Zonotopes, toric arrangements, labeled graphs, and the arithmetic Tutte polynomial

(joint work with Michele D'Adderio)

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The vector partition function

Let be $X = \{a_1, \ldots, a_h\} \subseteq \mathbb{Z}^n$.
For every $\lambda \in \mathbb{Z}^n$, we define $\mathcal{P}_X(\lambda)$ as the number of ways we can write

$$\lambda = \sum_{i=1}^{h} x_i a_i \qquad x_i \in \mathbb{N}.$$

(Since we want this number to be finite, we require that all the a_i lie on the same side of a hyperplane).
Fixed X, this is a function of λ, which we denote by $\mathcal{P}_X(\lambda)$ and we call the (vector) partition function.

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Example $X = \{20, 50, 100\}$

Let $n = 1, X = \{20, 50, 100\}$. Then we have the equation

$$20x + 50y + 100z = \lambda, \qquad x, y, z \geq 0$$

defining a variable triangle $P_X(\lambda)$ in $\mathbb{R}^3$, obtained by intersecting the

positive octant of $\mathbb{R}^3$ with a plane.

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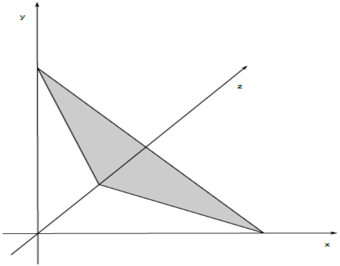

positive octant of $\mathbb{R}^3$ with a plane.

$\mathcal{P}_X(\lambda)$ is the number of integer points in $P_X(\lambda)$.

On every coset of $100\mathbb{Z} \subseteq \mathbb{Z}$, $\mathcal{P}_X(\lambda)$ is a polynomial.

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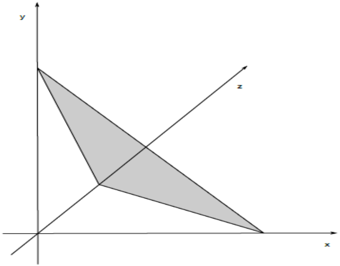

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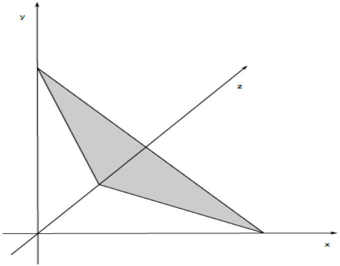

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A variable polytope

In general, we intersect a subspace with the positive orthant, thus we get a variable polytope $P_X(\lambda)$.

The partition function $\mathcal{P}_X(\underline{\lambda})$ counts the integer points in this polytope, hence it is related with another function, the multivariate spline, defined as the volume the same polytope:

$$\mathcal{S}_X(\underline{\lambda}) = vol\left(P_X(\underline{\lambda})\right).$$

Facts (Dahmen and Micchelli):
(1) $\mathcal{S}_X$ is piecewise polynomial; its local pieces span a space $D(X)$ of polynomials, defined by nice differential equations.
(2) $\mathcal{P}_X$ is piecewise quasipolynomial; its local pieces span a space $DM(X)$ of quasipolynomials, defined by nice difference equations.
(A function $q : \mathbb{Z}^n \to \mathbb{C}$ is a quasipolynomial if there is a finite index subgroup of $\mathbb{Z}^n$ such that the restriction of q to every coset is polynomial).

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Relation with arrangements

De Concini-Procesi's approach: applying "Laplace transform" L we get

$$L\mathcal{S}_X = \prod_{i=1}^{h} \frac{1}{a_i} \qquad L\mathcal{P}_X = \prod_{i=1}^{h} \frac{1}{1 - e^{2\pi\imath a_i}}$$

where we view every a_i as a linear functional on the dual space.

Strategy: develop these expressions as a sum of simpler fractions, then apply L^{-1} and get formulae for $\mathcal{S}_X$ and $\mathcal{P}_X$.

Notice that $L\mathcal{S}_X$ is rational function defined on the complement of a hyperplane arrangement $\mathcal{H}_X$.

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An example of arrangements

Take $V = \mathbb{C}^2$ with coordinates (x, y), $T = \mathbb{C}^{*2}$ with coordinates (t, s), and

$$X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2.$$

We associate to X three objects:

1. a finite hyperplane arrangement $\mathcal{H}_X$ given in V by the equations

$$2x = 0,\ 3y = 0,\ -x + y = 0;$$

2. a periodic hyperplane arrangement $\mathcal{A}_X$ given in in V by the conditions

$$2x \in \mathbb{Z},\ 3y \in \mathbb{Z},\ -x + y \in \mathbb{Z};$$

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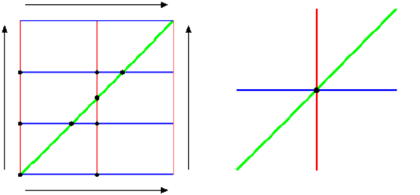

toric arrangement   hyperplane arrangement

If we replace $(0,3)$ by $(0,1)$ or $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. Then $\mathcal{H}_X$ depends only on the linear algebra of X, whereas $\mathcal{T}_X$ also depends on its arithmetics.

In fact $\mathcal{H}_X$ is related to a number of differentiable problems and objects (e.g. splines), $\mathcal{T}_X$ with their discrete counterparts (e.g. partition functions).

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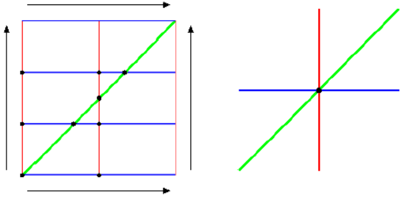

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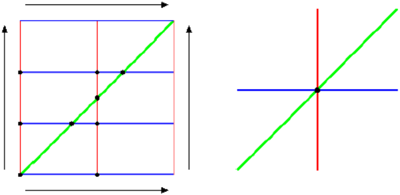

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Tutte polynomial

The Tutte polynomial associated to a list of vectors X is

$$T_X(x,y) \doteq \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}.$$

This polynomial embodies a lot of information on $\mathcal{H}_X$ and $D(X)$:

1. The number of regions of the complement in $\mathbb{R}^n$ is $T_X(2,0)$;
2. the Poincaré polynomial of the complement in $\mathbb{C}^n$ is $q^n T_X(\frac{q+1}{q}, 0)$
3. the Hilbert series of $D(X)$ is $T_X(1,y)$.

(Follows from work of Zaslawsky, Orlik and Solomon, De Boor and Hollig, ...)

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Arithmetic Tutte polynomial

Problem

Define a "Tutte polynomial" for $\mathcal{T}_X$ and $DM(X)$.

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}] .$$

Then we define an arithmetic Tutte polynomial $M_X(x, y)$:

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Theorem (M.)

1. *The number of regions of the complement in ${\mathbb{S}_1}^n$ is $M_X(1, 0)$;*
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Then we define an arithmetic Tutte polynomial $M_X(x, y)$:

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1. *The number of regions of the complement in ${\mathbb{S}_1}^n$ is $M_X(1, 0)$;*
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Arithmetic Tutte polynomial

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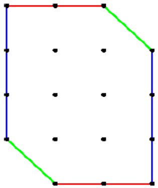
The zonotope

Let $U_{\mathbb{R}}$ be the real vector space spanned by the elements of X. Then we define in $U_{\mathbb{R}}$ the zonotope

$$\mathcal{Z}(X) \doteq \left\{ \sum_{a_i \in X} t_i a_i, 0 \leq t_i \leq 1 \right\}.$$

In our example $X = \{(2,0), (0,3), (1,-1)\}$, we have:



This convex polytope plays a central role both in the theory of arrangements and in that of partition functions.

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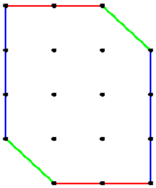

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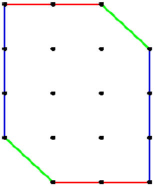

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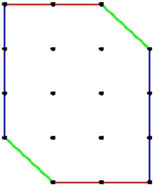

Theorem (M.-D'Adderio)

1. $M_X(1,1)$ *equals the volume of the zonotope* $\mathcal{Z}(X)$;
2. $M_X(2,1)$ *is the number of integer points in* $\mathcal{Z}(X)$;
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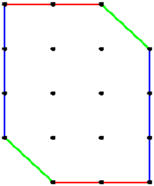

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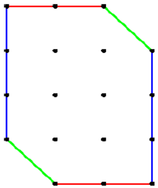

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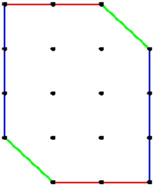

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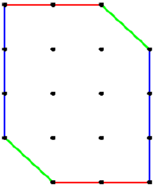

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Deletion-contraction

Furthermore, the polynomial $M_X(x, y)$ satisfies a deletion-contraction formula.

This requires to extend its definition to the case of a list X in a finitely generated abelian group G.

The classical Tutte polynomial was originally introduced for graphs: many invariants like the chromatic polynomial and the flow polynomial are computed by deletion-contraction. The Tutte polynomial is the most general deletion-contraction invariant of a graph.

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Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

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For example, let $(\mathcal{G}, \ell)$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
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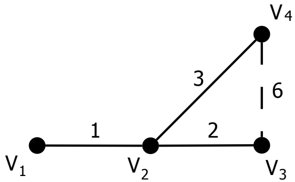

Oriented labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

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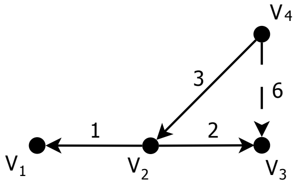
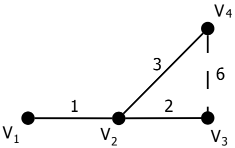


Deletion and contraction



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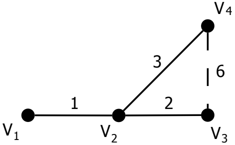
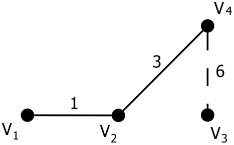
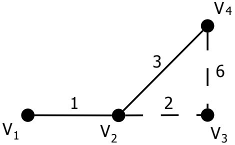




Deletion of $\{v_2, v_3\}$.



Contraction of $\{v_2, v_3\}$.

Arithmetic colorings

For our results we will consider only positive integers q such that $\ell(e)$ divides q for all $e \in E$. We will call such an integer admissible.

A (proper)arithmetic q-coloring of a labelled graph $(\mathcal{G}, \ell)$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G},\ell}(q)$ of $(\mathcal{G}, \ell)$ is defined as the number of (proper) arithmetic q-colorings of $(\mathcal{G}, \ell)$.

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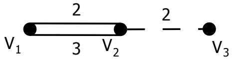
We have:

$$2c(v_1) \neq 2c(v_2)\ ,\ 3c(v_1) \neq 3c(v_2)$$



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We can color v_1 in q ways, then v_2 in $q - 3 - 2 + 1$ ways, then v_3 in 2 ways, so $\chi_{\mathcal{G},\ell}(q) = 2q(q-4) = 2q^2 - 8q$.

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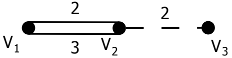

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Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{\substack{e^+=v \\ e\in E_\theta}} \ell(e)\cdot w(e) - \sum_{\substack{e^-=v \\ e\in E_\theta}} \ell(e)\cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.
The *arithmetic flow polynomial* $\chi^*_{\mathcal{G},\ell}(q)$ of $(\mathcal{G}, \ell)$ is defined as the number of (nowhere zero) arithmetic q-flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

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$$(1)\forall v \in V, \sum_{\substack{e^+=v \\ e\in E_\theta}} \ell(e)\cdot w(e) - \sum_{\substack{e^-=v \\ e\in E_\theta}} \ell(e)\cdot w(e) = \overline{0} \in \mathbb{Z}/q\mathbb{Z}$$

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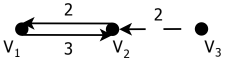
The equation $2x - 3y = 0$ has q solutions, but 4 of them are not nowhere zero.



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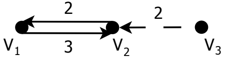

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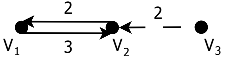

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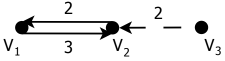

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Graphical toric arrangements

We associate to each labelled graph $(\mathcal{G}, \ell)$ a list of elements of a group in the following way.

To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$$

Then we look at the image of the list X_R in the group $G := \mathbb{Z}^n / \langle X_D \rangle$

We denote by $M_{\mathcal{G},\ell}(x, y)$ the associated arithmetic Tutte polynomial.

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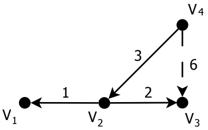

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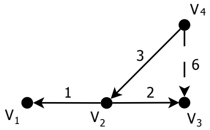

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Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

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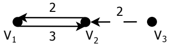

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