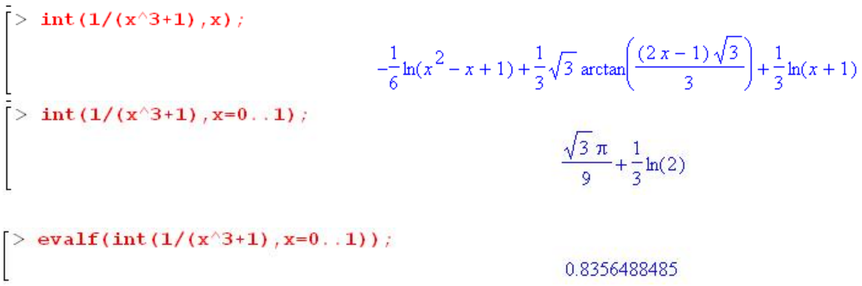

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By Maple

Symbolic methods $\Rightarrow$ a set of manipulation techniques aiming to perform algebraic calculations (possibly) through an algorithmic approach in order to find efficient mechanical processes to pass to a computer.

Computing

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Systems which *implement* symbolic methods are called symbolic manipulation systems: aka Symbolic computation, Computer algebra...

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Statistical manipulations

> "The idea of augmented symmetric functions we believe to be ours "
> David F.N., Kendall M.G., Barton D.E. - Tables (1966) - Pearson

A fundamental expectation result

$$E\left[\sum \underbrace{X_s X_t \cdots}_{r_1} \underbrace{X_q^2 X_p^2 \cdots}_{r_2} \underbrace{X_u^m X_v^m \cdots}_{r_m}\right] = (n)_{\nu_\lambda} \, a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}$$

- $(X_1, X_2, \ldots, X_n)$ i.i.d.r.s. with n sample size;
- $E[X_i^j] = a_j$ for $j = 1, 2, \ldots, k$ and $k \leq n$;
- with $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m}) \vdash k \leq n$ in ν_λ parts that is $r_1 + 2r_2 + \cdots + mr_m = k$ and $r_1 + r_2 + \cdots + r_m = \nu_\lambda$.

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Is it attractive enough?

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assume that $\Uparrow$ could be "symbolically represented " by $\Downarrow$

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$$\prod_{j=1}^{m} (\chi_1 X_1^j + \chi_2 X_2^j + \cdots + \chi_n X_n^j)^{r_j}$$

Is it attractive enough?

$$\sum \underbrace{X_s X_t \cdots}_{r_1} \underbrace{X_q^2 X_p^2 \cdots}_{r_2} \underbrace{X_u^m X_v^m \cdots}_{r_m}$$

assume that ⇑ could be "symbolically represented " by ⇓

$$\prod_{j=1}^m (\chi_1 X_1^j + \chi_2 X_2^j + \cdots + \chi_n X_n^j)^{r_j}$$

with a structure very similar to $a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}$

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How?

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How? $\triangleright\, E[\chi_j^i] = \begin{cases} 1 & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$

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How?

- $\triangleright\; E[\chi_j^i] = \begin{cases} 1 & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$
- $\triangleright\; E[\chi_1^{i_i}\chi_2^{i_2}\cdots\chi_n^{i_n}] = E[\chi_1^{i_i}]E[\chi_2^{i_2}]\cdots E[\chi_n^{i_n}]$

Is it attractive enough?

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Are $\{\chi_i\}_{i=1}^n$ r.v.'s?

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Are $\{\chi_i\}_{i=1}^n$ r.v.'s? $\boxed{\overset{\text{No}}{\Rightarrow}\ E[\chi_i^2] = 0}$

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Are $\{\chi_i\}_{i=1}^n$ r.v.'s? $\boxed{\overset{\text{No}}{\Rightarrow} E[\chi_i^2] = 0}$ **"Umbral Calculus"**

Is it attractive enough?

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$E\left[\left(\sum_{i\neq j}^{n} X_i^2 X_j\right)\left(\sum_{i=1}^{n} X_i^2 Y_i\right)^2\right]$ with $(X_1, Y_1), \ldots, (X_n, Y_n)$ separately i.i.d.r.v.'s

Is it attractive enough?

$$E\left[\left(\sum_{i\neq j}^{n} X_i^2 X_j\right)\left(\sum_{i=1}^{n} X_i^2 Y_i\right)^2\right] \quad \text{with } (X_1, Y_1), \ldots, (X_n, Y_n)$$

$$g_{i,j} = E[X^i Y^j] \quad \text{separately i.i.d.r.v.'s}$$

$$\Downarrow$$

$$2(n)_2[2g_{4,1}\, g_{3,1} + g_{5,2}\, g_{2,0} + g_{6,2}\, g_{1,0}] + 2(n)_3 g_{3,1}\, g_{2,1}\, g_{2,0} +$$
$$(n)_3[2g_{4,1}\, g_{2,1}\, g_{1,0} + g_{4,2}\, g_{2,0}\, g_{1,0}] + (n)_4 g_{2,1}^2\, g_{2,0}\, g_{1,0}$$

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$$E\left[\left(\sum_{i\neq j}^{n} X_i^2 X_j\right)\left(\sum_{i=1}^{n} X_i^2 Y_i\right)^2\right] \qquad \text{with } (X_1, Y_1), \ldots, (X_n, Y_n)$$

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$$\Downarrow$$

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$$(n)_3[2g_{4,1}\,g_{2,1}\,g_{1,0} + g_{4,2}\,g_{2,0}\,g_{1,0}] + (n)_4 g_{2,1}^2\,g_{2,0}\,g_{1,0}$$

One more advantage

When symbolic methods are used properly, they can give us more insights to problems.

In the literature

- **(1983) Speed T.** *Cumulants and partition lattices.* **Austr. J. Stat.**
- (1987) McCullagh P. *Tensor methods in statistics.* Chapman and Hall
- (1993) Kendall W. S. *Computer Algebra in probability and statistics,* Stat. Neerl.
- (2000) Andrews D. F. and Stafford J. E. *Symbolic computation for statistical inference.* Oxford University Press.
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$\Rightarrow$ The same sequence $1, a_1, a_2, \ldots$ (in the following $\{a_i\}$) could be represented by using *distinct* umbrae.

Similar umbrae

A second device

Two umbrae α and γ are said to be *similar* when

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In [SIAM] two umbrae such that $\alpha \equiv \gamma$ are called exchangeable.

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Moments

The sequence $\{a_i\}$, such that $E[\alpha^i] = a_i$ for all nonnegative integers i, is the sequence of *moments* of α.

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By the multinomial theorem

$$E[(\underbrace{\alpha + \gamma + \cdots + \zeta}_{m})^n] = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} a_{k_1}\, g_{k_2} \cdots z_{k_m}$$

with $\{a_i\}, \{g_i\}, \ldots, \{z_i\}$ moments of $\alpha, \gamma, \ldots, \zeta$ respectively. Therefore

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$$0.\alpha \equiv \varepsilon, \text{ (augmentation umbra) with } E[\varepsilon^i] = \delta_{i,0}.$$

(2008) Kowalski J. and Tu X. M. *Modern Applied U-Statistics.* Wiley.

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Ex: $E[\epsilon^k] = \begin{cases} e_k(x_1, x_2, \ldots, x_n) & k \leq n \\ 0 & k \geq n+1 \end{cases}$

Umbral equivalence

On the elementary symmetric polynomial umbra ϵ

$$\chi_1 x_1 + \cdots + \chi_n x_n \equiv \bar{u}\epsilon$$

$\{\chi_i\}_{i=1}^{n}$ uncorrelated umbrae similar to the singleton umbra χ and $\bar{u}$ the boolean unity such that $E[\bar{u}^k] = k!$ for all nonnegative integers k.

Umbral equivalence

▷ Working with umbral polynomials $p, q \in K[x_1, x_2, \ldots, x_n][A]$, then $\boxed{p \simeq q \Leftrightarrow E[p] = E[q].}$ (umbral equivalence)

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  (umbral symmetric power sum polynomial)

A fundamental expectation result

$$E[\sum \underbrace{X_s X_t \cdots}_{r_1} \underbrace{X_q^2 X_r^2 \cdots}_{r_2} \underbrace{X_u^m X_\nu^m \cdots}_{r_m}] = (n)_{\nu_\lambda} \, a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}$$

- $(X_1, X_2, \ldots, X_n)$ i.i.d.r.s. with n sample size;
- $E[X_i^j] = a_j$ for $j = 1, 2, \ldots, k$ and $k \leq n$;
- with $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m}) \vdash k \leq n$ in ν_λ parts that is $r_1 + 2r_2 + \cdots + mr_m = k$ and $r_1 + r_2 + \cdots + r_m = \nu_\lambda$.

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α_λ is an auxiliary symbol denoting the product $(\alpha_{j_1})^{\cdot r_1}(\alpha_{j_2}^2)^{\cdot r_2}\cdots$ with $j_1, j_2, \ldots \in [n]$.

U-statistics

$$\frac{[n_{\bullet}(\chi\alpha)]^{r_1}[n_{\bullet}(\chi\alpha^2)]^{r_2}\cdots}{(n)_{\nu_\lambda}} \simeq \alpha_\lambda$$

- $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ uncorrelated umbrae similar to α;
- $E[\alpha_i^j] = a_j$ for $j = 1, 2, \ldots, k$ and $k \leq n$;
- with $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m}) \vdash k \leq n$ in ν_λ parts that is $r_1 + 2r_2 + \cdots + mr_m = k$ and $r_1 + r_2 + \cdots + r_m = \nu_\lambda$.

Dot-power

$\alpha^{\cdot n}$ is an auxiliary symbol denoting the product $\underbrace{\alpha'\alpha''\cdots\alpha'''}_{n}$.

α_λ is an auxiliary symbol denoting the product $(\alpha_{j_1})^{\cdot r_1}(\alpha_{j_2}^2)^{\cdot r_2}\cdots$ with $j_1, j_2, \ldots \in [n]$.

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$Ex: a_1^2$

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Augmented and power sums

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A very general formula

$$M = \{\underbrace{\mu_1, \ldots, \mu_1}_{f(\mu_1)}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{f(\mu_k)}\}$$

with $\{\mu_1, \mu_2, \ldots, \mu_k\}$ umbral monomials.

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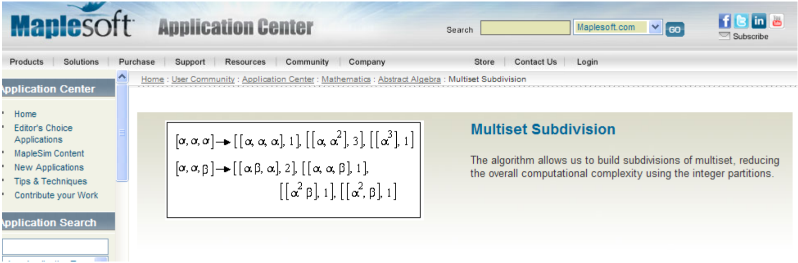
Computing multiset subdivision

Computing multiset subdivision



Home : User Community : Application Center : Mathematics : Abstract Algebra : Multiset Subdivision

$$[\alpha,\alpha,\alpha] \rightarrow [[\alpha,\alpha,\alpha],1], [[\alpha,\alpha^2],3], [[\alpha^3],1]$$

$$[\alpha,\alpha,\beta] \rightarrow [[\alpha\beta,\alpha],2], [[\alpha,\alpha,\beta],1], [[\alpha^2\beta],1], [[\alpha^2,\beta],1]$$

Multiset Subdivision

The algorithm allows us to build subdivisions of multiset, reducing the overall computational complexity using the integer partitions.

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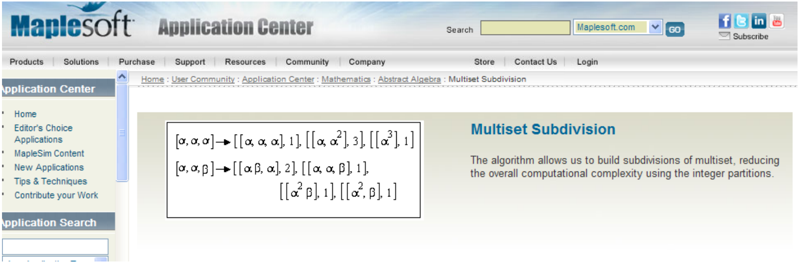

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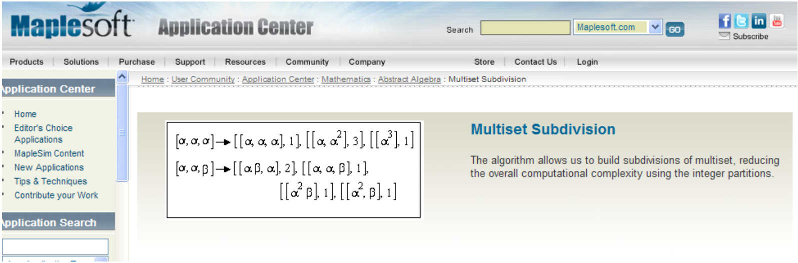

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- ▷ This strategy is speedier than the iterated full partition of Andrews and Stafford, given that it takes into account the multiplicity of all elements of M.
- ▷ The higher this multiplicity is, the more this procedure gives efficient results.

Computational results

The change of bases from augmented symmetric polynomials to power sums.

$[1^i 2^j 3^k \cdots]$	`SF`	`MathStatica`	`Umbral`
$[1^5\, 2^3\, 3^2]$	0.78	0.18	0.13
$[1^6\, 2^3]$	0.08	0.01	0.01
$[2^{10}]$	2.57	0.03	0.01
$[1^5\, 2^7\, 3^1]$	6.15	1.20	0.65
$[1^2\, 2^2\, 3^2\, 4^2]$	2.75	0.11	0.09

Table 1: Comparison of computational times.

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Differently from the umbral algorithm, `MathStatica` and `SF` do not work on multiple sets of variables.

Multivariate r.v.'s

Ex: If $M = \{\mu_1, \mu_1, \mu_2\}$ then $(n . \mu)_M = (n . \mu_1)^2 (n . \mu_2)$.

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S	d_S	$[n.(\chi\mu)]_S$
$\{\{\mu_1, \mu_1, \mu_2\}\}$	1	$n.(\chi\mu_1^2\mu_2)$
$\{\{\mu_1\}, \{\mu_1, \mu_2\}\}$	2	$[n.(\chi\mu_1)][n.(\chi\mu_1\mu_2)]$
$\{\{\mu_2\}, \{\mu_1, \mu_1\}\}$	1	$[n.(\chi\mu_2)][n.(\chi\mu_1^2)]$
$\{\{\mu_1\}, \{\mu_1\}, \{\mu_2\}\}$	1	$[n.(\chi\mu_1)]^2[n.(\chi\mu_2)]$

Table 2: Subdivisions of $M = \{\mu_1, \mu_1, \mu_2\}$.

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S	d_S	$[n.(\chi\mu)]_S$
$\{\{\mu_1, \mu_1, \mu_2\}\}$	1	$n.(\chi\mu_1^2\mu_2)$
$\{\{\mu_1\}, \{\mu_1, \mu_2\}\}$	2	$[n.(\chi\mu_1)][n.(\chi\mu_1\mu_2)]$
$\{\{\mu_2\}, \{\mu_1, \mu_1\}\}$	1	$[n.(\chi\mu_2)][n.(\chi\mu_1^2)]$
$\{\{\mu_1\}, \{\mu_1\}, \{\mu_2\}\}$	1	$[n.(\chi\mu_1)]^2[n.(\chi\mu_2)]$

Table 2: Subdivisions of $M = \{\mu_1, \mu_1, \mu_2\}$.

▷ $(n.\mu_1)^2(n.\mu_2) \simeq n.(\chi\mu_1^2\mu_2) + 2[n.(\chi\mu_1)][n.(\chi\mu_1\mu_2)] + \cdots ,$

Multivariate r.v.'s

Ex: If $M = \{\mu_1, \mu_1, \mu_2\}$ then $(n.\mu)_M = (n.\mu_1)^2(n.\mu_2)$. In statistical terminology, moments of $(n.\mu)_M$ correspond to moments of the product of sums $\left(\sum_{i=1}^n X_i\right)^2 \left(\sum_{i=1}^n Y_i\right)$, where $(X_1, Y_1), \ldots, (X_n, Y_n)$ are separately i.i.d.r.v.'s.

S	d_S	$[n.(\chi\mu)]_S$
$\{\{\mu_1, \mu_1, \mu_2\}\}$	1	$n.(\chi\mu_1^2\mu_2)$
$\{\{\mu_1\}, \{\mu_1, \mu_2\}\}$	2	$[n.(\chi\mu_1)][n.(\chi\mu_1\mu_2)]$
$\{\{\mu_2\}, \{\mu_1, \mu_1\}\}$	1	$[n.(\chi\mu_2)][n.(\chi\mu_1^2)]$
$\{\{\mu_1\}, \{\mu_1\}, \{\mu_2\}\}$	1	$[n.(\chi\mu_1)]^2[n.(\chi\mu_2)]$

Table 2: Subdivisions of $M = \{\mu_1, \mu_1, \mu_2\}$.

- ▷ $(n.\mu_1)^2(n.\mu_2) \simeq n.(\chi\mu_1^2\mu_2) + 2[n.(\chi\mu_1)][n.(\chi\mu_1\mu_2)] + \cdots$,
- ▷ $E[\left(\sum_{i=1}^n X_i\right)^2 \left(\sum_{i=1}^n Y_i\right)] = n\, g_{2,1} + 2\,(n)_2\, g_{1,0}\, g_{1,1} + (n)_2\, g_{2,0}\, g_{0,1} + \cdots$.

Moments of sampling Distributions

(2005) Vrbik J. *Populations Moments of Sampling Distributions* Comput. Stat.

$$E\left[\left(\sum_{i\neq j}^{n} X_i^2 X_j\right)\left(\sum_{i=1}^{n} X_i^2 Y_i\right)^2\right] \qquad g_{i,j} = E[X^i Y^j]$$

with $(X_1, Y_1), \ldots, (X_n, Y_n)$ separately i.i.d.r.v.'s

$$\boxed{E[n{\bf .}(\chi_1\mu_1^2)\, n{\bf .}(\chi_1\mu_1)\, n{\bf .}(\chi_2\mu_1^2\mu_2)\, n{\bf .}(\chi_3\mu_1^2\mu_2)] =}$$

$$[n{\bf .}(\chi_1\mu_1^2) n{\bf .}(\chi_1\mu_1)][n{\bf .}(\chi_2\mu_1^2\mu_2)][n{\bf .}(\chi_3\mu_1^2\mu_2)] \simeq (n{\bf .}\nu)_M \simeq \sum_{\pi\in\Pi}[n{\bf .}(\chi\nu)]_\pi.$$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}$.

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with $(X_1, Y_1), \ldots, (X_n, Y_n)$ separately i.i.d.r.v.'s

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_{\pi}.$

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$$g_{i,j} = E[X^i Y^j]$$ separately i.i.d.r.v.'s

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi.$

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with $(X_1, Y_1), \ldots, (X_n, Y_n)$ separately i.i.d.r.v.'s

- $(\sum_{i\neq j}^{n} X_i^2 X_j)$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_{\pi}.$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}$.

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$$g_{i,j} = E[X^i Y^j] \qquad \text{separately i.i.d.r.v.'s}$$

- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow}$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi.$$

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- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2) n.(\chi\mu_1)]$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi.$$

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$$g_{i,j} = E[X^i Y^j] \quad \text{separately i.i.d.r.v.'s}$$

- $(\sum_{i\neq j}^{n} X_i^2 X_j) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2) n.(\chi\mu_1)]$
- $(\sum_{i=1}^{n} X_i^2 Y_i)$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi.$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}.$

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- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2) n.(\chi\mu_1)]$
- $\left(\sum_{i=1}^{n} X_i^2 Y_i\right) \overset{\text{umbral}}{\Rightarrow}$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

$$[n.(\chi_1\mu_1^2)n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi.$$

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- $(\sum_{i\neq j}^{n} X_i^2 X_j) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2) n.(\chi\mu_1)]$
- $(\sum_{i=1}^{n} X_i^2 Y_i) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2\mu_2)]$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] =}$$

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with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}$.

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- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n\boldsymbol{.}(\chi\mu_1^2)n\boldsymbol{.}(\chi\mu_1)]$
- $\left(\sum_{i=1}^{n} X_i^2 Y_i\right) \overset{\text{umbral}}{\Rightarrow} [n\boldsymbol{.}(\chi\mu_1^2\mu_2)]$

$$\boxed{E[n\boldsymbol{.}(\chi_1\mu_1^2)\, n\boldsymbol{.}(\chi_1\mu_1)\, n\boldsymbol{.}(\chi_2\mu_1^2\mu_2)\, n\boldsymbol{.}(\chi_3\mu_1^2\mu_2)] =}$$

$$[n\boldsymbol{.}(\chi_1\mu_1^2)n\boldsymbol{.}(\chi_1\mu_1)][n\boldsymbol{.}(\chi_2\mu_1^2\mu_2)][n\boldsymbol{.}(\chi_3\mu_1^2\mu_2)] \simeq (n\boldsymbol{.}\nu)_M \simeq \sum_{\pi\in\Pi}[n\boldsymbol{.}(\chi\nu)]_{\pi}.$$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}$.

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$$g_{i,j} = E[X^i Y^j] \quad \text{separately i.i.d.r.v.'s}$$

- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n_{\bullet}(\chi\mu_1^2) n_{\bullet}(\chi\mu_1)]$
- $\left(\sum_{i=1}^{n} X_i^2 Y_i\right) \overset{\text{umbral}}{\Rightarrow} [n_{\bullet}(\chi\mu_1^2\mu_2)]$

$$\boxed{E[n_{\bullet}(\chi_1\mu_1^2)\, n_{\bullet}(\chi_1\mu_1)\, n_{\bullet}(\chi_2\mu_1^2\mu_2)\, n_{\bullet}(\chi_3\mu_1^2\mu_2)] = ?}$$

$[n_{\bullet}(\chi_1\mu_1^2) n_{\bullet}(\chi_1\mu_1)][n_{\bullet}(\chi_2\mu_1^2\mu_2)][n_{\bullet}(\chi_3\mu_1^2\mu_2)] \simeq (n_{\bullet}\nu)_M \simeq \sum_{\pi\in\Pi}[n_{\bullet}(\chi\nu)]_{\pi}.$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}.$

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with $(X_1, Y_1), \ldots, (X_n, Y_n)$ separately i.i.d.r.v.'s

- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2)n.(\chi\mu_1)]$
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$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] = ?}$$

$$[n.(\chi_1\mu_1^2)n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi .$$

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- $\left(\sum_{i\neq j}^n X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2) n.(\chi\mu_1)]$
- $\left(\sum_{i=1}^n X_i^2 Y_i\right) \overset{\text{umbral}}{\Rightarrow} [n.(\chi\mu_1^2\mu_2)]$

$$\boxed{E[n.(\chi_1\mu_1^2)\, n.(\chi_1\mu_1)\, n.(\chi_2\mu_1^2\mu_2)\, n.(\chi_3\mu_1^2\mu_2)] = ?}$$

$$[n.(\chi_1\mu_1^2) n.(\chi_1\mu_1)][n.(\chi_2\mu_1^2\mu_2)][n.(\chi_3\mu_1^2\mu_2)] \simeq (n.\nu)_M \simeq \sum_{\pi\in\Pi}[n.(\chi\nu)]_\pi .$$

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- $\left(\sum_{i\neq j}^{n} X_i^2 X_j\right) \overset{\text{umbral}}{\Rightarrow} [n{\bullet}(\chi\mu_1^2)n{\bullet}(\chi\mu_1)]$
- $\left(\sum_{i=1}^{n} X_i^2 Y_i\right) \overset{\text{umbral}}{\Rightarrow} [n{\bullet}(\chi\mu_1^2\mu_2)]$

$$\boxed{E[n{\bullet}(\chi_1\mu_1^2)\, n{\bullet}(\chi_1\mu_1)\, n{\bullet}(\chi_2\mu_1^2\mu_2)\, n{\bullet}(\chi_3\mu_1^2\mu_2)] = ?}$$

$$[n{\bullet}(\chi_1\mu_1^2)n{\bullet}(\chi_1\mu_1)][n{\bullet}(\chi_2\mu_1^2\mu_2)][n{\bullet}(\chi_3\mu_1^2\mu_2)] \simeq (n{\bullet}\nu)_M \simeq \sum_{\pi\in\Pi}[n{\bullet}(\chi\nu)]_\pi.$$

with $M = \{\chi_1\mu_1^2, \chi_1\mu_1, \chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}$.

For $\pi_1 = \{\{\chi_1\mu_1^2, \chi_2\mu_1^2\mu_2\}, \{\chi_1\mu_1, \chi_3\mu_1^2\mu_2\}\}$

$$[n\centerdot(\chi\mu)]_{\pi_1} = n\centerdot(\chi\,\chi_1\chi_2\mu_1^4\mu_2)n\centerdot(\chi\,\chi_1\chi_3\mu_1^3\mu_2) \simeq n\centerdot(\chi\mu_1^4\mu_2)\,n\centerdot(\chi\mu_1^3\mu_2)$$

For $\pi_2 = \big\{\{\chi_1\mu_1^2, \chi_1\mu_1\}, \{\chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}\big\}$

$$[n\centerdot(\chi\mu)]_{\pi_2} = n\centerdot(\chi\,\chi_1^2\,\mu_1^3)\,n\centerdot(\chi\,\chi_2\chi_3\,\mu_1^4\mu_2^2) \simeq 0 \ \text{ as } [n\centerdot(\chi\mu)]_{\pi_2} \simeq 0$$

$[1^i 2^j 3^k \cdots]$	SIP	MAPLE
$[5^3\,8\,9\,10][1\,2\,3\,4\,5]$	5.6	0.4
$[6\,7\,8\,9\,10][1\,2\,3\,4\,5]$	2.2	0.1
$[6\,7\,8\,9\,10][1\,2][3\,4\,5]$	3.1	0.4
$[6\,7][8\,9\,10][1\,2][3\,4\,5]$	4.7	1.3
$[5\,6\,7\,8\,9\,10][1\,2\,3\,4\,5]$	16.7	0.3
$[5\,6\,7\,8\,9\,10][1\,2\,3\,4\,5\,6]$	348.7	1.5
$[6\,7\,8\,9\,10][6\,7][3\,4\,5][1\,2]$	125.6	16.4

Table 3: Computational times.

For $\pi_1 = \{\{\chi_1\mu_1^2, \chi_2\mu_1^2\mu_2\}, \{\chi_1\mu_1, \chi_3\mu_1^2\mu_2\}\}$

$$[n\boldsymbol{.}(\chi\mu)]_{\pi_1} = n\boldsymbol{.}(\chi\,\chi_1\chi_2\mu_1^4\mu_2)n\boldsymbol{.}(\chi\,\chi_1\chi_3\mu_1^3\mu_2) \simeq n\boldsymbol{.}(\chi\mu_1^4\mu_2)\,n\boldsymbol{.}(\chi\mu_1^3\mu_2)$$

For $\pi_2 = \{\{\chi_1\mu_1^2, \chi_1\mu_1\}, \{\chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\}\}$

$$[n\boldsymbol{.}(\chi\mu)]_{\pi_2} = n\boldsymbol{.}(\chi\,\chi_1^2\,\mu_1^3)\,n\boldsymbol{.}(\chi\,\chi_2\chi_3\,\mu_1^4\mu_2^2) \simeq 0 \quad \text{as } [n\boldsymbol{.}(\chi\mu)]_{\pi_2} \simeq 0$$

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Table 3: Computational times.

For $\pi_1 = \{\{\chi_1\mu_1^2, \chi_2\mu_1^2\mu_2\}, \{\chi_1\mu_1, \chi_3\mu_1^2\mu_2\}\}$

$$[n_\bullet(\chi\mu)]_{\pi_1} = n_\bullet(\chi\,\chi_1\chi_2\mu_1^4\mu_2) n_\bullet(\chi\,\chi_1\chi_3\mu_1^3\mu_2) \simeq n_\bullet(\chi\mu_1^4\mu_2)\, n_\bullet(\chi\mu_1^3\mu_2)$$

For $\pi_2 = \big\{\{\chi_1\mu_1^2, \chi_1\mu_1\}, \big\{\chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\big\}\big\}$

$[n_\bullet(\chi\mu)]_{\pi_2} = n_\bullet(\chi\,\chi_1^2\,\mu_1^3)\, n_\bullet(\chi\,\chi_2\chi_3\,\mu_1^4\mu_2^2) \simeq 0$ as $[n_\bullet(\chi\mu)]_{\pi_2} \simeq 0$

$[1^i 2^j 3^k \cdots]$	SIP	MAPLE
$[5^3\, 8\, 9\, 10][1\, 2\, 3\, 4\, 5]$	5.6	0.4
$[6\, 7\, 8\, 9\, 10][1\, 2\, 3\, 4\, 5]$	2.2	0.1
$[6\, 7\, 8\, 9\, 10][1\, 2][3\, 4\, 5]$	3.1	0.4
$[6\, 7][8\, 9\, 10][1\, 2][3\, 4\, 5]$	4.7	1.3
$[5\, 6\, 7\, 8\, 9\, 10][1\, 2\, 3\, 4\, 5]$	16.7	0.3
$[5\, 6\, 7\, 8\, 9\, 10][1\, 2\, 3\, 4\, 5\, 6]$	348.7	1.5
$[6\, 7\, 8\, 9\, 10][6\, 7][3\, 4\, 5][1\, 2]$	125.6	16.4

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For $\pi_1 = \{\{\chi_1\mu_1^2, \chi_2\mu_1^2\mu_2\}, \{\chi_1\mu_1, \chi_3\mu_1^2\mu_2\}\}$

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$$[n_{\bullet}(\chi\mu)]_{\pi_2} = n_{\bullet}(\chi\,\chi_1^2\,\mu_1^3)\, n_{\bullet}(\chi\,\chi_2\chi_3\,\mu_1^4\mu_2^2) \simeq 0 \quad \text{as}\, [n_{\bullet}(\chi\mu)]_{\pi_2} \simeq 0$$

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For $\pi_2 = \big\{\{\chi_1\mu_1^2, \chi_1\mu_1\}, \big\{\chi_2\mu_1^2\mu_2, \chi_3\mu_1^2\mu_2\big\}\big\}$

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$[6\, 7\, 8\, 9\, 10][6\, 7][3\, 4\, 5][1\, 2]$	125.6	16.4

Table 3: Computational times.

Grouped data

Grouped data

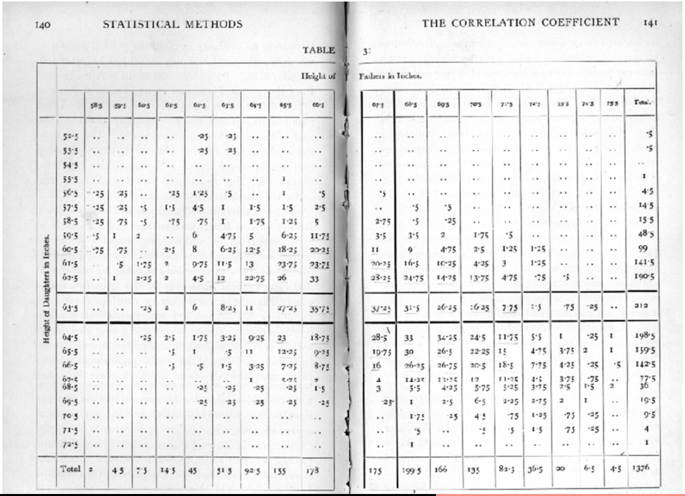


TABLE 31

Height of Fathers in Inches.

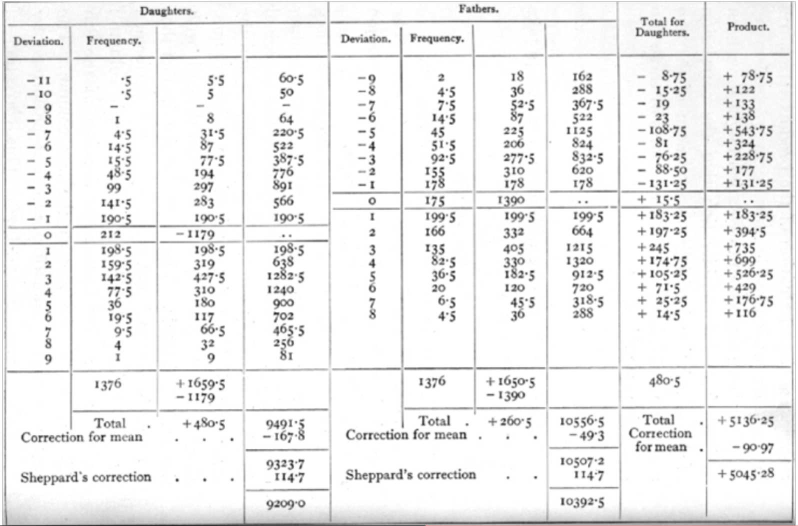
Height of Daughters in Inches.

	58·5	59·5	60·5	61·5	62·5	63·5	64·5	65·5	66·5	67·5	68·5	69·5	70·5	71·5	72·5	73·5	74·5	75·5	Total.
52·5	..	..	..	..	·25	·25	..	..	..	..	..	..	..	..	..	..	..	..	·5
53·5	..	..	..	..	·25	·25	..	..	..	..	..	..	..	..	..	..	..	..	·5
54·5	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
55·5	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	1
56·5	·25	·25	..	·25	1·25	·5	..	1	·5	·5	..	..	..	..	..	..	..	..	4·5
57·5	·25	·25	·5	1·5	4·5	1	1·5	1·5	2·5	..	·5	·5	..	..	..	..	..	..	14·5
58·5	·25	·75	·5	·75	·75	1	1·75	1·25	5	2·75	·5	·25	..	..	..	..	..	..	15·5
59·5	·5	1	2	..	6	4·75	5	6·25	11·75	3·5	3·5	2	1·75	·5	..	..	..	..	48·5
60·5	·75	·75	..	2·5	8	6·25	12·5	18·25	20·25	11	9	4·75	2·5	1·25	1·25	..	..	..	99
61·5	..	·5	1·75	2	9·75	11·5	13	23·75	23·75	20·25	16·5	10·25	4·25	3	1·25	..	..	..	141·5
62·5	..	1	2·25	2	4·5	12	22·75	26	33	28·25	24·75	14·25	13·75	4·75	·75	·5	..	..	190·5
63·5	..	..	·25	2	6	8·25	11	27·25	35·75	37·25	31·5	26·25	16·25	7·75	1·5	·75	·25	..	212
64·5	..	..	·25	2·5	1·75	3·25	9·25	23	18·75	28·5	33	34·25	24·5	11·75	5·5	1	·25	1	198·5
65·5	..	..	..	·5	1	·5	11	12·25	9·25	19·75	30	26·5	22·25	15	4·75	3·75	2	1	159·5
66·5	..	..	..	·5	·5	1·5	3·25	7·25	8·75	16	26·25	26·75	20·5	18·5	7·75	4·25	·25	·5	142·5
67·5	..	..	..	..	..	..	1	5·75	7	4	14·25	13·25	12	11·25	4·5	3·75	·75	..	77·5
68·5	..	..	..	..	·25	·25	·25	·25	1·5	3	5·5	4·25	5·75	5·25	3·75	2·5	1·5	2	36
69·5	..	..	..	..	·25	·25	·25	·25	·25	·25	1	2·5	6·5	2·25	2·75	2	1	..	19·5
70·5	..	..	..	..	..	..	..	..	..	..	1·75	·25	4·5	·75	1·25	·75	·25	..	9·5
71·5	..	..	..	..	..	..	..	..	..	..	·5	..	·5	·5	1·5	·75	·25	..	4
72·5	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	1
Total	2	4·5	7·5	14·5	45	51·5	92·5	155	178	175	199·5	166	135	82·5	36·5	20	6·5	4·5	1376

A record in table form of the stature of 1376 fathers and daughters

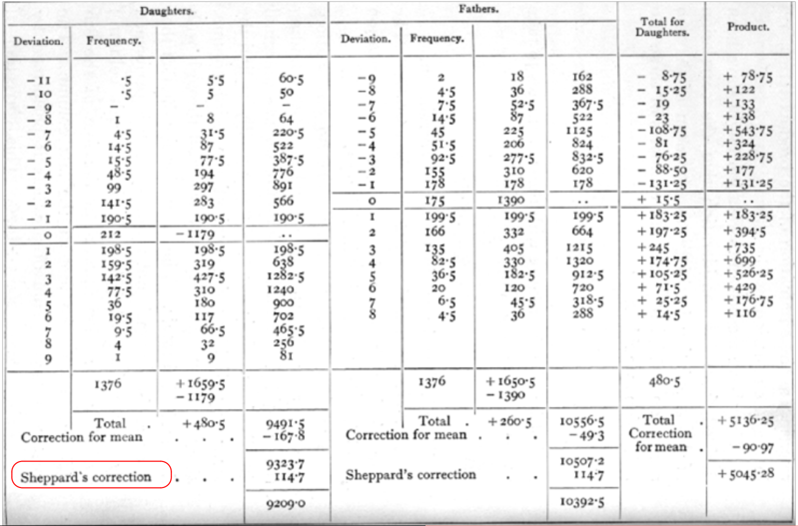
Sheppard's corrections

Sheppard's corrections



Daughters.				Fathers.				Total for Daughters.	Product.
Deviation.	Frequency.			Deviation.	Frequency.				
−11	·5	5·5	60·5	−9	2	18	162	− 8·75	+ 78·75
−10	·5	5	50	−8	4·5	36	288	− 15·25	+122
− 9	–	–	–	−7	7·5	52·5	367·5	− 19	+133
− 8	1	8	64	−6	14·5	87	522	− 23	+138
− 7	4·5	31·5	220·5	−5	45	225	1125	−108·75	+543·75
− 6	14·5	87	522	−4	51·5	206	824	− 81	+324
− 5	15·5	77·5	387·5	−3	92·5	277·5	832·5	− 76·25	+228·75
− 4	48·5	194	776	−2	155	310	620	− 88·50	+177
− 3	99	297	891	−1	178	178	178	−131·25	+131·25
− 2	141·5	283	566	0	175	1390	..	+ 15·5	..
− 1	190·5	190·5	190·5	1	199·5	199·5	199·5	+183·25	+183·25
0	212	−1179	..	2	166	332	664	+197·25	+394·5
1	198·5	198·5	198·5	3	135	405	1215	+245	+735
2	159·5	319	638	4	82·5	330	1320	+174·75	+699
3	142·5	427·5	1282·5	5	36·5	182·5	912·5	+105·25	+526·25
4	77·5	310	1240	6	20	120	720	+ 71·5	+429
5	36	180	900	7	6·5	45·5	318·5	+ 25·25	+176·75
6	19·5	117	702	8	4·5	36	288	+ 14·5	+116
7	9·5	66·5	465·5						
8	4	32	256						
9	1	9	81						
	1376	+1659·5 −1179			1376	+1650·5 −1390		480·5	
	Total .	+480·5	9491·5		Total .	+260·5	10556·5	Total .	+5136·25
Correction for mean			−167·8	Correction for mean . . .			−49·3	Correction for mean .	−90·97
			9323·7				10507·2		
Sheppard's correction . . .			114·7	Sheppard's correction . .			114·7		+5045·28
			9209·0				10392·5		

Sheppard's corrections



Daughters.			
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−11	·5	5·5	60·5
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− 7	4·5	31·5	220·5
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− 3	99	297	891
− 2	141·5	283	566
− 1	190·5	190·5	190·5
0	212	−1179	..
1	198·5	198·5	198·5
2	159·5	319	638
3	142·5	427·5	1282·5
4	77·5	310	1240
5	36	180	900
6	19·5	117	702
7	9·5	66·5	465·5
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	Total .	+480·5	9491·5
Correction for mean			−167·8
			9323·7
Sheppard's correction . . .			114·7
			9209·0

Fathers.				Total for Daughters.	Product.
Deviation.	Frequency.				
−9	2	18	162	− 8·75	+ 78·75
−8	4·5	36	288	− 15·25	+122
−7	7·5	52·5	367·5	− 19	+133
−6	14·5	87	522	− 23	+138
−5	45	225	1125	−108·75	+543·75
−4	51·5	206	824	− 81	+324
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−2	155	310	620	− 88·50	+177
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1	199·5	199·5	199·5	+183·25	+183·25
2	166	332	664	+197·25	+394·5
3	135	405	1215	+245	+735
4	82·5	330	1320	+174·75	+699
5	36·5	182·5	912·5	+105·25	+526·25
6	20	120	720	+ 71·5	+429
7	6·5	45·5	318·5	+ 25·25	+176·75
8	4·5	36	288	+ 14·5	+116
	1376	+1650·5 −1390		480·5	
	Total .	+260·5	10556·5	Total .	+5136·25
Correction for mean . . .			−49·3	Correction for mean .	−90·97
			10507·2		
Sheppard's correction . .			114·7		+5045·28
			10392·5		

(2005) Vardeman S. B. *Sheppard's correction for variances and "the quantization noise model"*. IEEE Trans. Instr. Meas.

To multiply summations like

$$a_n = \sum_{j=0}^{n} \binom{n}{j} \left(2^{1-j} - 1\right) B_j \, h^j \, \tilde{a}_{n-j}$$

corresponding to the n-th power of $\tilde{\mu}_k + h_k \left(\iota_k + \frac{1}{2}\right)$, and then to replace products like $\tilde{\mu}_1^{s_1} \tilde{\mu}_2^{s_2} \cdots \tilde{\mu}_j^{s_j}$ with $\tilde{m}_{s_1 \ldots s_j}$.

$\alpha \equiv \tilde{\alpha} + h \left(\iota + \frac{1}{2}\right)$ ι the Bernoulli umbra

$$\mu_M \simeq \left[\tilde{\mu} + h\left(\iota + \frac{1}{2}\right)\right]_M = \prod_{k=1}^{j} \left[\tilde{\mu}_k + h_k\left(\iota_k + \frac{1}{2}\right)\right]^{t_k}$$

with $E[\mu_M] = m_{t_1 \ldots t_j} = \int_{\mathbb{R}^j} \mathbf{x} f_{\mathbf{X}}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$

(2010) Di Nardo E. *A new approach to Sheppard's corrections* Math.Meth.Stat.

(2005) Vardeman S. B. *Sheppard's correction for variances and "the quantization noise model"*. IEEE Trans. Instr. Meas.

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From univariate to multivariate ...

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From univariate to multivariate ...

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To multiply summations like

$$a_n = \sum_{j=0}^{n} \binom{n}{j} \left(2^{1-j} - 1\right) B_j \, h^j \, \tilde{a}_{n-j}$$

where $\{B_j\}$ are the *Bernoulli numbers* and h is the length of the grouping intervals and then to replace products like $\tilde{\mu}_1^{s_1} \tilde{\mu}_2^{s_2} \cdots \tilde{\mu}_j^{s_j}$ with $\tilde{m}_{s_1 \ldots s_j}$.

$$\alpha \equiv \tilde{\alpha} + h \left(\iota + \tfrac{1}{2}\right) \qquad \iota \text{ the Bernoulli umbra}$$

$$\mu_M \simeq \left[\tilde{\mu} + h\left(\iota + \frac{1}{2}\right)\right]_M = \prod_{k=1}^{j} \left[\tilde{\mu}_k + h_k\left(\iota_k + \frac{1}{2}\right)\right]^{t_k}$$

with $E[\mu_M] = m_{t_1 \ldots t_j} = \int_{\mathbb{R}^j} \mathbf{x} f_{\mathbf{X}}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$

From univariate to multivariate ...

(2005) Vardeman S. B. *Sheppard's correction for variances and "the quantization noise model"*. IEEE Trans. Instr. Meas.

To multiply summations like

$$\sum_{s_k=1}^{n} \binom{n}{s_k} \tilde{\mu}_k^{s_k} h_k^{n-s_k} \left(2^{1-n+s_k} - 1\right) B_{n-s_k}$$

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(2010) Di Nardo E. *A new approach to Sheppard's corrections* Math.Meth.Stat.

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Infinite divisibility property

A r.v. X has an *infinitely divisible distribution* if for each $n = 1, 2, \ldots$ there exist a sequence of i.i.d.r.v.'s $X_{1,n}, \ldots, X_{n,n}$ such that $X \overset{d}{=} X_{1,n} + \cdots + X_{n,n}$ where $\overset{d}{=}$ means equal in distribution.

▷

Stochastic process with independent and stationary increments

$$X_t \overset{d}{=} \underbrace{\Delta X_{t/n} + \Delta X_{t/n} + \cdots + \Delta X_{t/n}}_{n}$$

Infinite divisibility property $\Leftarrow (n.\alpha)$

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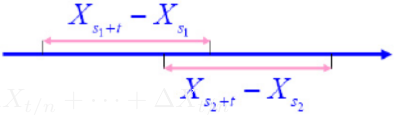
Stochastic process with independent and stationary increments



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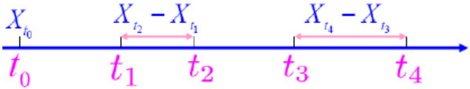

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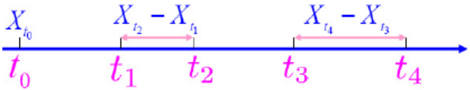

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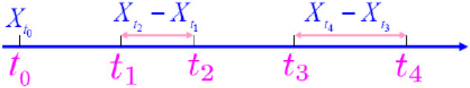

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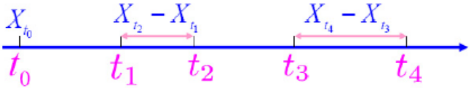

$$X_t \overset{d}{=} \underbrace{\Delta X_{t/n} + \Delta X_{t/n} + \cdots + \Delta X_{t/n}}_{n}$$

Lévy (1940) referred to a sub-class of additive processes

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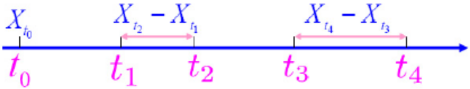

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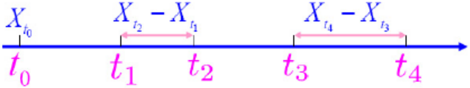

$$X_t \stackrel{d}{=} \underbrace{\Delta X_{t/n} + \Delta X_{t/n} + \cdots + \Delta X_{t/n}}_{n}$$

$$E[e^{izX_t}] = E[e^{iz\Delta X_{t/n}}]^n \Rightarrow E[e^{izX_t}] = E[e^{iz\Delta X_1}]^t$$

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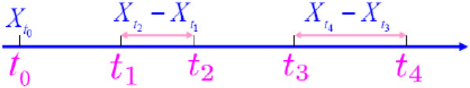
Stochastic process with independent and stationary increments



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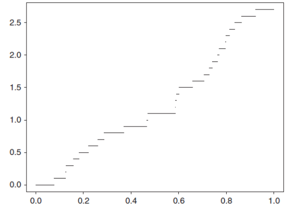
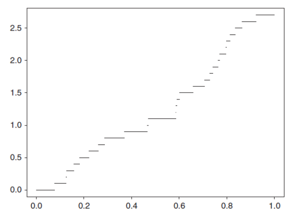


A sample path of a Poisson process



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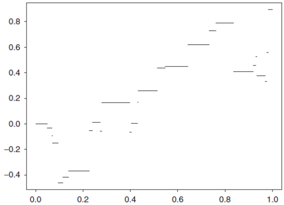
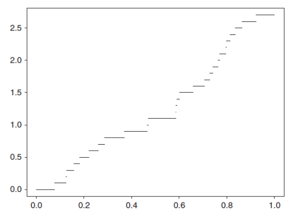


A sample path of compound Poisson process



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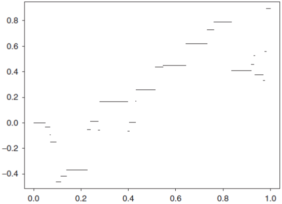

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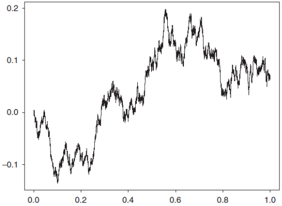
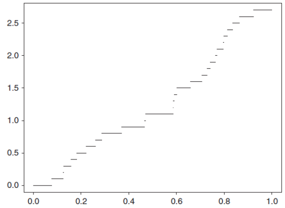


A sample path of standard Brownian motion



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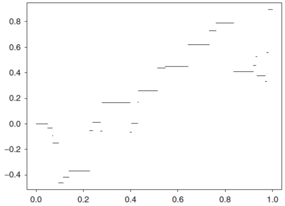

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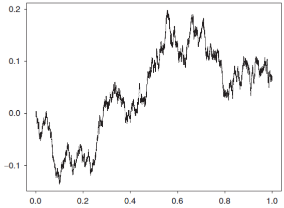

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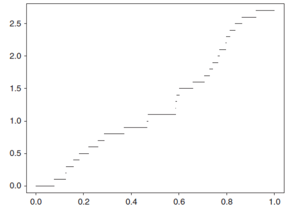

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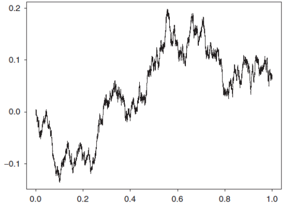

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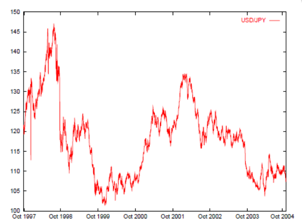
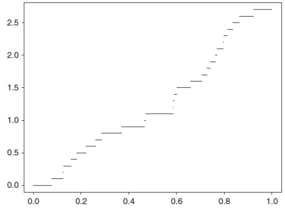


USD/JPY foreign exchange rate during October 1997-2004



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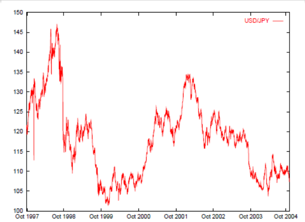
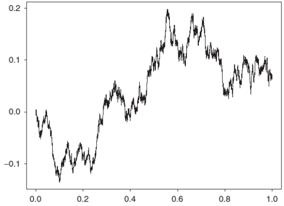
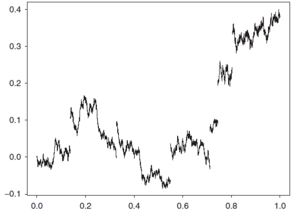


USD/JPY foreign exchange rate during October 1997-2004



A sample path of standard Brownian motion



A sample path of a summation of a Brownian motion and compound Poisson process

A new auxiliary umbra

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We shall denote by the symbol $t.\alpha$ the umbra representing the sequence $\{q_i(t)\}$.

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More issues

In any Lévy process with finite moments, the n-th moment $\mu_n(t) = E(X_t^n)$ is a *polynomial function* of t such that

$$\mu_n(t+s) = \sum_{k=0}^{n} \binom{n}{k} \mu_k(t) \mu_{n-k}(s)$$

Additivity property

If $\{W_t\}$ and $\{Z_t\}$ are two independent Lévy processes, then the process $\{X_t\}$ with $X_t = W_t + Z_t$ is a Lévy process.

$$t.(c\alpha) \equiv c(t.\alpha)$$

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If $\{X_t\}$ is a Lévy process, then $\{(X_t)_s\}$ is a Lévy process.

A drawback

A Lévy process is not necessarily a martingale.

A discrete-time martingale is a discrete-time stochastic process, i.e. a sequence of r.v.'s $\{X_i\}$, such that $E[|X_i|] < \infty$ for all i and

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Efficient markets are equated to the existence of a martingale!

▷ $E[M_t] = 1$

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A family of polynomials $\{P(x,t)\}_{t \geq 0}$ is said to be *time-space harmonic* with respect to a stochastic process $\{X_t\}_{t \geq 0}$ if $\{P(X_t,t)\}$ is a martingale.

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(2004) Hammouch H. *Umbral calculus, Martingales and Associated Polynomials.* Stoch. Anal. Appl.

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For all nonnegative integers k, the family of polynomials

$$Q_k(x,t) = E[(-t.\alpha + x)^k] \in \mathbb{R}[x]$$

is time-space harmonic with respect to a Lévy process X_t with the moments of X_1 umbrally represented by the umbra α.

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A family of polynomials $\{P(x,t)\}_{t \geq 0}$ is said to be *time-space harmonic* with respect to a stochastic process $\{X_t\}_{t \geq 0}$ if $\{P(X_t, t)\}$ is a martingale.

The inverse of an umbra:

- $-1.\alpha$ such that $-1.\alpha + \alpha \equiv \varepsilon$;
- $-t.\alpha \equiv t.(-1.\alpha)$ such that $-t.\alpha + t.\alpha \equiv \varepsilon$;

The main result

For all nonnegative integers k, the family of polynomials

$$Q_k(x,t) = E[(-t.\alpha + x)^k] \in \mathbb{R}[x]$$

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▷ $E[Q_k(X_t, t)] = 0$;

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is time-space harmonic with respect to a Lévy process X_t with the moments of X_1 umbrally represented by the umbra α.

▷ $E[Q_k(X_t,t)] = 0$; ▷ The proof involves the moments of X_t up to k.

The sequence of polynomials $\{Q_k(x,t)\}$ is an Appell sequence.

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A polynomial $P(x,t) = \sum_{j=0}^{k} p_j(t)\, x^j$, of degree k for all $t \geq 0$, is a time-space harmonic polynomial with respect to a Lévy process X_t if and only if $p_j(t) = \sum_{i=j}^{k} \binom{i}{j} p_i(0)\, E[(-t{\boldsymbol .}\alpha)^{i-j}]$, for $j = 0, \ldots, k$.

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- Krawtchouk polynomials (Random walk with $B(p)$)

Wiener-Ito stochastic integral

$$I(Y) = \int_0^\infty Y_t \, \mathsf{d}\, W_t \qquad \Leftarrow \begin{cases} Y_t & \text{a stochastic process} \\ W_t & \text{a Wiener process} \end{cases}$$

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Umbral theory of stochastic integration

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Conclusions

A different choice of foundations can lead to a different way of thinking about the subject, and thus to ask a different set of questions and to discover a different set of proofs and solutions. Thus it is often of value to understand multiple foundational perspectives at once, to get a truly stereoscopic view of the subject.

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...to be continued...

Part II

Moment symbolic calculus in probability and in statistics (II)

⑦ Dot-product of two umbrae
  The Bell umbra
  The cumulant umbra

⑧ Sheffer umbrae

⑨ Topics on Sheffer umbrae
  The Lagrange inversion formula
  Generalized Bell polynomials
  Riordan arrays
  Connection constants

⑩ Parametrizations of cumulants
  Classical cumulants
  Boolean cumulants
  A more general class of cumulants
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  Computational issues

⑪ Solving some linear recurrences

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The story of “one-line” proof

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The Bell umbra Why factorial moments?

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Given a sequence $\{a_n\}$, the sequence of its cumulants $\{c_j\}$ is such that

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$$0 = \sum_{i=1}^{n} (-1)^{i-1} (i-1)! B_{n,i}(x, x^2, \ldots, x^{n-i+1}), n \geq 2$$

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Generatingfunctionology...

First step:

- $q_n(t) = E[(t.\alpha)^n] = \sum_{i=1}^{n} (t)_i B_{n,i}(a_1, a_2, \ldots, a_{n-i+1});$

$\gamma.\alpha \Leftarrow$ Random sum

The symbol $\gamma.\alpha$ denotes the umbra representing the sequence $\{E[q_i(\gamma)]\}$.

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A new auxiliary umbra

$$\sum_{n\geq 0} a_n \frac{z^n}{n!} = \exp\left(\sum_{j\geq 1} c_j \frac{z^j}{j!}\right) \Leftrightarrow \log\left(1+\sum_{n\geq 1} a_n \frac{z^n}{n!}\right) = \sum_{j\geq 1} c_j \frac{z^j}{j!}.$$

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Dot-product of two umbrae

Random sum

Let $\{X_n\}$ be a sequence of r.v.'s and N be a nonnegative integer-valued r.v. The r.v. $S_N = X_1 + X_2 + \cdots + X_N$ is a random sum.

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- $$\begin{aligned} E[(\beta{\boldsymbol .}\alpha)^n] &= \sum_{i=1}^{n} E[(\beta)_i] B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \\ &= \sum_{i=1}^{n} B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) = Y_n(a_1, a_2, \ldots, a_n) \end{aligned}$$

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Compositional umbra

$$\omega \equiv \gamma.\beta.\alpha$$

$$c_n = \sum_{i=1}^{n} (-1)^{i-1}(i-1)! B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) = E[(?.\beta.\alpha)^n]$$

$$\sum_{i \geq 1} (-1)^{i-1}(i-1)! \frac{z^i}{i!} = \log(1+z) \quad \text{i.c.} \quad f(u,z) = \exp(z)$$

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We need an umbra corresponding to the compositional inverse of a g.f.

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$$c_n = \sum_{i=1}^{n} (-1)^{i-1}(i-1)! B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) = E[(?\boldsymbol{.}\beta\boldsymbol{.}\alpha)^n]$$

$$\sum_{i\geq 1}(-1)^{i-1}(i-1)!\frac{z^i}{i!} = \log(1+z) \quad \text{i.c.} \quad f(u,z) = \exp(z)$$

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$\alpha^{<-1>}$ is an auxiliary symbol such that

$$u{\bullet}\beta{\bullet}u^{<-1>} \equiv u^{<-1>}{\bullet}\beta{\bullet}u \equiv \chi.$$

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The cumulant umbra

The α-cumulant is the umbra κ_α such that $\kappa_\alpha \equiv \chi{\boldsymbol{.}}\alpha$.

Additivity property

$$\chi{\boldsymbol{.}}(\alpha+\gamma) \equiv \chi{\boldsymbol{.}}\alpha \dot{+} \chi{\boldsymbol{.}}\gamma$$
$$\Uparrow$$
$$c_n(X+Y) = c_n(X) + c_n(Y)$$

Homogeneity property

$$\chi{\boldsymbol{.}}(a\alpha) \equiv a(\chi{\boldsymbol{.}}\alpha)$$
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$$c_n(aX) = a^n\, c_n(X)$$

A recurrence relation

$$\alpha^n \simeq \kappa_\alpha(\kappa_\alpha+\alpha)^{n-1}$$
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Semi-invariance property

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$$\begin{cases} c_1(X+a) &= c_1(X)+a \\ c_n(X+a) &= c_n(X), \; n \geq 2 \end{cases}$$

(2006) Giovanni P. and Wynn H.P. *Cumulant varieties* Journal of Symbolic Computation

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Cumulants or factorial moments?

$$\boxed{\chi{\boldsymbol .}\beta \equiv u \equiv \beta{\boldsymbol .}\chi} \qquad \Rightarrow \boxed{(\alpha{\boldsymbol .}\chi)^n \simeq (\alpha)_n}$$

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Compound Poisson process

Let $\{X_n\}$ be a sequence of r.v.'s and $N(t)$ a Poisson r.v. of parameter t. The r.v. $S_N = X_1 + X_2 + \cdots + X_N$ is a compound Poisson r.v.

$$t.\alpha \equiv t.\beta.\kappa_\alpha \Rightarrow \boxed{f(t.\alpha, z) = \{\exp\left[f(\kappa_\alpha, z) - 1\right]\}^t}$$

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A polynomial umbra σ_x is said to be a Sheffer umbra for (α, γ) if

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Set $p_k(x) = E[(x_{\bullet}\gamma^*)^k]$ then $p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$.

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$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, p_{n-k}(y).$$

$$\boxed{\sigma_x^{(\varepsilon,\gamma)} \equiv x \boldsymbol{.} \gamma^* \text{ (associated to } \gamma)}$$

$$(x+y) \boldsymbol{.} \gamma^* \equiv x \boldsymbol{.} \gamma^* + y \boldsymbol{.} \gamma^* \text{ Binomial identity}$$

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Set $p_k(x) = E[(x \boldsymbol{.} \gamma^*)^k]$ then $p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y)$.

- $x \boldsymbol{.} \chi^* \equiv x \boldsymbol{.} u \Rightarrow p_k(x) = x^k$;
- $x \boldsymbol{.} u^* \equiv x \boldsymbol{.} \chi \Rightarrow p_k(x) = (x)_k$;
- $x \boldsymbol{.} (u^{<-1>})^* \equiv x \boldsymbol{.} \beta \Rightarrow p_k(x) = \Phi_k(x)$

Generalized Sheffer identity

A polynomial umbra σ_x is a Sheffer umbra iff there exists an umbra γ, provided with a compositional inverse, such that

$$\sigma_{\eta+\zeta} \equiv \sigma_\eta + \zeta{\boldsymbol .}\gamma^*, \quad \text{for any } \eta, \zeta \in A.$$

$$s_n(\eta+\zeta) \simeq \sum_{k=0}^n \binom{n}{k} s_k(\eta)\, p_{n-k}(\zeta).$$

$$\boxed{\sigma_x^{(\varepsilon,\gamma)} \equiv x{\boldsymbol .}\gamma^* \ (\text{associated to } \gamma)}$$

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$$\boxed{\sigma_x^{(\alpha,\chi)} \equiv \alpha + x \qquad (\text{Appell umbra})}$$

$$\sigma_{x+y}^{(\alpha,\chi)} \equiv \alpha + (x+y){\boldsymbol .}u \equiv \sigma_x^{(\alpha,\chi)} + y{\boldsymbol .}u \qquad \text{Appell identity}$$

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Set $q_k(x) = E[(\alpha + x{\boldsymbol .}u)^k]$ then $q_n(x+y) = \sum_{k=0}^n \binom{n}{k} q_k(x) y^{n-k}$.

$\sigma_{\gamma+x{\boldsymbol .}u} \equiv \chi + \sigma_x \Rightarrow$ $\sigma^n_{\chi+x{\boldsymbol .}u} - \sigma^n_x \simeq n\sigma^{n-1}_x$

$$D_x\left[(\alpha + x{\boldsymbol .}u)^n\right] \simeq (\alpha + \chi + x{\boldsymbol .}u)^n - (\alpha + x{\boldsymbol .}u)^n$$

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$$\tfrac{d}{dx} q_n(x) = n q_{n-1}(x) \quad n = 1, 2, \ldots . \ (\text{Appell propriety})$$

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Umbral Abel polynomials

$$x(x - n.\gamma)^{n-1}$$

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Derivative umbra

The derivative umbra α_{D} is such that $(\alpha_{D})^n \simeq n\alpha^{n-1}(\partial_\alpha \alpha^n)$ for $n = 1, 2, \ldots$.

$E[\alpha_D] = 1 \neq 0$

Abel representation of binomial sequences

$(x.\gamma_{D}^{*})^n \simeq x(x - n.\gamma)^{n-1}$, for $n = 1, 2, \ldots$.

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$$g_1 = 1 \Rightarrow (\gamma_{_D}^{<-1>})^n \simeq (-n . \gamma)^{n-1}$$

Umbral Abel polynomials

$$x(x - n{\boldsymbol .}\gamma)^{n-1}$$

(2009) Petrullo, P. *A Symbolic Treatment of Abel Polynomials.* From Combinatorics to Philosophy: the legacy of G.C. Rota. (E. Damiani et al. eds)

Derivative umbra

The derivative umbra α_{D} is such that $(\alpha_{D})^{n} \simeq n\alpha^{n-1}(\partial_{\alpha}\alpha^{n})$ for $n = 1, 2, \ldots$.
$E[\alpha_{D}] = 1 \neq 0$

Abel representation of binomial sequences

$(x{\boldsymbol .}\gamma_{D}^{*})^{n} \simeq x(x - n{\boldsymbol .}\gamma)^{n-1}$, for $n = 1, 2, \ldots$.

$$g_1 = 1 \Rightarrow (\gamma_{D}^{<-1>})^{n} \simeq (-n{\boldsymbol .}\gamma)^{n-1}$$

$(g_1 \neq 1) \Rightarrow \gamma^{{\boldsymbol .}n}(\gamma^{<-1>})^{n} \simeq (-n{\boldsymbol .}\tilde{\gamma})^{n-1}$ with $E[\tilde{\gamma}^{n}] = \frac{g_n}{g_1(n+1)}$

Abel identity - Generalized Bell polynomials

$$(x+y)^n \simeq \sum_{k\geq 0}\binom{n}{k}(x+k.\gamma)^{n-k}\; y(y-k.\gamma)^{k-1}$$

$$(x+\beta.\gamma_D)^n \simeq \sum_{k\geq 0}\binom{n}{k}(x+k.\gamma)^{n-k}.$$

Umbral representation of Bell exponential polynomials

$$(x.\beta.\gamma_D)^n \simeq \sum_{k\geq 0}\binom{n}{k}(k.\gamma)^{n-k}x^k$$

$$\boxed{s(n,k) \simeq \tbinom{n}{k}(k.\iota.\chi)^{n-k}} \qquad \boxed{S(n,k) \simeq \tbinom{n}{k}(-k.\iota)^{n-k}}$$

Closed formulae for Stirling numbers

Umbral representation of Bell exponential polynomials

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(2011) Petrullo, P. *Outcomes of the Abel Identity.* ArXiv

Umbral representation of Bell exponential polynomials

$$(x.\beta.\gamma_D)^n \simeq \sum_{k\geq 0} \binom{n}{k} (k.\gamma)^{n-k} x^k \simeq \sum_{k\geq 0}$$ [illegible]

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Umbral representation of Sheffer polynomials via generalized Bell polynomials

If $\{s_n(x)\}$ are moments of a Sheffer umbra for (α, γ), then for all nonnegative integers n

$$s_n(x) \simeq \sum_{k=0}^{n} \binom{n}{k} \delta^{.k} (\alpha + k.\tilde{\delta})^{n-k} x^k \quad \text{with} \quad \delta \equiv \gamma^{<-1>}.$$

Umbral representation of exponential Riordan arrays

The elements of the exponential Riordan array $(f(\alpha, t), f^{<-1>}(\gamma, t) - 1)$ are umbrally represented by

$$d_{n,k} \simeq \binom{n}{k} \delta^{.k} (\alpha + k.\tilde{\delta})^{n-k} \quad \text{with} \quad \delta \equiv \gamma^{<-1>}.$$

An (exponential) *Riordan array* is a pair $(g(t), f(t))$ of (exponential) formal power series, where $g(t)$ is an invertible series and $f(0) = 0$.

The pair defines an infinite lower triangular array according to the rule:

$d_{n,k}$ = n-th coefficient of $g(t)\frac{f(t)^k}{k!}$ for $0 \leq k \leq n$

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Umbral representation of connection constants

Let $\{s_n(x)\}$ be the moments of a Sheffer umbra for (α, γ) and $\{r_n(x)\}$ be the moments of a Sheffer umbra for (δ, ζ). The connection constants $\{c_{n,k}\}$ are such that

$$c_{n,k} \simeq \binom{n}{k} \xi^{.k} (\varsigma + k.\xi)^{n-k}$$

with $\varsigma \equiv (-1.\alpha.\beta.\gamma + \delta.\beta.\zeta).\zeta^{<-1>}$ and $\xi \equiv \zeta.\gamma^{<-1>}$.

$\Uparrow$

If $\{s_n(x)\}$ and $\{r_n(x)\}$ are Sheffer sequences, the constants $c_{n,k}$ in the expression $r_n(x) = \sum_{k=0}^{n} c_{n,k} s_k(x)$ are known as *connection constants*.

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More issues

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Umbral Abel polynomials

$$x(x+m.\alpha)^{n-1}$$

Is it possible to “invert” the recurrence relation?

$$\kappa_{\alpha}^{n} \simeq (\chi.\alpha)^{n} \simeq (u^{<-1>}.\beta.\alpha)^{n} \simeq \alpha(\alpha-1.\alpha)^{n-1} \simeq \mathfrak{a}_{n}^{(-1)}(\alpha,\alpha)$$

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First Abel Inversion Theorem

$\alpha^{n} \simeq \mathfrak{a}_{n}^{(m)}(\gamma,\gamma)$

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Within stochastic differential equations, boolean cumulants are also known as "partial cumulants ".

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$\bar{\eta}_{\alpha} \equiv \bar{u}^{<-1>}.\beta.\bar{\alpha}$ compare with $\kappa_{\alpha} \equiv \chi.\alpha \equiv u^{<-1>}.\beta.\alpha$

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Homogeneity $\bar{\eta}_{c\alpha} \equiv c\bar{\eta}_{\alpha}$

Additivity $\bar{\eta}_{\xi} \equiv \bar{\eta}_{\alpha} \dot{+} \bar{\eta}_{\gamma} \Leftrightarrow -1.\bar{\xi} \equiv -1.\bar{\alpha} \dot{+} -1.\bar{\gamma}$

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$E[(\bar{u}.\beta.\bar{u})^n] = 2^{n-1} = |\mathcal{I}_n|$ compare with $E[\beta^n] = B_n = |\Pi_n|$

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A more general class of cumulants

Abel-type cumulant of α

An Abel-type cumulant $c_{n,m}(\alpha)$ of α is such that $c_{n,m}(\alpha) \simeq \mathfrak{a}_n^{(-m)}(\alpha,\alpha) \simeq \alpha(\alpha - m.\alpha)^{n-1}$ for all $n, m \geq 1$.

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(2010) Di Nardo E., Petrullo P. and Senato D. *Cumulants and convolutions via Abel polynomials.* Europ. Jour. Combinatorics.

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What about the others columns?

- (2000) Torney D. C. *Binary cumulants* Adv.Appl.Math.
- (2011) Zwiernik P. and Smith J. *Tree-cumulants and the geometry of binary tree models* Bernoulli

Π_T lattice of tree partitions: let $T=(V,E)$ be a fixed tree with n leaves labelled by $[n]$. Removing a subset of edges E' induces a forest. Restricting $[n]$ to the connected components of this forest, gives a tree partition induced by T.

- (2011) Zwiernik P. L-cumulants, L-cumulants embeddings and algebraic statistics ArXiv

$E[(X-a_1)^n]$ central moments are cumulants of $\{a_n\}_n$ on the $C([n])$ lattice of one-cluster partitions: a partition $\pi \in \Pi_n$ is a one-cluster partition if at most one block of π has size greater than one.

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Cyclic cumulants

Set $\hat{\alpha} \equiv \alpha \kappa_{\bar{u}}$, where $\kappa_{\bar{u}} \equiv \chi . \bar{u}$ is such that $E[(\kappa_{\bar{u}})^n] = (n-1)!$.

On the symmetric group

$$(\hat{\gamma}.\beta.\hat{\alpha})^n \simeq \sum_{\sigma \in S_n} [\nu(\sigma) - 1]! \gamma^{\nu(\sigma)} \alpha_{C(\sigma)}$$

with $\nu(\sigma)$ = number of cycles of σ, $\alpha_{C(\sigma)}$ is a symbol denoting the product $\alpha_{j_1}^{l(c_1)} \alpha_{j_2}^{l(c_2)} \cdots$ with $j_1, j_2, \ldots \in [n]$ and $l(c)$ = length of the cycle c.

(classical) $\kappa_\alpha \equiv u^{<-1>}.\beta.\alpha \Leftrightarrow \alpha \equiv u.\beta.\kappa_\alpha$. (boolean) $\bar{\eta}_\alpha \equiv \bar{u}^{<-1>}.\beta.\bar{\alpha} \Leftrightarrow \bar{\alpha} \equiv \bar{u}.\beta.\bar{\eta}_\alpha$.

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Cumulants of a Wishart matrix $\Rightarrow E[(\beta.\hat{\alpha})^n] = C_n(a_1, a_2, \ldots, a_n)$ *cycle index polynomial of* S_n

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- Additivity $\hat{\delta}_{\zeta} \equiv \hat{\delta}_{\alpha} \dot{+} \hat{\delta}_{\gamma} \Leftrightarrow -1.\beta.\hat{\zeta} \equiv -1.\beta.\hat{\alpha} \dot{+} -1.\beta.\hat{\gamma}$
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$$C_\alpha = \begin{pmatrix} \mathfrak{a}_1^{(-1)}(\alpha,\alpha) & \mathfrak{a}_1^{(-2)}(\alpha,\alpha) & \ldots \\ \mathfrak{a}_2^{(-1)}(\alpha,\alpha) & \mathfrak{a}_2^{(-2)}(\alpha,\alpha) & \ldots \\ \vdots & \vdots & \\ \mathfrak{a}_n^{(-1)}(\alpha,\alpha) & \mathfrak{a}_n^{(-2)}(\alpha,\alpha) & \ldots \\ \vdots & \vdots & \end{pmatrix}$$

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$$C_{\alpha} = \begin{pmatrix} \mathfrak{a}_1^{(-1)}(\alpha,\alpha) & \mathfrak{a}_1^{(-2)}(\alpha,\alpha) & \ldots \\ \mathfrak{a}_2^{(-1)}(\alpha,\alpha) & \mathfrak{a}_2^{(-2)}(\alpha,\alpha) & \ldots \\ \vdots & \vdots & \\ \mathfrak{a}_n^{(-1)}(\alpha,\alpha) & \mathfrak{a}_n^{(-2)}(\alpha,\alpha) & \ldots \\ \vdots & \vdots & \end{pmatrix}$$

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Free cumulants

If $M(z) = 1 + \sum_{n \geq 1} m_n z^n$ is the (formal) ordinary moment generating function of X, the *noncrossing (or free) cumulants* of X are the coefficients r_n of the ordinary power series $R(t) = 1 + \sum_{n \geq 1} r_n z^n$ such that $M(z) = R[z\, M(z)]$.

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(1994) Speicher R. *Multiplicative functions on the lattice of non-crossing partitions and free convolution* Math. Ann.

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To find ϕ s.t. $a = \phi(za) \Leftrightarrow$ to find f s.t. $f(za) = z$ with $f(z) = z/\phi(z)$.

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$$\Rightarrow \bar{\alpha} \equiv \bar{\mathfrak{K}}_\alpha . \beta . \bar{\alpha}_D$$

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Free cumulant umbra

The umbra $\bar{\mathfrak{K}}_{\alpha}$ such that $(-1.\bar{\mathfrak{K}}_{\alpha})_D \equiv \bar{\alpha}_D^{<-1>}$ is called the *free cumulant umbra of* α.

$$\bar{\mathfrak{K}}_{\alpha} \equiv \bar{\alpha}.\beta.\bar{\alpha}_D^{<-1>} \text{ and } \bar{\alpha} \equiv \bar{\mathfrak{K}}_{\alpha}.\beta.(-1.\bar{\mathfrak{K}}_{\alpha})_D^{<-1>}$$

Compare with

$$\text{(classical) } \kappa_{\alpha} \equiv u^{<-1>}.\beta.\alpha \Leftrightarrow \alpha \equiv u.\beta.\kappa_{\alpha}.$$

Abel parametrization

$$\bar{\mathfrak{K}}_{\alpha}^n \simeq \mathfrak{a}_n^{(-n)}(\bar{\alpha}, \bar{\alpha}) \simeq \bar{\alpha}(\bar{\alpha} - n.\bar{\alpha})^{n-1}$$

Lagrange Involution: If we define the umbra δ_P such that

$$(\delta_P)^n \simeq \frac{\delta^{n+1}}{n+1} \Rightarrow$$

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$$\boxed{\bar{\alpha}^n \simeq \bar{\mathfrak{K}}_\alpha(\bar{\mathfrak{K}}_\alpha + n\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)^{n-1} \simeq V_n(\bar{\mathfrak{K}}_1, \bar{\mathfrak{K}}_2, \ldots, \bar{\mathfrak{K}}_n)}$$

Compare with $\alpha^n \simeq Y_n(\kappa_1, \kappa_2, \ldots, \kappa_n)$

Catalan umbra $\varsigma^n \simeq \mathcal{C}_n$.

The *Catalan umbra* is the unique umbra ς such that $\bar{\mathfrak{K}}_{\varsigma} \equiv u$, that is $\bar{\varsigma} \equiv \bar{u}{\boldsymbol .}\beta{\boldsymbol .}(-1{\boldsymbol .}\bar{u})_D^{<-1>}$.

The n-volume polynomial $V_n(x)$,

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The proposed algorithm relied on a suitable expansion of the umbral polynomial $\gamma(\gamma + \delta.\gamma)^{i-1}$ for $i = 1, 2, \ldots$ and δ, γ umbrae.

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The proposed algorithm relied on a suitable expansion of the umbral polynomial $\gamma(\gamma + \delta.\gamma)^{i-1}$ for $i = 1, 2, \ldots$ and δ, γ umbrae.

$$\boxed{\gamma(\gamma + \delta.\gamma)^{i-1} \simeq \sum_{\lambda \vdash i} (\delta)_{\nu_\lambda - 1} \, d_\lambda \, \gamma_\lambda}$$

(2009) Di Nardo E. and Oliva I. *On the computation of classical, boolean and free cumulants* Appl. Math. Comp.

Computing cumulants and moments

(2007) Bryc W. *Computing moments of free additive convolution of measures.* Appl. Math. Comp.

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If δ^i are known, then use $(\delta)_i \simeq \sum_{k=1}^{i} s(i,k)\delta^k$, with $\{s(i,k)\}$ the Stirling numbers of I kind.

(2009) Di Nardo E. and Oliva I. *On the computation of classical, boolean and free cumulants* Appl. Math. Comp.

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i	`MAPLE` (umbral)	`MAPLE` (Bryc)
15	0.015	0.016
18	0.031	0.062
21	0.078	0.141
24	0.172	0.266
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Table 2: Free cumulants in terms of moments

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Sheffer Polynomials. In: Encyclopedia of Statistical Sciences. Niederhausen H.

Ex: We ask for the solution of the difference equation

$\boxed{q_n(x+1) = q_n(x) + q_{n-1}(x)}$ under the condition $\boxed{\int_0^1 q_n(x)dx = 1}$

for all nonnegative integers n. Set $q_n(x) = s_n(x)/n!$. Then

$$s_n(x+1) = s_n(x) + n s_{n-1}(x).$$

Compare with $\sigma_{\gamma + x.u} \equiv \chi + \sigma_x \quad \Rightarrow s_n(\gamma + x.u) = s_n(x) + n s_{n-1}(x)$, then $s_n(x) = E[(\sigma_x^{(\alpha,u)})^n]$. Since $\chi \equiv u^*$

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A ballot path

A ballot path takes up steps (*u*) and right steps (*r*), starting at the origin and staying weakly above the diagonal.

Ex: *ururuur* is a ballot path to $(3, 4)$.

$D(n, m) =$ the number of ballot paths to (n, m) such that

- no path goes below the diagonal (Dyck paths);
- no path contains the pattern (substring) *urru*.

m	1	7	22	46	82	132
6	1	6	16	29	46	63
5	1	5	11	17	23	23
4	1	4	7	9	9	
3	1	3	4	4		
2	1	2	2			
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from the Sheffer identity with $y = -1$. Recurrence (1) can be rewritten as

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from the Sheffer identity with $y = -1$. Recurrence (1) can be rewritten as

$$(\sigma_{x-1} + \gamma^* - \chi)^n \simeq \sigma_{x-1}^n + n \sum_{i=0}^{n-1} \binom{n-1}{i} \sigma_{x-1}^{n-i-1} \left[\frac{(\gamma^*)^{i+1}}{i+1} - (\gamma^*)^i \right]$$

by which a characterization of the moments of γ^* is available.

$D(n,m)$ is the solution of the following recurrence relation

$$\boxed{D(n,m) - D(n-1,m) = D(n,m-1) - D(n-2,m-1) + D(n-3,m-1)}$$

under the *initial condition* $\left\{ \begin{array}{lcl} D(n,n) & = & D(n-1,n) \\ D(0,0) & = & 1 \end{array} \right.$

Set $D(n,m) = s_n(m)/n!$ then

$$s_n(m) - n s_{n-1}(m) = s_n(m-1) - (n)_2 \, s_{n-2}(m-1) + (n)_3 \, s_{n-3}(m-1),$$

with the initial condition $s_n(n) = n s_{n-1}(n)$. Replace m with x

$$s_n(x) - n s_{n-1}(x) = s_n(x-1) - (n)_2 \, s_{n-2}(x-1) + (n)_3 \, s_{n-3}(x-1), \qquad (1)$$

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$$f(\gamma^*, t) = 1 + t + \sum_{k \geq 3} t^k = \frac{1 - t^2 + t^3}{1 - t} \Rightarrow f(x_{\bullet}\gamma^*, t) = \left[\frac{1 - t^2 + t^3}{1 - t}\right]^x.$$

For the umbra α, the initial condition $s_n(n) = n s_{n-1}(n)$, in umbral terms gives

$$(\alpha + n_{\bullet}\gamma^*)^n \simeq n(\alpha + n_{\bullet}\gamma^*)^{n-1}.$$

Assume $\alpha \equiv \bar{u} + \zeta$. Then

$$(\zeta + n_{\bullet}\gamma^*)^n \simeq \epsilon^n \Rightarrow (n_{\bullet}\gamma^*)^n \simeq \sum_{k=1}^{n} \binom{n}{k} (-\zeta^k)(n_{\bullet}\gamma^*)^{n-k}.$$

Since

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$$(n_{\bullet}\gamma^*)^n \simeq \sum_{k=1}^{n} \binom{n}{k} k[\gamma^{<-1>}]^k (n_{\bullet}\gamma^*)^{n-k}$$

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The following result turns out to be useful in solving the class of recursions involving Sheffer sequences with the initial condition $s_n(-c\,n) = \delta_{0,n}$, with $c \in R$.

If γ is an umbra with $E[\gamma] = 1$, then for $n \geq 1$ we have

$$x(\chi{\bf .}c{\bf .}\beta{\bf .}\eta_{D} + x{\bf .}\gamma^{*})^{n} \simeq (x + c\,n)(x{\bf .}\gamma^{*})^{n}$$

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- (2010) Niederhausen H. *Finite Operator Calculus With Applications to Linear Recursions* www.math.fau.edu/Niederhausen

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Conclusions

The following result turns out to be useful in solving the class of recursions involving Sheffer sequences with the initial condition $s_n(-c\,n) = \delta_{0,n}$, with $c \in R$.

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Part III

Moment symbolic calculus in probability and in statistics (III)

⑫ **k-statistics**
  - A new formula
  - Polykays

⑬ **Multivariate r.v.'s**
  - Joint cumulants
  - Multivariate cumulants
  - Multivariate polykays

⑭ **Multivariate Faà di Bruno Formula**
  - Partitions of a multi-index
  - Multivariate Hermite polynomials
  - Multivariate TSH polynomials

⑮ **Work in progress**
  - Non-asymptotic theory of random matrices
  - Wishart distribution
  - Spectral polykays

Computational results

PC Pentium(R)4, Intel(R) CPU 2.08 Ghz, 512MB Ram

Maple 10.0

Mathematica 4.2

Times in seconds

k-statistics	A&S	MathStat I	Umbral	MathStat II*
k_5	$0,06$	$0,01$	$0,01$	$0,008$
k_7	$0,31$	$0,02$	$0,01$	$0,017$
k_9	$1,44$	$0,04$	$0,01$	$0,039$
k_{11}	$8,36$	$0,14$	$0,01$	$0,084$
k_{14}	$396,39$	$0,64$	$0,02$	$0,329$
k_{16}	$57982,4$	$2,63$	$0,08$	$0,917$
k_{18}	$-$	$6,90$	$0,16$	$2,804$
k_{20}	$-$	$25,15$	$0,33$	$9,363$
k_{22}	$-$	$81,70$	$0,80$	$32,11$
k_{24}	$-$	$359,40$	$1,62$	$\ldots$
k_{26}	$-$	$1581,05$	$2,51$	$\ldots$
k_{28}	$-$	$6505,45$	$4,83$	$\ldots$

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k_{11}	8, 36	0, 14	0, 01	0, 084
k_{14}	396, 39	0, 64	0, 02	0, 329
k_{16}	57982, 4	2, 63	0, 08	0, 917
k_{18}	–	6, 90	0, 16	2, 804
k_{20}	–	25, 15	0, 33	9, 363
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* (2008) Rose C. *MathStatica: a symbolic approach to computational mathematical statistics.* (Singapore)

k-statistics

Definition

The n-th k-statistic k_n is the unique symmetric unbiased estimator of the cumulant c_n of a given statistical distribution, that is $E[k_n] = c_n$.

- $k_1 \Rightarrow$
- $\Leftarrow k_2$
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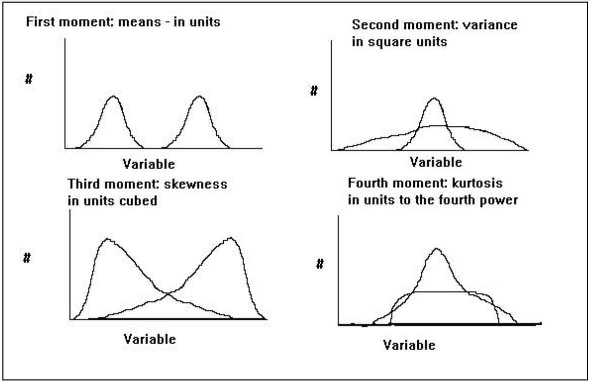

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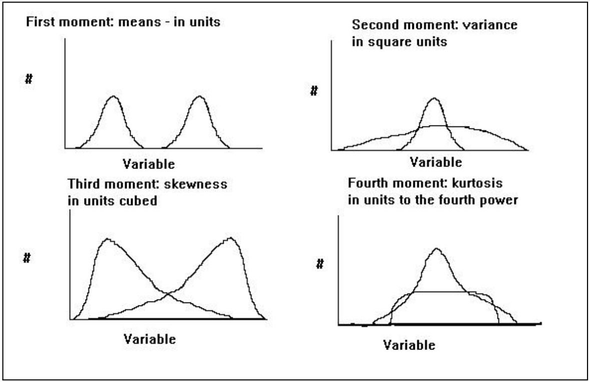

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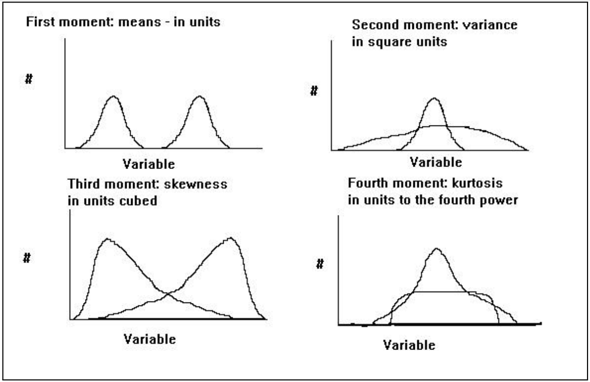
- $k_3 \Rightarrow$
- $\Leftarrow k_4$



k-statistics

Definition

The n-th k-statistic k_n is the unique symmetric unbiased estimator of the cumulant c_n of a given statistical distribution, that is $E[k_n] = c_n$.

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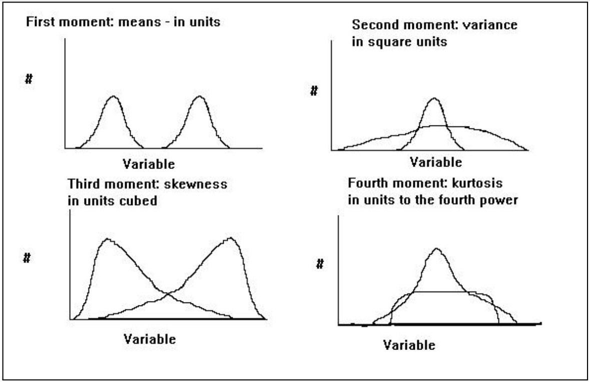

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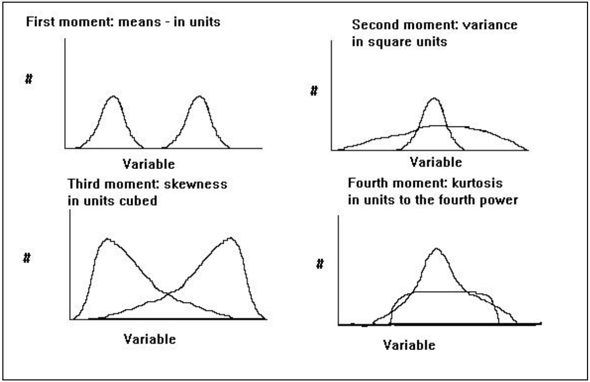
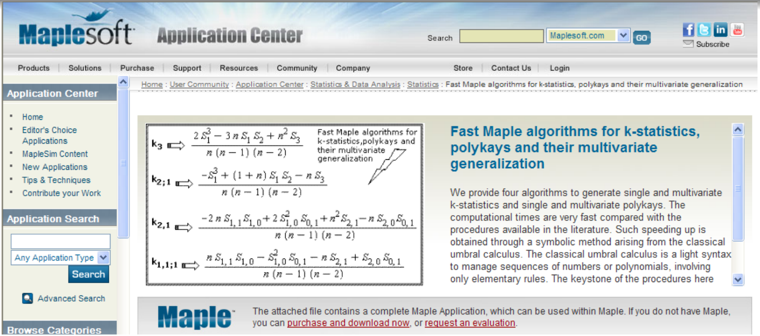


- $\Leftarrow k_2$
- $\Leftarrow k_4$

The algorithm



$$k_3 \Longrightarrow \frac{2\,S_1^3 - 3\,n\,S_1\,S_2 + n^2\,S_3}{n\,(n-1)\,(n-2)}$$

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Fast Maple algorithms for k-statistics,polykays and their multivariate generalization

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We provide four algorithms to generate single and multivariate k-statistics and single and multivariate polykays. The computational times are very fast compared with the procedures available in the literature. Such speeding up is obtained through a symbolic method arising from the classical umbral calculus. The classical umbral calculus is a light syntax to manage sequences of numbers or polynomials, involving only elementary rules. The keystone of the procedures here

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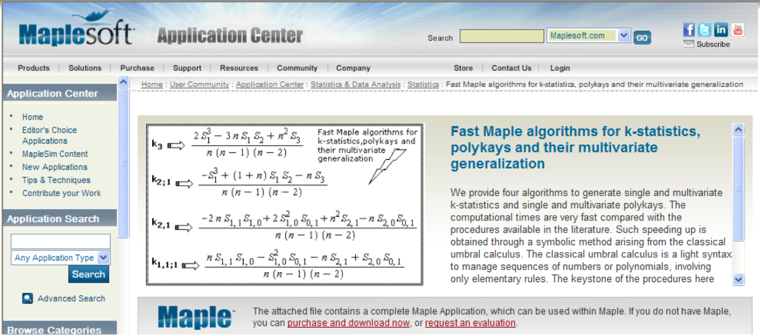

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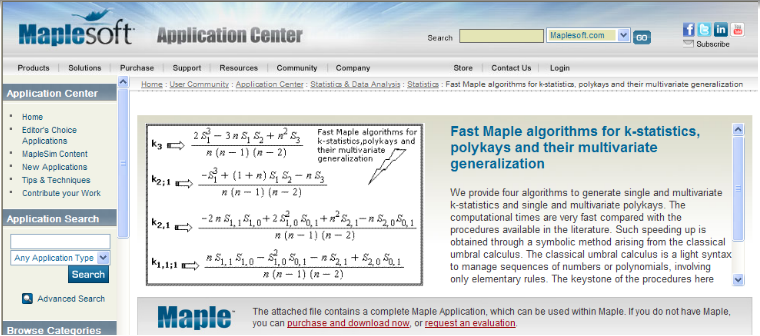

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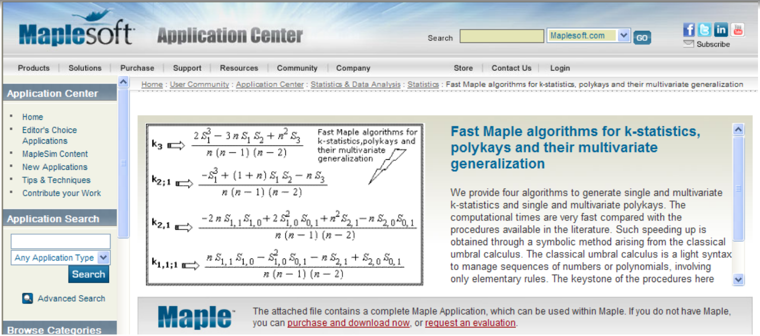

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Examples

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k_1 &= \frac{S_1}{n} \\
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k-statistics

The nice formula: cumulants in terms of moments

If $r_1 + 2r_2 + \cdots + mr_m = i$ and $r_1 + r_2 + \cdots + r_m = l$, then

$$c_i = i! \sum_{m=1}^{i} \sum \frac{(-1)^{l-1}(l-1)!}{r_1! r_2! \cdots r_m!} \frac{a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}}{(1!)^{r_1}(2!)^{r_2} \cdots (m!)^{r_m}}$$

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▷ Too heavy by a computational point of view!

AugToPs

PC
Pentium(R)4,
Intel(R)
CPU 2.08
Ghz, 512MB
Ram
Maple 10.0
Mathematica
4.2
Times in
seconds

k-statistics	A&S	MathStat I	Umbral	AugToPs
k_5	$0,06$	$0,01$	$0,01$	$0,08$
k_7	$0,31$	$0,02$	$0,01$	$0,03$
k_9	$1,44$	$0,04$	$0,01$	$0,16$
k_{11}	$8,36$	$0,14$	$0,01$	$0,23$
k_{14}	$396,39$	$0,64$	$0,02$	$1,33$
k_{16}	$57982,4$	$2,63$	$0,08$	$4,25$
k_{18}	–	$6,90$	$0,16$	$13,70$
k_{20}	–	$25,15$	$0,33$	$42,26$
k_{22}	–	$81,70$	$0,80$	$172,59$
k_{24}	–	$359,40$	$1,62$	$647,56$
k_{26}	–	$1581,05$	$2,51$	$3906,19$
k_{28}	–	$6505,45$	$4,83$	$21314,65$

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A speeder way of computing = a new formula and a new insight

$$\kappa_i = E[(C_{1,Z} + \cdots + C_{n,Z})^i]$$

- $\{C_{j,y}\}_{j=1}^{n}$ are i.i.d. cumulant r.v.'s of randomized compound Poisson r.v.'s with parameter Z;
- $E[Z^m] = (-1)^{m-1}(m-1)!/(n)_m$ for $m = 0, 1, \ldots, n$;

The new formula...written in *“umbral” way*

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The umbral algorithm

$$(\chi\centerdot\alpha)^i \simeq \sum_{\lambda\vdash i} d_\lambda p_\lambda\left(\frac{\chi\centerdot\chi}{n\centerdot\chi}\right)(n\centerdot\alpha)^{r_1}(n\centerdot\alpha^2)^{r_2}\cdots,$$

where $\lambda=(1^{r_1},2^{r_2},\ldots)\vdash i$ and $p_\lambda(y)=[p_1(y)]^{r_1}[p_2(y)]^{r_2}\cdots$ and

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where $S(n,k)$ are the Stirling numbers of second kind (ϕ_n are the exponential umbral polynomials)

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- $(\chi\boldsymbol{.}y\boldsymbol{.}\beta)^i \simeq y$ for $i = 1, 2, \ldots$.
- $(\chi\boldsymbol{.}y\boldsymbol{.}\beta)\boldsymbol{.}\alpha \equiv \chi\boldsymbol{.}(y\boldsymbol{.}\beta\boldsymbol{.}\alpha)$ (cumulant of compound Poisson r.v.)
- $E[(n\boldsymbol{.}\chi\boldsymbol{.}(y\boldsymbol{.}\beta\boldsymbol{.}\alpha))^i] = c_i(y) = \sum_{\lambda \vdash i} y^{\nu_\lambda} (n)_{\nu_\lambda} d_\lambda a_\lambda$
- $(\chi\boldsymbol{.}\alpha)^i \simeq c_i\left(\frac{\chi\boldsymbol{.}\chi}{n\boldsymbol{.}\chi}\right), i = 1, 2, \ldots$. this means $y^k \leftarrow \frac{(-1)^{k-1}(k-1)!}{(n)_k}$
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- $\chi\boldsymbol{.}(\gamma_1 x_1 + \cdots + \gamma_n x_n) \equiv (\chi\boldsymbol{.}\gamma)\sigma$, where σ is the power sum polynomial umbra and $\{\gamma_i\}_{i=1}^n$ are uncorrelated umbrae similar to an umbra γ.
- $(\gamma_1 x_1 + \cdots + \gamma_n x_n)^i \simeq \sum_{\lambda \vdash i} d_\lambda (\chi\boldsymbol{.}\gamma)_\lambda \sigma_\lambda. \simeq \sum_{\lambda \vdash i} d_\lambda (\chi\boldsymbol{.}\gamma)_\lambda s_1^{r_1} s_2^{r_2} \cdots$
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- ▷ $(\chi . y . \beta)^i \simeq y$ for $i = 1, 2, \ldots$.
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*If $a_n \neq 0$ for all nonnegative integers n, then by the symbol $1/\alpha$ we denote the umbra whose moments are $1/a_n$ (inverse moments).

A sketch of the proof

- ▷ $(\chi.y.\beta)^i \simeq y$ for $i = 1, 2, \ldots.$
- ▷ $(\chi.y.\beta).\alpha \equiv \chi.(y.\beta.\alpha)$ (cumulant of compound Poisson r.v.)
- ▷ $E[(n.\chi.(y.\beta.\alpha))^i] = c_i(y) = \sum_{\lambda \vdash i} y^{\nu_\lambda} (n)_{\nu_\lambda} d_\lambda a_\lambda$
- ▷ $(\chi.\alpha)^i \simeq c_i\left(\dfrac{\chi.\chi}{n.\chi}\right), i = 1, 2, \ldots.$ this means $y^k \leftarrow \dfrac{(-1)^{k-1}(k-1)!}{(n)_k}$
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Polykays

Definition

A polykay $k_{r,\ldots,s}$ is an unbiased estimator of a product of cumulants, that is $E[\kappa_{r,\ldots,s}] = c_r \cdots c_s$.

" This is an attempt to present as simple as possible the best tools we know today for keeping computations simple when dealing with samples from general populations. Such computations seem inevitably be made in terms of two systems of multi-index quantities: moments and cumulants. (...) Every polynomial symmetric function can also be expressed in terms of polykays" "a natural bases"

(1956) Tukey J. *Keeping moment-like sample computations simple.* Ann.Math.Stat.

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Being a product of cumulants, the umbral expression of a polykay is

Umbral polykays

$$c_r \cdots c_s = E[k_{r,\ldots,s}] = E[(\chi.\alpha)^r \cdots (\chi'.\alpha')^s]$$

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In terms of moments...

$$k_{r,\ldots,s} = \sum_{(\lambda \vdash r, \ldots, \eta \vdash s)} (\chi'.\chi')^{\nu_\lambda} \cdots (\chi''.\chi'')^{\nu_\eta} \, d_\lambda \cdots d_\eta \alpha_\lambda \cdots \alpha'_\eta$$

$k_{r,\ldots,s}$	AS Algorithms	`MathStatica`	`Fast-algorithms`	`Polyk-algorithm`
$k_{3,2}$	0.06	0.02	0.01	0.02
$k_{4,4}$	0.67	0.06	0.02	0.06
$k_{5,3}$	0.69	0.08	0.02	0.07
$k_{7,5}$	34.23	0.79	0.11	0.70
$k_{7,7}$	435.67	2.52	0.26	2.43
$k_{9,9}$	-	27.41	2.26	23.32
$k_{10,8}$	-	30.24	2.98	25.06
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(2009) Di Nardo E., Guarino G., Senato D. it A new method for fast computing unbiased estimators of cumulants. Statistics and Computing

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Polykays: why a "natural" bases?

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of n uncorrelated umbrae similar to α.

Set $\alpha^{\cdot \pi} = \alpha_{i_1}^{|B_1|} \alpha_{i_2}^{|B_2|} \cdots \alpha_{i_k}^{|B_k|}$, where $\pi = \{B_1, B_2, \ldots, B_k\}$ is a partition of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $i_1, i_2, \ldots, i_k$ are distinct integers chosen in $[n]$.

$\alpha^{\cdot \pi} \simeq \alpha_{\lambda}$, where λ is the type of the partition π.

Polykays to moments

$$\alpha_{\lambda} \simeq \alpha^{\cdot \pi} \simeq \sum_{\sigma \leq \pi} (\chi . \alpha)^{\cdot \sigma}$$

Moments to polykays

$$(\chi . \alpha)^{\cdot \pi} \simeq \sum_{\sigma \leq \pi} \mu(\sigma, \pi) \, \alpha^{\cdot \sigma}$$

If $\pi = 1_n$

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Polykays to moments

$$\frac{(n.\chi\alpha)^{r_1}(n.\chi\alpha^2)^{r_2}\cdots}{(n)_{\nu_\lambda}} \simeq \alpha_\lambda \simeq \alpha^{\cdot\pi} \simeq \sum_{\sigma \leq \pi} (\chi.\alpha)^{\cdot\sigma}$$

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Polykays to moments (and U-statistics!)

$$\frac{(n.\chi\alpha)^{r_1}(n.\chi\alpha^2)^{r_2}\cdots}{(n)_{\nu_\lambda}} \simeq \alpha_\lambda \simeq \alpha^{\cdot\pi} \simeq \sum_{\sigma\leq\pi}(\chi.\alpha)^{\cdot\sigma}$$

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Set $\alpha^{\cdot\pi} = \alpha_{i_1}^{|B_1|} \alpha_{i_2}^{|B_2|} \cdots \alpha_{i_k}^{|B_k|}$, where $\pi = \{B_1, B_2, \ldots, B_k\}$ is a partition of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $i_1, i_2, \ldots, i_k$ are distinct integers chosen in $[n]$.

$\alpha^{\cdot\pi} \simeq \alpha_\lambda$, where λ is the type of the partition π.

Polykays to moments (and U-statistics!)

$$\frac{(n{\boldsymbol .}\chi\alpha)^{r_1}(n{\boldsymbol .}\chi\alpha^2)^{r_2}\cdots}{(n)_{\nu_\lambda}} \simeq \alpha_\lambda \simeq \alpha^{\cdot\pi} \simeq \sum_{\sigma \leq \pi} (\chi{\boldsymbol .}\alpha)^{\cdot\sigma}$$

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▷ Speed T. (1983) *Cumulants and partition lattices.* Austr. J. Stat.*

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$$\kappa(\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n) = \sum_{\sigma \leq \pi} (-1)^{|\sigma|-1} (|\sigma| - 1)!\, m_\sigma$$

▷ Speed T. (1983) *Cumulants and partition lattices.* Austr. J. Stat.*

$(\chi . \tilde{\mu})_\sigma = \prod_{B \in \sigma} \chi . [\prod_{j \in B} \tilde{\mu}_j]$. generalized cumulant (McCullagh)

Ex: $\{ \underbrace{\{\tilde{\mu}_1, \tilde{\mu}_2\}}_{}, \underbrace{\{\tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5, \tilde{\mu}_6, \}}_{}, \underbrace{\{\tilde{\mu}_7\}}_{} \}$

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$$\alpha^{\cdot\pi} \simeq \sum_{\sigma\leq\pi} (\chi\boldsymbol{.}\alpha)^{\cdot\sigma} \qquad (\chi\boldsymbol{.}\alpha)^{\cdot\pi} \simeq \sum_{\sigma\leq\pi} \mu(\sigma,\pi)\,\alpha^{\cdot\sigma}$$

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Multivariate cumulants

If $M = \{\underbrace{\tilde{\mu}_1, \ldots, \tilde{\mu}_1}_{s_1}, \underbrace{\tilde{\mu}_2, \ldots, \tilde{\mu}_2}_{s_2}, \ldots, \underbrace{\tilde{\mu}_r, \ldots, \tilde{\mu}_r}_{s_r}\}$ then

Multivariate moment $m_{t_1 \ldots t_r}$

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with $\tilde{\mu}_M = \prod_{\tilde{\mu} \in \bar{M}} \tilde{\mu}^{f(\tilde{\mu})}$ and $(\chi.\tilde{\mu})_M = \prod_{\tilde{\mu} \in \bar{M}} (\chi.\tilde{\mu})^{f(\tilde{\mu})}$.

Products of cumulants - Kendall and Stuart

For example if $M = \{\tilde{\mu}_1^{(1)}, \tilde{\mu}_2^{(2)}, \tilde{\mu}_3^{(1)}\}$ we have $E[\tilde{\mu}_M] = m_{121}$.

When the umbral monomials $\tilde{\mu}_i$ have disjoint supports $m_{t_1 \ldots t_r}$ becomes the products of moments of $m_{t_1} \cdots m_{t_r}$.

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Computational times

For `AS` Algorithms, missed computational times means “greater than 20 hours”. For `MathStatica`, missed computational times means “procedures not available”.

$k_{s_1 \ldots s_r; l_1 \ldots l_m}$	`AS` Algorithms	`MathStatica`	`Fast-algorithms`
$k_{3\,2}$	0.25	0.03	0.01
$k_{4\,4}$	28.36	0.16	0.02
$k_{5\,5}$	259.16	0.55	0.06
$k_{6\,5}$	959.67	1.01	0.16
$k_{7\,7}$	-	8.49	1.04
$k_{8\,7}$	-	14.92	2.19
$k_{3\,3\,3}$	1180.03	0.88	0.47
$k_{4\,4\,3}$	-	4.80	0.94
$k_{4\,4\,4}$	-	13.53	2.30
$k_{2\,1;\,1\,1}$	0.20	-	0.01
$k_{2\,2;\,2\,1}$	6.30	-	0.08
$k_{2\,2;\,2\,2}$	33.75	-	0.14
$k_{2\,2;\,2\,1;\,1\,1}$	126.19	-	0.28
$k_{2\,2;\,2\,1;\,2\,1}$	398.42	-	0.55
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A last look to AugToPs

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$$\left\{ \begin{array}{rcl} [n_{\bullet}(\chi\mu_1)] \cdots [n_{\bullet}(\chi\mu_n)] & \simeq & \sum_{\pi} (\chi_{\bullet}\chi)^{\cdot\pi} \, (n_{\bullet}\mu)_{\pi} \\ (n_{\bullet}\mu_1) \cdots (n_{\bullet}\mu_n) & \simeq & \sum_{\pi} [n_{\bullet}(\chi\mu)]_{\pi} \end{array} \right. \quad \Rightarrow E[(\chi_{\bullet}\chi)^{\cdot\pi}] = \mu(\mathbf{0}_n, \pi)$$

A last look to AugToPs

(1990) Streitberg B. *Lancaster Interactions Revisited* Ann. Statist.

Interaction measure

A signed measure ΔF is a function of the joint distribution $F_{1,2,\ldots,n}$ (and its marginals) of a random vector $(X_1, X_2, \ldots, X_n)$ that is supposed to be identically zero whenever the random vector is decomposable into two mutually independent subvectors.

(1997) Rota G.-C., Wallstrom T.C. *Stochastic integrals: a combinatorial approach*

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Univariate Faà di Bruno Formula

If $h(z) = f[g(z) - 1]$ with h, f and g exponential formal power series, then

$$h_n = \sum_{k \geq 1} f_k B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})$$

A new algorithm based on a suitable generalization of the well-known multinomial theorem:

$$(x_1 + x_2 + \cdots + x_n)^i = \sum_{k_1+k_2+\cdots+k_n=i} \binom{i}{k_1, k_2, \ldots, k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

where the indeterminates are replaced by symbolic objects.

$$\boxed{f(\boldsymbol{\mu}, \boldsymbol{z}) = 1 + \sum_{k=1}^{\infty} \sum_{\substack{\boldsymbol{i} \in \mathbb{N}_0^n \\ |\boldsymbol{i}|=k}} g_{\boldsymbol{i}} \frac{\boldsymbol{z}^{\boldsymbol{i}}}{\boldsymbol{i}!}}$$

by extending the action of E coefficient wise to g.f.'s

$$e^{\boldsymbol{\mu}\, \boldsymbol{z}^T} = e^{\mu_1 z_1 + \mu_2 z_2 + \cdots + \mu_n z_n} = u + \sum_{k=1}^{\infty} \sum_{\substack{\boldsymbol{i} \in \mathbb{N}_0^n \\ |\boldsymbol{i}|=k}} \boldsymbol{\mu}^{\boldsymbol{i}} \frac{\boldsymbol{z}^{\boldsymbol{i}}}{\boldsymbol{i}!}$$

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- (Univariate composite Multivariate)

$$f(\alpha.\beta.\boldsymbol{\mu},\boldsymbol{z}) = f[\alpha, f(\boldsymbol{\mu},\boldsymbol{z})-1].$$

- (Multivariate composite Univariate)

$$f[(\mu_1+\cdots+\mu_n).\beta.\gamma.z] = f[\boldsymbol{\mu},(f(\gamma,z)-1,\ldots,f(\gamma,z)-1)].$$

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$$f(\alpha . \beta . \boldsymbol{\mu}, \boldsymbol{z}) \, = \, f[\alpha, f(\boldsymbol{\mu}, \boldsymbol{z}) - 1].$$

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$$f[(\mu_1 + \cdots + \mu_n) . \beta . \gamma, z] \, = \, f[\boldsymbol{\mu}, (f(\gamma, z) - 1, \ldots, f(\gamma, z) - 1)].$$

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Multivariate Generalized Bell polynomials

The $\boldsymbol{i}$-th coefficient $B_{\boldsymbol{i}}(x_1, \ldots, x_n)$ of $\exp\left\{\sum_{i=1}^{n} x_i [f(\boldsymbol{\nu}_i, \boldsymbol{z}_{(m)}) - 1]\right\}$ is such that $B_{\boldsymbol{i}}(x_1, \ldots, x_n) \simeq (x_1.\beta.\boldsymbol{\nu}_1 + \cdots + x_n.\beta.\boldsymbol{\nu}_n)^{\boldsymbol{i}}$. The $\boldsymbol{i}$-th coefficient of (*) is such that

$$B_{\boldsymbol{i}}(\mu_1, \ldots, \mu_n) \simeq (\mu_1.\beta.\boldsymbol{\nu}_1 + \cdots + \mu_n.\beta.\boldsymbol{\nu}_n)^{\boldsymbol{i}}$$

(2011) Di Nardo E., G. Guarino, D. Senato A new algorithm for computing the multivariate Faá di Brunos formula Appl. Math. Comp.

⊙ (Multivariate composite different multivariates)

$$f[\mu_1 . \beta . \boldsymbol{\nu}_1 + \cdots + \mu_n . \beta . \boldsymbol{\nu}_n, \boldsymbol{z}_{(m)}] = f[\boldsymbol{\mu}, (f(\boldsymbol{\nu}_1, \boldsymbol{z}_{(m)}) - 1, \ldots, f(\boldsymbol{\nu}_n, \boldsymbol{z}_{(m)}) - 1)]. (*)$$

Multivariate Generalized Bell polynomials

The $\boldsymbol{i}$-th coefficient $B_{\boldsymbol{i}}(x_1, \ldots, x_n)$ of $\exp\left\{\sum_{i=1}^{n} x_i[f(\boldsymbol{\nu}_i, \boldsymbol{z}_{(m)}) - 1]\right\}$ is such that $B_{\boldsymbol{i}}(x_1, \ldots, x_n) \simeq (x_1 . \beta . \boldsymbol{\nu}_1 + \cdots + x_n . \beta . \boldsymbol{\nu}_n)^{\boldsymbol{i}}$. The $\boldsymbol{i}$-th coefficient of (*) is such that

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Subdivisions in terms of integer partitions

$l(\boldsymbol{\lambda})$ the length of $\boldsymbol{\lambda}$ = the number of columns of $\boldsymbol{\lambda}$

$$\boxed{(\alpha.\beta.\boldsymbol{\mu})^{\boldsymbol{i}} \simeq \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{i}} \frac{\boldsymbol{i}!}{\mathfrak{m}(\boldsymbol{\lambda})!\,\boldsymbol{\lambda}!}\, \alpha^{l(\boldsymbol{\lambda})} \boldsymbol{\mu}_{\boldsymbol{\lambda}}}$$

$\boldsymbol{\mu}_{\boldsymbol{\lambda}} = (\boldsymbol{\mu}'^{\boldsymbol{\lambda}_1})^{\cdot r_1} (\boldsymbol{\mu}''^{\boldsymbol{\lambda}_2})^{\cdot r_2} \ldots$

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A composition $\boldsymbol{\lambda}$ of a multi-index $\boldsymbol{i}$, in symbols $\boldsymbol{\lambda} \models \boldsymbol{i}$, is a matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ of nonnegative integers and with no zero columns such that $\lambda_{r1} + \cdots + \lambda_{rk} = i_r$ for $r = 1, \ldots, n$.

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Any subdivision of a multiset M, having $\boldsymbol{i}$ as vector of multiplicities, corresponds to a suitable partition of the multi-index $\boldsymbol{i}$.

Ex: $M = \{\mu_1, \mu_1, \mu_2\}$ corresponds to the multi-index $(2, 1)$

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$\boldsymbol{i}$	m	# terms in output	Time (`UMFB`)	Time (`diff`)
$(6, 5)$	2	14089	0.7	1.6
$(7, 6)$	2	60190	3.2	29.8
$(7, 7)$	2	123134	8.1	75.4
$(5, 4)$	3	20208	0.7	2.3
$(6, 5)$	3	122034	6.3	62.5
$(5, 4)$	4	86768	4.5	26.9
$(5, 4)$	5	288370	25.9	130.9
$(4, 4, 3)$	2	95138	6.3	12.3
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$(5,4)$	4	86768	4.5	26.9
$(5,4)$	5	288370	25.9	130.9
$(4,4,3)$	2	95138	6.3	12.3
$(4,4,4)$	2	257854	22.8	41.1
$(4,3,3)$	3	313866	22.5	54.5
$(4,2,2)$	4	106912	6.5	17.3

Comparison of computational times in seconds.

Any subdivision of a multiset M, having $\boldsymbol{i}$ as vector of multiplicities, corresponds to a suitable partition of the multi-index $\boldsymbol{i}$.

Ex: $M = \{\mu_1, \mu_1, \mu_2\}$ corresponds to the multi-index $(2, 1)$

$\{\{\mu_1, \mu_1, \mu_2\}\}; \{\{\mu_1, \mu_1\}, \{\mu_2\}\};$
$\{\{\mu_1, \mu_2, \}\{\mu_1\}\}; \{\{\mu_1\}, \{\mu_1\}, \{\mu_2\}\}.$

$$\binom{2}{1}, \binom{2,0}{0,1}, \ \binom{1,1}{1,0}, \binom{1,1,0}{0,0,1}.$$

$\boldsymbol{i}$	m	# terms in output	Time (`UMFB`)	Time (`diff`)
$(6,5)$	2	14089	0.7	1.6
$(7,6)$	2	60190	3.2	29.8
$(7,7)$	2	123134	8.1	75.4
$(5,4)$	3	20208	0.7	2.3
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Comparison of computational times in seconds.

The i-th Hermite polynomial

$$H_{\boldsymbol{i}}(\boldsymbol{x}, \Sigma) = (-1)^{|\boldsymbol{i}|} \frac{D_{\boldsymbol{x}}^{(\boldsymbol{i})} \phi(\boldsymbol{x}; \boldsymbol{0}, \Sigma)}{\phi(\boldsymbol{x}; \boldsymbol{0}, \Sigma)}$$

where $\phi(\boldsymbol{x}; \boldsymbol{0}, \Sigma)$ denotes the multivariate Gaussian density with $\boldsymbol{0}$ mean and covariance matrix Σ of full rank n.

$$H_{\boldsymbol{i}}(\boldsymbol{x}, \Sigma) \simeq (-1.\beta.\boldsymbol{\nu} + \boldsymbol{x}\Sigma^{-1})^{\boldsymbol{i}}$$

$$\Downarrow$$

$$\begin{aligned} \tilde{H}_{\boldsymbol{i}}(\boldsymbol{x}, \Sigma) &= H_{\boldsymbol{i}}(\boldsymbol{x}\Sigma^{-1}, \Sigma^{-1}) \\ &\simeq (-1.\beta.\boldsymbol{\mu} + \boldsymbol{x})^{\boldsymbol{i}} \end{aligned}$$

with $f(\boldsymbol{\nu}, \boldsymbol{z}) = 1 + \frac{1}{2}\boldsymbol{z}\Sigma^{-1}\boldsymbol{z}^T$
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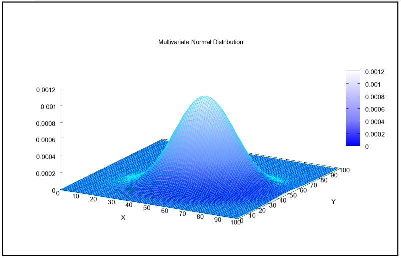

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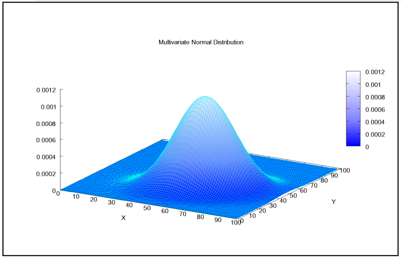

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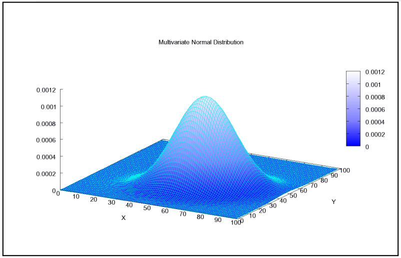

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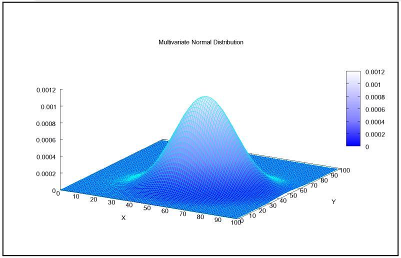

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As special generalized Bell polynomials

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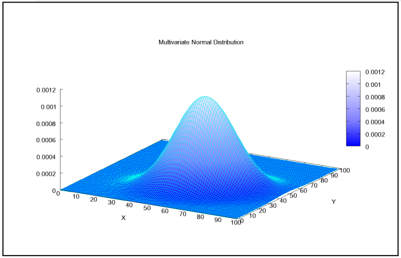

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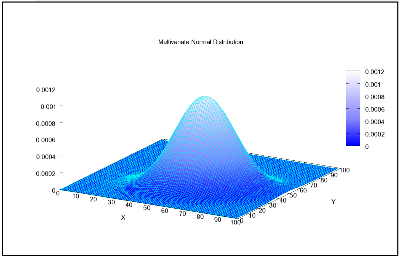

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Any connection with multivariate Brownian motion?

Let $\{\boldsymbol{X}_t\}_{t \geq 0} = \{(X_1^{(t)}, \ldots, X_d^{(t)})\}_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$, with $\{X_j^{(t)}\}_{j=1}^d$ univariate r.v.'s all defined on $(\Omega, \mathcal{F}, \mathcal{P})$.

$$\{t . \boldsymbol{\mu}\}_{t \geq 0}$$

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Symbolic multivariate Lévy-Khintchine formula

$$\{t . \beta . (\boldsymbol{\delta} C^T + \chi . \boldsymbol{m} + \boldsymbol{\eta})\}_{t \geq 0}$$

- $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)$ is a d-tuple of uncorrelated umbrae such that $f(\boldsymbol{\delta}, \boldsymbol{z}) = 1 + \boldsymbol{z}\boldsymbol{z}^T/2$;
- the matrix C is the square root of the covariance matrix Σ;
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Symbolic multivariate Lévy-Khintchine formula

$$\{t\boldsymbol{.}\beta\boldsymbol{.}(\boldsymbol{\delta} C^T \dot{+} \chi\boldsymbol{.}\boldsymbol{m} \dot{+} \boldsymbol{\gamma})\}_{t\geq 0}$$

▷ $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)$ is a d-tuple of uncorrelated umbrae such that $f(\boldsymbol{\delta}, \boldsymbol{z}) = 1 + \boldsymbol{z}\boldsymbol{z}^T/2$;

▷ the matrix C is the square root of the covariance matrix Σ;

▷ $f(\boldsymbol{\gamma}, \boldsymbol{z}) = 1 + \int_{\mathbb{R}^d} (e^{\boldsymbol{z}\boldsymbol{x}^T} - 1 - \boldsymbol{z}\boldsymbol{x}^T)\nu(d\boldsymbol{x})$, *Lévy measure*

A theory of multivariate TSH polynomials

A conjecture

$$Q_i(x,t) = E[(-t.\mu + x)^i]$$

Multivariate Poisson-Charlier polynomials

Every multivariate compound Poisson process with intensity parameter $\lambda > 0$ is umbrally represented by the family of auxiliary umbrae $\{(\lambda t).\beta.\gamma\}_{t \geq 0}$

Why?

Multivariate Hermite polynomials

$$\tilde{H}_i \quad (x\,\Sigma) \simeq (-t.\beta.\mu + x)^i$$

⊙$\{t.\beta.(\delta C^T)\}_{t\geq 0}$ is a symbolic version of a standard multivariate Brownian motion $\{\mathbf{B}_t\}_{t\geq 0}$

⊙$\{t.\beta.(\delta C^T \dot{+} \chi.m)\}_{t\geq 0}$ is a symbolic version of a non standard multivariate Brownian motion $X_t = mt + C\mathbf{B}_t$, where $m \in \mathbb{R}^d$.

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$$E[\tilde{H}_{\boldsymbol{i}}^{(t)}(t.\beta.(\delta C^T), \Sigma)] = 0? \text{ Yes, } \boldsymbol{\mu} = \delta C^T$$

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Multivariate Bernoulli polynomials

The Multivariate Bernoulli umbra

$\boldsymbol{\iota} = (\iota, \ldots, \iota)$, with ι the Bernoulli umbra (moments = Bernoulli numbers).

Therefore $-t.\boldsymbol{\iota} \equiv t.(-1.\boldsymbol{\iota})$ is a Lévy process.

Random variable counterpart

The inverse $-1.\boldsymbol{\iota}$ of the multivariate Bernoulli umbra is the umbral counterpart of a d-tuple identically distributed to $(U, \ldots, U)$, with $U(0,1)$.

Symbolic representation of multivariate Bernoulli polynomials

- ▷ $\mathcal{B}_{\boldsymbol{v}}^{(t)}(-t.\boldsymbol{\iota}) = \mathcal{B}_{\boldsymbol{v}}^{(t)}[t.(-1.\boldsymbol{\iota})] = 0$; (TSH polynomials?)
- ▷ $\mathcal{B}_{\boldsymbol{v}}^{(t)}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \leq \boldsymbol{v}} \binom{\boldsymbol{v}}{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{v}-\boldsymbol{k}} B_{\boldsymbol{k}}^{(t)}$;
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$$B_{\boldsymbol{v}}^{(t)}(\boldsymbol{x}) = E[(\boldsymbol{x} + t . \boldsymbol{\iota})^{\boldsymbol{v}}]$$

for all $\boldsymbol{v} \in \mathbb{N}_0^d, t \in \mathbb{R}$

- ▷ $\mathcal{B}_{\boldsymbol{v}}^{(t)}(-t . \boldsymbol{\iota}) = \mathcal{B}_{\boldsymbol{v}}^{(t)}[t . (-1 . \boldsymbol{\iota})] = 0$; (TSH polynomials?)
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(2011) Hasabe T. and Saigo H. *Joint cumulants for natural independence* Electronic Communications in Probability - a first draft of a non-commutative definition of $n.\alpha$.

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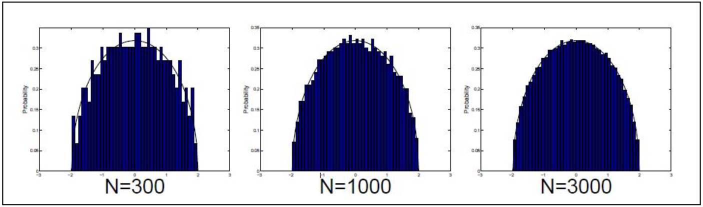

Wireless Communications

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x is a K- vector, y is a N-vector, n is a N-vector Gaussian noise, H is a $N \times K$ random matrix.

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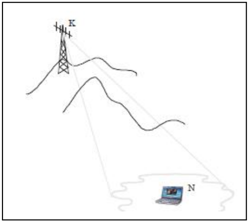

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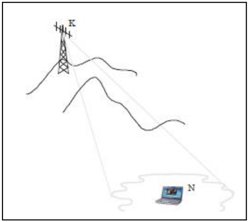

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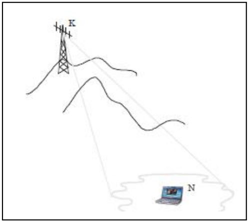

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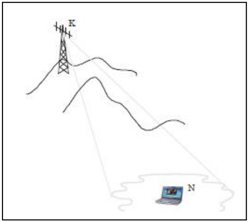

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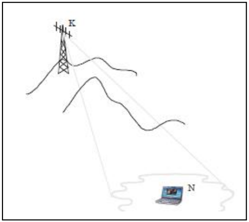

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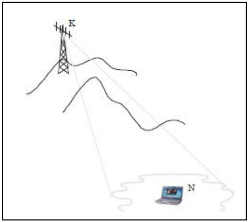
Wireless Communications



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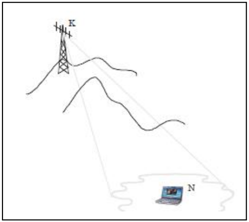

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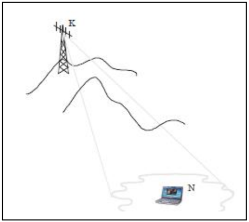

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Any help from symbolic methods?

▷ Diaconis, P., Shahshahani, M. (1994) *On the Eigenvalues of Random Matrices* J. Appl. Prob.

▷ Hanlon P.J., Stanley R., Stembridge J. (1992) *Some combinatorial aspects of the Spectra of Normally distributed random matrices* Contemporary Mathematics.

Wishart distribution S of parameter p

The distribution of the sample variance-covariance matrix of a multivariate Gaussian model.

Let $X = (X_1, X_2, \cdots, X_p) \approx N(\mathbf{0}, \Sigma)$ with Σ of full rank p. Set

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Symbolic representation of the Wishart distribution S

Let $\{E[\text{Tr}^k(S)]\}$ be umbrally represented by the umbra ψ. Then

$$\psi \equiv \quad \beta.\tilde{\varrho}$$

where $\tilde{\varrho} = \varrho \kappa_{\bar{u}}$ and $E[\varrho^k] = \text{Tr}(\Sigma^k)$ for all $k \geq 1$.

- Any help from recursions? From Abel representation?
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- Any help from recursions? From Abel representation?
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$$S = X_{(1)}^T X_{(1)} + \cdots + X_{(n)}^T X_{(n)}$$

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$$fg(\tau) = \sum_{\omega\in S_k} f(\omega)g(\omega^{-1}\tau) = \sum_{\omega\in S_k} f(\tau\omega^{-1})g(\omega)$$

What about the noncentral Wishart distributions?

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Let H be a random unitary matrix uniformly distributed with respect to the Haar measure on the group $\mathcal{U}_n$ of $n \times n$ unitary matrices.

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Inheritance on the average

A statistic T is said to be natural relative to spectral sampling if, for each $(X_1, \ldots, X_n)$ we have

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It is this property that gives to these functions a common interpretation independent of the sample size.

For $m = n$ and for simple random samples, T is a symmetric polynomial.

Therefore assume $m = n$

- ▷ $\sigma \equiv \beta.(\chi.\sigma)$ with $E[\sigma^i] = s_i$ in n indeterminates;
- ▷ $E[\sigma^i] = \sum\limits_{\lambda \vdash i} n^{\nu_\lambda} d_\lambda \prod\limits_{i=1}^{\nu_\lambda} E\left[\frac{(\chi.\sigma)^i}{n}\right]^{r_i}$;
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▷ $E[\sigma^i] = \sum_{\lambda \vdash i} n^{\nu_\lambda} d_\lambda \prod_{i=1}^{\nu_\lambda} E\left[\frac{(\chi.\sigma)^i}{n}\right]^{r_i}$;

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Inheritance on the average

A statistic T is said to be natural relative to spectral sampling if, for each $(X_1, \ldots, X_n)$ we have

$$E\left[T_m(\lambda_{1,Y}, \ldots, \lambda_{m,Y}) | (X_1, \ldots, X_n)\right] = T_n(X_1, \ldots, X_n)$$

It is this property that gives to these functions a common interpretation independent of the sample size.

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▷ Replace the indeterminates with the umbrae corresponding to the eigenvalues of Y and introduce permutations

$$\mathsf{Tr}(Y^i) = \sum_{\tau \in S_i} n^{l(\tau)} \prod_{c \in C(\tau)} q_{l(c)}$$

▷ By using convolutions

$$E\left[\prod_{c \in C(\omega)} \mathsf{Tr}(Y^{l(c)})\right] = \sum_{\tau \in S_i} n^{l(\omega\tau^{-1})} E\left[\prod_{c \in C(\tau)} q_{l(c)}\right]$$

▷ The unitary element respect to this convolution is $\delta_e(\tau) = \delta_{\tau,e}$ and so under suitable hypothesis, invert the previous equation by using $n^{(-1)}$

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Natural statistics! depending on conjugacy classes

In agreement with some computations made by hand starting from simple k-statistics for $i = 2, 3, 4$. Starting from polykays, more natural statistics have been found:

$$\boxed{l_\pi = \sum_{\tilde{\pi} \geq \pi} \mu(\pi, \tilde{\pi}) \kappa_{\tilde{\pi}}, \qquad \kappa_\pi = \sum_{\tilde{\pi} \geq \pi} l_{\tilde{\pi}}}$$

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Thank you for your attention!

A parking function is a sequence of non negative integers $(u_1, u_2, \ldots, u_n)$ such that there exists a permutation $(a_1, a_2, \ldots, a_n)$ satisfying $u_i < a_i$ for all i.

For example, $3, 0, 1, 3, 1$ is parking function, use the permutation $4, 1, 3, 5, 2$; but $1, 4, 2, 0, 4$ is not.

We denote by $park(n)$ the set of all parking functions of length n, its cardinality is $(n+1)^{n-1}$.

▸ return P.F.