Analytic approach for Brenti conjecture on R-polynomials

Masato Kobayashi

September 19, 2011

Plan

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- R-polynomials? – History

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- Proof

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[Keywords]

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Coxeter groups, Bruhat order

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"Like Boolean"

Boolean vs Bruhat intervals

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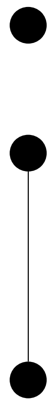

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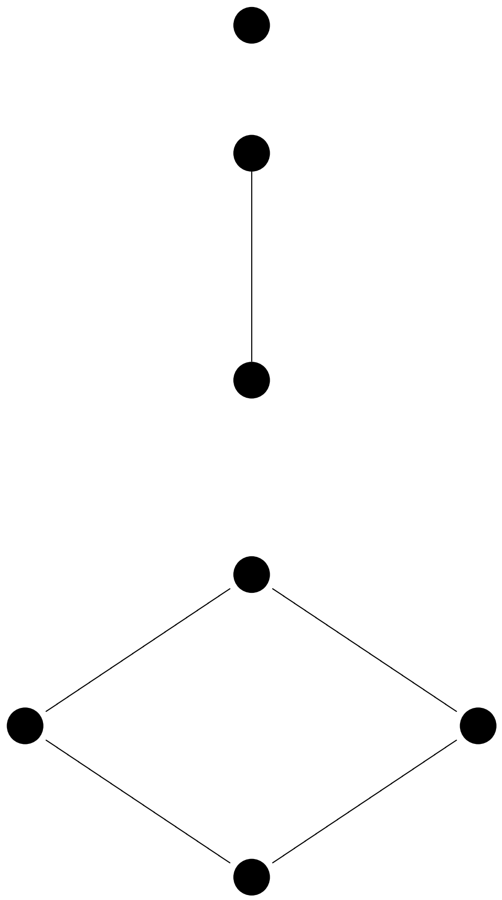

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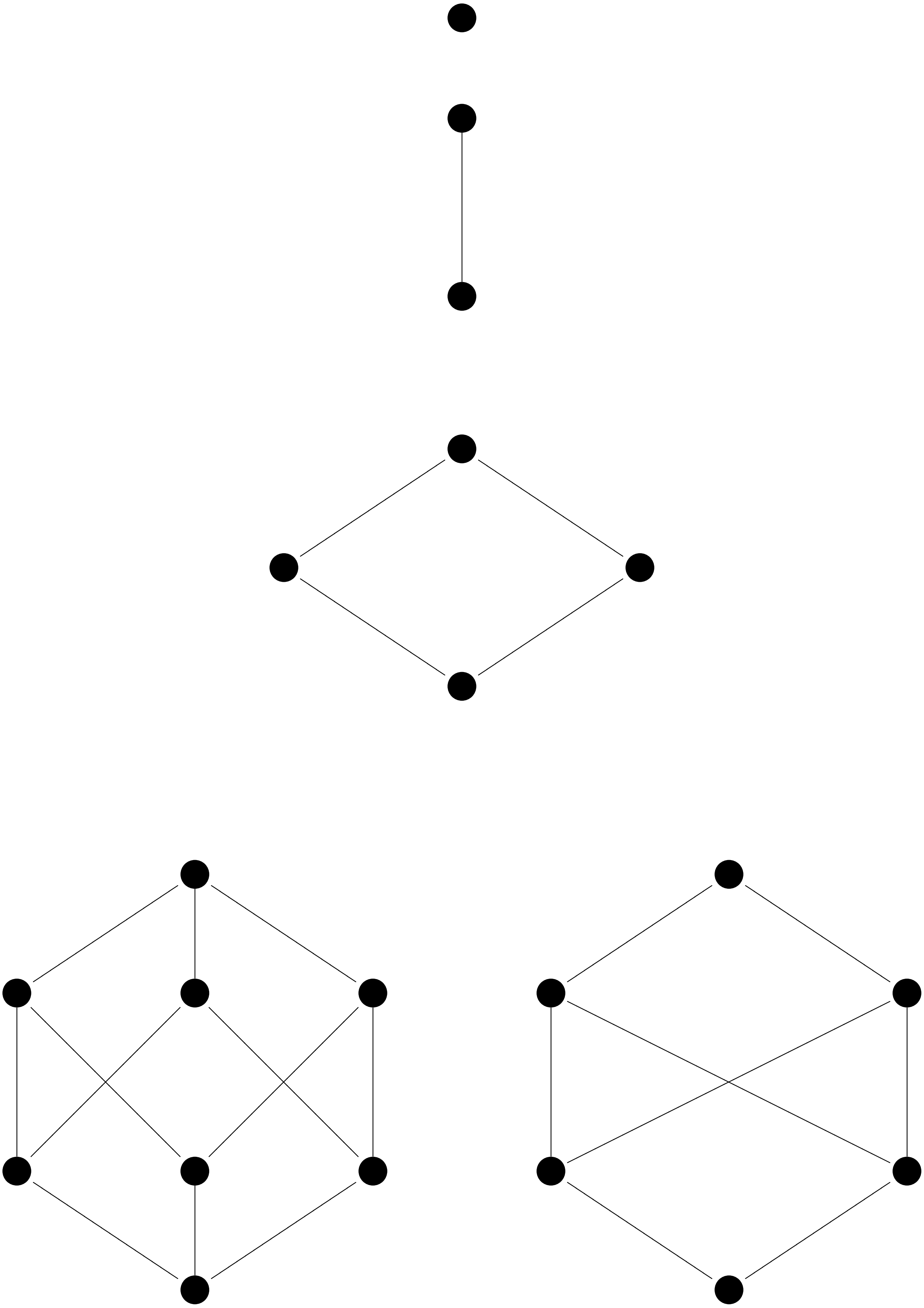

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$$R_{uw} = \begin{cases} R_{us,ws} & \text{if } us < u, \\ qR_{us,ws} + (q-1)R_{u,ws} & \text{if } u < us. \end{cases}$$

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Guess:
for all roots α,

$$\alpha^{-1} = \overline{\alpha} \quad ???$$

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My idea: Differentiate!

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Thank you!