Codes for Trees

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Joint work with:
Ignacio M. Pelayo (Univ. Politècnica da Catalunya)

A sharp lower bound for the *locating-domination number* of a tree. A realization result for this code. A conjectured upper for the identifying code and some open problems.

Using distance: Locating sets/ Metric dimension

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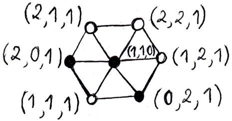

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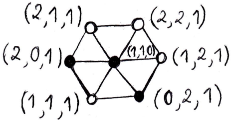

- Introduced by Harary and Melter (1976)

Using neighbors: Dominating sets/ Domination number

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- Domination in Graphs : Haynes, Hedetniemi, Slater (1998)

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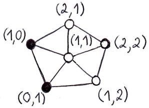

Only binary: Location-domination number

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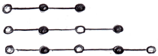
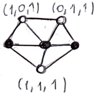
- *location domination number* $\lambda(G) =$ minimum cardinality of a locating-dominating set,
- $\lambda(W_{1,5}) = \eta(W_{1,5})$ $\lambda(P_n) = \lceil \frac{2n}{5} \rceil \neq \eta(P_n) = \lceil \frac{n}{3} \rceil$





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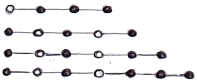
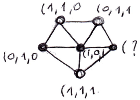
- $4 = \iota(W_{1,5}) \neq \lambda(W_{1,5}) = 3$ $\iota(P_n) = \lceil \frac{n+1}{2} \rceil$





Known for trees

- The calculation of the metric dimension of a tree is a well studied problem with different contributions, since the refered paper of Harary and Melter. (eg,. (*Landmarks in graphs*, Khuller et. al.(1996)) . There is closed formula for $\beta(T)$.

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- Covering codes for trees and $\gamma(T)$ are completely studied.
- In 2004 Henning and Oellermann showed that $\eta(T)$ can be calculated using the covering code of the tree T:

$$\eta(T) = \gamma(T) + l(T) - s(T)$$

$l(T)$ number of leaves (any degree one vertex is a *leaf*)
$s(T)$ number of support vertices (any vertex adjacent to a leaf is a *support vertex*)

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- In the same paper of Henning and Oellermann they relate the binary parameter with the metric one, proving that: $\eta(T) \leq \lambda(T) \leq 2\eta(T) - 2$.
- All the values on the previous interval can occur: take the star with r branches with 3 vertices and s branches of 4 vertices. Then $\eta(T) = r + s + 1$ and $\lambda(T) = r + 2s$ (extremal cases with $s = 0$ making $\lambda = \eta = r + 1$ and $s = 0$ making $\lambda = 2\eta - 2 = r + 2s$
- Blidia et. al., in 2007, showed that $\frac{|V(T)|+l(T)-s(T)}{2}$ is a sharp upper bound for $\lambda(T)$.

A good lower bound for λ

- Slater, in 1987 , showed that $\frac{|V(T)|+1}{3}$ is a lower bound for $\lambda(T)$ and constructed an infinite family of trees with this value of λ, all them with $l(T) = s(T)$

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- In general, $\lambda(T) \geq \frac{(|V(T)|+2(l(T)-s(T))+1}{3}$ and the bound is sharp.
- Given a tree T_2 with $c = l(T_2) - s(T_2) > 0$ build another tree T_1 by deleting all but one of the leaves on each support vertex and apply Slater result to this one.
  $|V(T_2)| = |V(T_1)| + c \leq (3\lambda(T_1) - 1) + c =$
  $(3(\lambda(T_2) - c) - 1) + c = 3\lambda(T_2) - 2c - 1$

A realization result with trees for λ

- Theorem

  $\forall a, b, c \in \mathbb{N}$ such that:

  - $0 \leq c < b < a$
  - $2b - c \leq a \leq 3b - 2c - 1$

  There is a tree $T = T(a, b, c)$ such that:

  - $|V(T)| = a$
  - $\lambda(T) = b$
  - $l(T) - s(T) = c$

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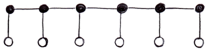
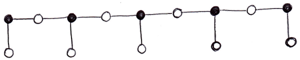
- $c = 0 \;\; a = 2b$



$T(12, 6)$

A realization result with trees for λ(cont.)

- $c = 0, \ a = 3b - 1$



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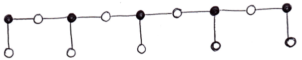

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- $c = 0,\ 2b < a < 3b - 1$ make convenient subdivision of the edges connecting support vertices of $T(2b, b)$

A realization result with trees for λ(cont.)

- $c > 0$ $0 < c < b < a$ the pair $(a - c, b - c)$ verifies
  $0 < b - c < a - c$
  as $2b - c \leq a \leq 3b - 2c - 1$ then
  $2(b - c) \leq a - c \leq 3(b - c) - 1$

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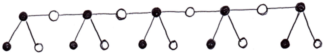
  as $2b - c \leq a \leq 3b - 2c - 1$ then
  $2(b - c) \leq a - c \leq 3(b - c) - 1$
- Construct $T_0 = T(a - c, b - c)$ ($l(T_0) = s(T_0)$)

A realization result with trees for λ(cont.)

- $T = T(a, b, c)$ can be obtained from T_0 adding c new leaves connected to the support vertices of T_0



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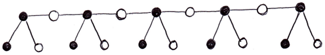

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- $|V(T)| = |V(T_0)| + c = (a - c) + c = a$,
  $\lambda(T)| = \lambda(T_0) + c = (b - c) + c = b$ and
  $l(T) = l(T_0) + c = s(T_0) + c = s(T) + c$.

A conjecture and some open problems

- Solve the equations:
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- Given "good" $a, b \in \mathbb{N}$ construct a tree with $|V(T)| = a$ and $\iota(T) = b$.

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- Solve the equation $\iota(T) = \lambda(T)$

Bibliography

- M. Blidia, M Chellali, F. Maffray, J. Moncel, A. Semri, *Locating-domination and identifying codes in trees*, Australian J. of Combinatorics, 39 (2007), 219-232.

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