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SCHUR-WEYL DUALITY FOR THE ROOK MONOID - COMBINATORIAL ASPECTS

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Motivation

Let n and m be positive integers. Let S_n be the symmetric group on $[n] = \{1, \ldots, n\}$.

Let $V \cong \mathbb{C}^m$ be an m-dimensional vector space over $\mathbb{C}$ with basis $\{e_1, \cdots, e_m\}$.

There is a right action of $\mathbb{C}[S_n]$ on $\otimes^n V$ given by place permutation

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

where $\sigma \in S_n$ and $v_1, \cdots, v_n \in V$.

Let λ be a partition of n and let χ^λ be the irreducible character of S_n corresponding to λ.

For $v_1, \cdots, v_n \in V$, set $v^\otimes = v_1 \otimes \cdots \otimes v_n$.

Let π_λ be the linear operator of $\otimes^n V$ given by

$$\pi_\lambda(v^\otimes) = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}).$$

Let $v_1, \cdots v_n \in V$, and $v^{\bigotimes} = v_1 \otimes \cdots \otimes v_n$. The image $\pi_\lambda(\otimes^n V)$ is a *symmetry class of tensors* and $\pi_\lambda(v^{\bigotimes})$ is called a *symmetrized tensor*.

Classic problems are to determine necessary and sufficient conditions for the annulment and equality of symmetrized tensors [C. Gamas; J. Dias da Silva]. For example,

Theorem 1 (Gamas, 1988) *Let λ be a partition of n and let $v_1, \cdots, v_n$ be vectors in V. Then*

$$\pi_\lambda(v_1 \otimes \cdots \otimes v_n) \neq 0$$

if and only if there is a tableau T of shape λ whose columns index linearly independent subsets of $\{v_1, \cdots, v_n\}$.

Schur-Weyl Duality and Berget's approach

Let $G = GL_m(\mathbb{C})$. G acts diagonally on $\otimes^n V$ via, for $g \in G$ and $v_1, \cdots, v_n \in V$,

$$g(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n).$$

This action centralizes the right action of $\mathbb{C}[S_n]$ on $\otimes^n V$ by place permutation. We have

Theorem 2 (Schur-Weyl Duality)

$$\mathbb{C}[S_n] \cong End_{\mathbb{C}[G]}(\otimes^n V)$$

and

$$\mathbb{C}[G] \cong End_{\mathbb{C}[S_n]}(\otimes^n V).$$

The Rook Monoid

Definition 1 The *rook monoid* R_n is the set of all partial permutations of $[n]$ endowed with the usual composition of partial functions.

Equivalently, R_n is the set of all $n \times n$ matrices that contain at most one entry equal to 1 in each column and row and zeros elsewhere, under matrix multiplication.

Example Let $\sigma \in R_5$ be

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & - & 1 & 4 & - \end{pmatrix}.$$

The element σ can be represented as

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problems

(i) Is it possible to define the notion of *partial symmetry classes of tensors if we replace the action of* S_n *on* $\otimes^n V$ *by a suitable action of the rook monoid* R_n *on some tensor space?*

(ii) *What can we say about the annulment or equality of partially symmetrized tensors?*

(iii) *What combinatorics are involved in those problems (in particular, with relation with Matroid Theory)?*

Representation theory of $\mathbb{C}[R_n]$

Theorem 3 (Munn, 1957) *For* $1 \leq r \leq n$, *let* $A_r = \mathcal{M}_{\binom{n}{r}}(\mathbb{C}[S_r])$ *be the* $\mathbb{C}$*-algebra of all matrices with rows and columns indexed by subsets* I, $J \subseteq [n]$ *of size* r *and entries in* $\mathbb{C}[S_r]$. *For* $r = 0$, *let* $A_0 \cong \mathbb{C}$. *Then*

$$\mathbb{C}[R_n] \cong \bigoplus_{r=0}^{n} \mathcal{M}_{\binom{n}{r}}(\mathbb{C}[S_r]).$$

In particular, $\mathbb{C}[R_n]$ *is a semisimple algebra.*

Theorem 4 (Munn, 1957) Let $0 \leq r \leq n$. For each partition λ of r, let ρ^{λ} be the irreducible representation of $\mathbb{C}[S_r]$ corresponding to λ.The set

$$\{\rho^{\lambda^*} : \lambda \text{ is a partition of } r, r = 0, 1, \cdots n\}$$

is a full set of inequivalent irreducible representations of R_n.

Schur-Weyl duality for R_n and $GL_m(\mathbb{C})$

Let $V \cong \mathbb{C}^m$ be an m-dimensional vector space over $\mathbb{C}$ and $U = V \oplus \mathbb{C}$.

Theorem 5 (Solomon, 2002) Let $GL_m(\mathbb{C})$ act on $\otimes^n U$ by fixing $\mathbb{C}$ and $\phi : R_n \mapsto End_{\mathbb{C}}(\otimes^n U)$ defined by the right action of R_n over $\otimes^n U$. If $m \geq n$, then

$$\mathbb{C}[R_n] \cong End_{\mathbb{C}[GL_m(\mathbb{C})]}(\otimes^n U).$$

A naive application

Let λ be a partition of r, where $1 \leq r \leq n$. The primitive central idempotent of R_n corresponding to λ is given by

$$e_\lambda^* = \frac{\chi^\lambda(1_r)}{r!} \sum_{\substack{K \subseteq [n] \\ |K|=r}} \sum_{\substack{X \subseteq K \\ |K|=r}} \sum_{\tau \in S_r} (-1)^{|K|-|X|} \chi^\lambda(\tau)(p_K \tau p_K^{-1})_{|X}$$

Polynomial representations of $GL_m(\mathbb{C})$

Let $V \cong \mathbb{C}^m$ be an m-dimensional vector space over $\mathbb{C}$ and

$$U = V \oplus \mathbb{C}e_\infty$$

with basis $\{e_1, \cdots, e_m, e_\infty\}$ over $\mathbb{C}$.

For every $X \subseteq [n]$, set

$$\Gamma_X(m) = \{\alpha : X \mapsto [m]\}$$

and $\Gamma(m) = \bigcup_{X \subseteq [n]} \Gamma_X(m)$.

Example Let $m = 7$ and $n = 5$.

If $X = \{1, 3, 5\} \subseteq [5]$, then

$$\alpha = (\alpha(1), \alpha(3), \alpha(5)) = (7, 2, 2) \in \Gamma_X(7).$$

Polynomial representations of $GL_m(\mathbb{C})$

For $X \subseteq [n]$, let $\alpha \in \Gamma_X(m)$, $\alpha : X \mapsto [m]$. the element $e_\alpha^{\otimes} \in \otimes^n U$ will be defined by

$$e_\alpha^{\otimes} = e_{\beta(1)} \otimes \cdots \otimes e_{\beta(n)}$$

where $\beta : [n] \mapsto [m] \in \Gamma_{[n]}(m)$ and $\beta(i) = \alpha(i)$ if $i \in X$ and $e_{\beta(i)} = e_\infty$ if $i \notin X$.

Example As in the previous example, let $m = 7$, $n = 5$ and $X = \{1, 3, 5\} \subseteq [5]$. As before

$$\alpha = (\alpha(1), \alpha(3), \alpha(5)) = (7, 2, 2) \in \Gamma_X(7).$$

Then, the element $e_\alpha^{\otimes} \in \otimes^5 U$ is given by

$$e_\alpha^{\otimes} = e_7 \otimes e_\infty \otimes e_2 \otimes e_\infty \otimes e_2.$$

The set $\{e_\alpha^{\otimes} : \alpha \in \Gamma(m)\}$ is a $\mathbb{C}$-basis of $\otimes^n U$.

Polynomial representations of $GL_m(\mathbb{C})$

Let $G = GL_m(\mathbb{C})$. U can be regarded has a $\mathbb{C}[G]$-module with, for any $j = 1, \cdots, m$ and $g \in G$,

$$g.e_j = \sum_{i=1}^{m} c_{i,j}(g) e_i \quad \text{and} \quad g.e_\infty = e_\infty$$

where $c_{i,j} : G \mapsto \mathbb{C}$ is given by $c_{i,j}(g) = g_{i,j}$.

G acts diagonally on $\otimes^n U$ via

$$g(u_1 \otimes \cdots \otimes u_n) = g(u_1) \otimes \cdots \otimes g(u_n),$$

for $g \in G$ and $u_1, \cdots, u_n \in U$.

Equivalently, let $X = \{x_1, \cdots, x_r\} \subseteq [n]$, $\beta \in \Gamma_X(m)$, $e^{\otimes}_\beta \in \otimes^n U$ is the corresponding basis element and $g \in G$, then

$$g.e^{\otimes}_\beta = \sum_{\alpha \in \Gamma_X(m)} c_{\alpha,\beta}(g) e^{\otimes}_\alpha$$

where $c_{\alpha,\beta}(g) = c_{\alpha(x_1),\beta(x_1)}(g) \cdots c_{\alpha(x_r),\beta(x_r)}(g)$.

The Schur Algebra

$\mathcal{A} = \mathcal{A}_n(m) =< c_{\alpha,\beta} : \alpha, \beta \in \Gamma_X(m), X \subseteq [n] >$ is the $\mathbb{C}$-space generated be all the monomial functions $c_{\alpha,\beta} : G \mapsto \mathbb{C}$.

The *Schur algebra* $\mathcal{S}$ is the dual $\mathbb{C}$-space of $\mathcal{A}$

$$\mathcal{S} = \mathcal{A}^* = Hom_{\mathbb{C}}(\mathcal{A}; \mathbb{C}).$$

$\mathcal{S}$ is a finite-dimensional associative $\mathbb{C}$-algebra.

Every $\mathbb{C}[G]$-module whose coefficient space lies in $\mathcal{A}$ can be viewed as a $\mathcal{S}$-module.

Therefore, $\otimes^n U$ has the structure of a left $\mathcal{S}$-module. For any $\xi \in \mathcal{S}$, $X \subseteq [n]$ and $\beta \in \Gamma_X(m)$, we define

$$\xi . e_\beta^\otimes = \sum_{\alpha \in \Gamma_X(m)} \xi(c_{\alpha,\beta}) e_\alpha^\otimes$$

Let $\mathcal{R}_n = \bigoplus_{r=0}^{n} \mathcal{M}_{\binom{n}{r}}(\mathbb{C}[S_r])$ be the $\mathbb{C}$-algebra of matrices referred to in theorem 3.

It is possible to define an appropriate right $\mathcal{R}_n$-action on $\otimes^n U$ that commutes with the above left $\mathcal{S}$-action. Since $\mathcal{R}_n \cong \mathbb{C}[R_n]$ as $\mathbb{C}$-algebras, we have

Theorem 6 (Schur-Weyl Duality) Let $m \geq n$. The representation $\rho : \mathcal{S} \mapsto End_{\mathbb{C}}(\otimes^n U)$ afforded by the left action of $\mathcal{S}$ on $\otimes^n U$ induces an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{S} \cong End_{\mathbb{C}[R_n]}(\otimes^n U).$$

An application

Let $0 \leq r \leq n$ and let λ be a partition of r. Consider the linear operator of $\otimes^n U$ associated with λ, $\pi^*_\lambda \in End_{\mathcal{S}}(\otimes^n U)$.

Let $u_1, \cdots, u_n \in U$ and $u^{\otimes} = u_1 \otimes \cdots \otimes u_n \in \otimes^n U$.

$\mathcal{S}(u^{\otimes})$ is the $\mathcal{S}$-submodule of $\otimes^n U$ generated by $u^{\otimes}$,

$\mathcal{R}(u^{\otimes})$ is the $\mathbb{C}[R_n]$-submodule of $\otimes^n U$ generated by $u^{\otimes}$.

Proposition 1 *Let $0 \leq r \leq n$ and let λ be a partition of r. The following are equivalent*

(i) *The multiciplicity of λ is positive in $\mathcal{S}(u^{\otimes})$;*

(ii) *The multiciplicity of λ is positive in $\mathcal{R}(u^{\otimes})$;*

(iii) $\pi^*_\lambda(u^{\otimes}) \neq 0$.

Further directions

Let $0 \leq r \leq n$ and let λ be a partition of r. A λ_r^n-*tableau* is a Ferrers diagram of shape λ filled with r distinct entries from the set $\{1, 2, \cdots, n\}$.

In 2002, C. Grood showed that the irreducible $\mathbb{C}[R_n]$-modules can be realized in terms of λ_r^n-tableaux.

Using Schur algebras, we expect to provide a combinatorial condition for the annulment of a partial symmetrized tensor $\pi_\lambda^*(u^\otimes)$ analog to Gama's condition.

We also expect to study and solve open problems related to the linear matroid determined by a finite collection of vectors $u = \{u_1, \cdots, u_n\}$, where $u_i \in U$.

References

A. Berget, A short proof of Gama's theorem, *Linear Algebra and its Applications*, **430** (2009) 791-794.

A. Berget, Equality of symmetrized tensors and the coordinate ring of the flag variety, *submetido*.

H. da Cruz, J. A. Dias da Silva, Equality of immanantal decomposable tensors, *Linear Algebra and its Applications*, **395** (2005) 95-119.

H. da Cruz, J. A. Dias da Silva, Equality of immanantal decomposable tensors, *Linear Algebra and its Applications*, **401** (2005) 29-46.

J. A. Dias da Silva, New conditions for equality of decomposable symmetrized tensors, *Linear*

Algebra and its Applications **245** (1996) 353-372.

C. Gamas, Conditions for a symmetrized decomposable tensor to be zero, *Linear Algebra and its Applications* **108** (1988) 83-119.

J. A. Green, *Polynomial Representations of GLn*, Springer-Verlag, New York, 1980.

C. Grood, A Specht module analog for the rook monoid, *Electronic Journal of Combinatorics* **9** (2002), Research Paper 2.

M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

M. Malandro, D. Rockmore, Fast Fourier transforms for the rook monoid (2007).

W. D. Munn, The characters of the symmetric inverse semigroup, *Mathematical Proceedings of the Cambridge Philosophical Society* **53** (1957) 13-18.

W. D. Munn, Matrix representations of semigroups *Mathematical Proceedings of the Cambridge Philosophical Society* **53** (1957) 5-12.

L. Solomon, Representations of the rook monoid, *Journal of Algebra* **256** (2002) 309-342.