Combinatorial Properties of the Temperley–Lieb Algebra of a Coxeter Group

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Motivation

By study of the combinatorial properties of the Temperley–Lieb algebra we mean the study of two families of polynomials which arise naturally in the context of the Temperley–Lieb algebra associated to a Coxeter group. These polynomials are the analogous of the well–known R–polynomials and Kazhdan–Lusztig polynomials defined in the context of the Hecke algebra of a Coxeter group.

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This work was motivated by the fact that, on the one hand, the Kazhdan–Lusztig polynomials and the R–polynomials have been studied a lot, since they were first defined. On the other hand, no one has ever studied their analogous in the Temperley–Lieb algebra.

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The main purpose of this work is to highlight the analogies between these polynomials.

Outline

1. Preliminaries
   - Coxeter Groups
   - The Hecke Algebra
   - The Generalized Temperley–Lieb Algebra
   - Polynomials $D_{x,w}$
2. My Results
   - Combinatorial Properties of $D_{x,w}$
   - Combinatorial properties of $L_{x,w}$
   - Combinatorial properties of $a_{x,w}$

Outline

Basic Definitions

A Coxeter Matrix of order n is a symmetric matrix $m : [n] \times [n] \to \mathbb{P} \cup \{\infty\}$ such that

$$m(i,j) = 1 \iff i = j, \forall i,j \in [n].$$

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A Coxeter System associated to a Coxeter matrix m is a pair (W,S), where W is a group with set of generators $S = \{s_1, \ldots, s_n\}$ and relations

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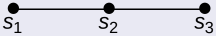
A Coxeter Graph of a Coxeter system (W, S) is the graph whose node set is S and whose edges are the unordered pairs $\{s_i, s_j\}$ such that $m(i,j) \geq 3$. The edges $\{s_i, s_j\}$ such that $m(i,j) \geq 4$ are labelled by the number $m(i,j)$.

An Example

Example

Coxeter matrix and corresponding Coxeter graph, which we will denote by A_3:

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \longleftrightarrow$$



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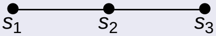


The previous Coxeter matrix determines a group $W = W(A_3)$ generated by s_1, s_2, and s_3 subject to the relations $s_i^2 = e$ and

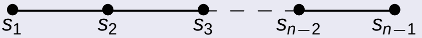
$$\begin{cases} s_1 s_2 s_1 = s_2 s_1 s_2, & \longleftrightarrow \quad m(1,2) = 3 \\ s_3 s_2 s_3 = s_2 s_3 s_2, & \longleftrightarrow \quad m(2,3) = 3 \\ s_1 s_3 = s_3 s_1 & \longleftrightarrow \quad m(1,3) = 2 \end{cases}$$

The Symmetric Group

Let (W, S) be the Coxeter system associated to the Coxeter graph X. Then we say that (W, S) has type X.

Theorem

The pair (S_n, S) is a Coxeter system of type



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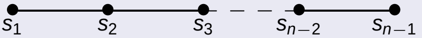

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denoted by A_{n-1}, with $(n \geq 1)$.

Group isomorphism: $s_i \mapsto (i, i+1)$. Hence, S_n is generated by $s_1, s_2, \cdots, s_{n-1}$ such that $s_i^2 = e$ and subject to

$$\begin{cases} s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{if } |i-j| \geq 2 \end{cases}$$

Length Function and Bruhat order

Any element $w \in W(X)$ can be written as product of generators. The length of w, denoted by $\ell(w)$, is the minimal k such that w can be written as the product of k generators. If $w = s_{i_1} \cdots s_{i_k}$ and $k = \ell(w)$ then $s_{i_1} \cdots s_{i_k}$ is called a reduced expression or a reduced word of w.

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We may define a partial order relation $\leq$ on $W(X)$, called the Bruhat order relation. The following is a characterization of the Bruhat order relation.

Theorem (Subword Property)

Let $x, w \in W(X)$ and let $s_1 s_2 \cdots s_q$ be a reduced expression of w. Then $x \leq w$ if and only if x admits a reduced expression of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq q$. In this case we say that x is a subword of w.

Fully Commutative Elements

Definition (J. R. Stembridge)

An element $w \in W(X)$ is fully commutative if any reduced expression for w can be obtained from any other by applying Coxeter relations that involve only commuting generators. Let

$$W_c(X) \stackrel{\text{def}}{=} \{w \in W(X) : w \text{ is a fully commutative element}\}.$$

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Therefore $W_c(A_{n-1})$ may be described as the set of elements of $W(A_{n-1})$ whose reduced expressions avoid substrings of the form $s_i s_{i+1} s_i$, for all $i \in [n-2]$.

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Theorem (S. C. Billey; W. Jockush; R. P. Stanley)

$$S_n(321) = W_c(A_{n-1}), \text{ for all } n \geq 2.$$

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Definition of Hecke Algebra

Let $\mathcal{A}$ be the ring of Laurent polynomials $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

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Definition

The Hecke algebra $\mathcal{H}(X)$ associated to $W(X)$ is an $\mathcal{A}$–algebra with linear basis $\{T_w : w \in W(X)\}$. For all $w \in W(X)$ and $s \in S(X)$ the multiplication law is determined by

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases}$$

We refer to $\{T_w : w \in W(X)\}$ as the *standard basis* for $\mathcal{H}(X)$.

Involution and R–polynomials in $\mathcal{H}(X)$

Define a map $\mathrm{j} : \mathcal{H} \to \mathcal{H}$ such that $\mathrm{j}(T_w) = (T_{w^{-1}})^{-1}$, $\mathrm{j}(q) = q^{-1}$ and linear extension. The map j is a ring homomorphism of order 2 on $\mathcal{H}(X)$.

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To express $\mathrm{j}(T_w)$ as a linear combination of elements in the standard basis, one defines the so–called R–polynomials.

Theorem (D. Kazhdan; G. Lusztig)

Let $\varepsilon_x \stackrel{\text{def}}{=} (-1)^{\ell(x)}$, for every $x \in W(X)$. There is a unique family of polynomials $\{R_{x,w}(q)\}_{x,w \in W(X)} \subseteq \mathbb{Z}[q]$ such that

$$T_{w^{-1}}^{-1} = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x,$$

where $R_{x,x}(q) = 1$ and $R_{x,w}(q) = 0$ if $x \not\leq w$.

Canonical Basis for $\mathcal{H}(X)$

Theorem (D. Kazhdan; G. Lusztig)

There exists a unique basis $\{C'_w : w \in W(X)\}$ for $\mathcal{H}(X)$ such that

(i) $\mathrm{j}(C'_w) = C'_w$,

(ii) $C'_w = q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x,w}(q) T_x$,

where $\deg(P_{x,w}(q)) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$, $P_{x,x}(q) = 1$ *and* $P_{x,w}(q) = 0$ *if* $x \not\leq w$.

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We will refer to the latter basis as the *Kazhdan–Lusztig basis* for $\mathcal{H}(X)$.

Outline

Definition of Generalized Temperley–Lieb Algebra

Consider the two–sided ideal $J(X)$ generated by all elements of $\mathcal{H}(X)$ of the form $\sum_{w \in \langle s_i, s_j \rangle} T_w$, where (s_i, s_j) runs over all pairs in $S(X)^2$ such that $2 < m(i,j) < \infty$.

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Definition (H. N. V. Temperley; E. H. Lieb)

Let X be a Coxeter graph of type A. The Temperley–Lieb algebra is

$$TL(X) \stackrel{\text{def}}{=} \mathcal{H}(X)/J(X).$$

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J. J. Graham extended this definition to arbitraty Coxeter graphs and he showed that the *generalized Temperley–Lieb algebra* is finite dimensional when X is a finite irreducible Coxeter graph.

Multiplication Law

Let $t_w = \sigma(T_w)$, where $\sigma : \mathcal{H} \to \mathcal{H}/J$ is the canonical projection.

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Proposition (J. J. Graham)

The generalized Temperley–Lieb algebra $TL(X)$ admits an $\mathcal{A}$–basis of the form $\{t_w : w \in W_c(X)\}$. It satisfies

$$t_w t_s = \begin{cases} t_{ws} & \text{if } \ell(ws) > \ell(w), \\ q t_{ws} + (q-1) t_w & \text{if } \ell(ws) < \ell(w). \end{cases}$$

We call $\{t_w : w \in W_c(X)\}$ the *t–basis* of $TL(X)$

Involution and Polynomials $a_{x,w}$

The map j induces an involution on $TL(X)$, which we still denote by j.Therefore $\mathrm{j}(t_w) = (t_{w^{-1}})^{-1}$ and $\mathrm{j}(q) = q^{-1}$.

Involution and Polynomials $a_{x,w}$

The map j induces an involution on $TL(X)$, which we still denote by j.Therefore $\mathrm{j}(t_w) = (t_{w^{-1}})^{-1}$ and $\mathrm{j}(q) = q^{-1}$.
We have seen that the R–polynomials express the coordinates of $\mathrm{j}(T_w)$ with respect to the standard basis of $\mathcal{H}(X)$. The polynomials $a_{x,w}$ play the same role in $TL(X)$.

Proposition (R. M. Green; J. Losonczy)

Let $w \in W_c(X)$. Then there exists a unique family of polynomials $\{a_{y,w}(q)\} \subset \mathbb{Z}[q]$ such that

$$(t_{w^{-1}})^{-1} = q^{-\ell(w)} \sum_{\substack{y \in W_c(X) \\ y \leq w}} a_{y,w}(q) t_y,$$

where $a_{w,w}(q) = 1$ and $a_{y,w}(q) = 0$ if $y \not\leq w$.

The *IC Basis*

The generalized Temperley–Lieb algebra admits a basis $\{c_w : w \in W_c(X)\}$, called *IC basis*, which is analogous to the Kazhdan–Lusztig basis $\{C'_w : w \in W(X)\}$ of $\mathcal{H}(X)$.

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Theorem (R. M. Green; J. Losonczy)

There exists a unique basis $\{c_w : w \in W_c(X)\}$ for $TL(X)$ such that

(i) $\mathrm{j}(c_w) = c_w$,

(ii) $c_w = \sum_{\substack{x \in W_c \\ x \leq w}} q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}) t_x$,

where $\{L_{x,w}(q^{-\frac{1}{2}})\} \subset q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$, $L_{x,x}(q^{-\frac{1}{2}}) = 1$ *and* $L_{x,w}(q^{-\frac{1}{2}}) = 0$ *if* $x \not\leq w$.

Analogies

We make clear the general setting by means of the following diagrams. The arrow $\xrightarrow{\sigma}$ denotes the canonical projection.

$$\begin{array}{ccccc}
\mathcal{H}(X) & \dashrightarrow & \{T_w : w \in W(X)\} & \dashrightarrow & \{C'_w : w \in W(X)\} \\
\downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma \\
TL(X) & \dashrightarrow & \{t_w : w \in W_c(X)\} & \dashrightarrow & \{c_w : w \in W_c(X)\}
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\begin{array}{ccccc}
\mathcal{H}(X) & \dashrightarrow & R\text{–polynomials} & \dashrightarrow & \text{K–L polynomials} \\
\sigma \downarrow & & \updownarrow & & \updownarrow \\
TL(X) & \dashrightarrow & \text{Polynomials } \{a_{x,w}\} & \dashrightarrow & \text{Polynomials } \{L_{x,w}\}
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Recursive Fromula for $D_{x,w}$

Proposition (A. Pesiri)

Let X be an arbitrary Coxeter graph. Let $w \notin W_c(X)$ and $s \in S(X)$ be such that $w > ws \notin W_c(X)$. Then, for all $x \in W_c(X)$, $x \leq w$, we have

$$D_{x,w}(q) = \tilde{D} + \sum_{\substack{y \in W_c(X),\, ys \notin W_c(X) \\ ys > y}} D_{x,ys}(q) D_{y,ws}(q),$$

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Observe that this recursion is similar to the one for the parabolic Kazhdan–Lusztig polynomials.

Branching Coxeter Graph

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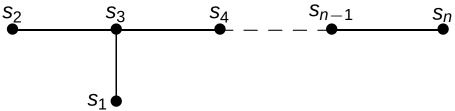
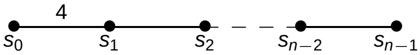


$D_n \quad (n \geq 4)$



$B_n \quad (n \geq 2)$

Non–recursive Formula for $D_{X,w}$

From now on, X will always denote a finite irreducible non–branching Coxeter graph.
The following theorem is the main result of this work.

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Theorem (A. Pesiri)

For all $x \in W_c(X)$ and $w \notin W_c(X)$ such that $x < w$, we have

$$D_{x,w}(q) = \sum \left((-1)^k \prod_{i=1}^{k} P_{x_{i-1},x_i}(q) \right),$$

where the sum is taken over all the chains $x = x_0 < x_1 < \cdots < x_k = w$ such that $x_i \notin W_c(X)$ if $i > 0$, and $1 \leq k \leq \ell(x,w)$.

Corollaries

Corollary (A. Pesiri)

Let $x \in W_c(X)$ and $w \notin W_c(X)$ be such that $x < w$. Then

- $D_{x,w}(q) = D_{x^{-1},w^{-1}}(q)$;
- $D_{x,w}(q) = D_{w_0xw_0,w_0ww_0}(q)$,

where w_0 denotes the maximum in $W(X)$.

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In [1], Green and Losonczy state that a degree bound on $D_{x,w}$ may be of interest. Here is the answer.

Corollary (A. Pesiri)

Let $x \in W_c(X)$ and $w \notin W_c(X)$ be such that $x < w$. Then

$$deg(D_{x,w}(q)) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1).$$

Explicit formulas

We obtain some explicit formulas for the polynomials $D_{x,w}$ such that the Bruhat interval $[x, w]$ has a particular structure.

Recall that $\varepsilon_x \stackrel{\text{def}}{=} (-1)^{\ell(x)}$, for every $x \in W(X)$.

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Proposition (A. Pesiri)

Let $s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1$ be a reduced expression for $w \in W(A_n)$ and let $x \in W(A_n)$ be a Coxeter element. Then

$$D_{x,w}(q) = \varepsilon_x \varepsilon_w .$$

Explicit formulas

We obtain some explicit formulas for the polynomials $D_{x,w}$ such that the Bruhat interval $[x, w]$ has a particular structure.
Recall that $\varepsilon_x \stackrel{\text{def}}{=} (-1)^{\ell(x)}$, for every $x \in W(X)$.

Proposition (A. Pesiri)

Let $s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1$ be a reduced expression for $w \in W(A_n)$ and let $x \in W(A_n)$ be a Coxeter element. Then

$$D_{x,w}(q) = \varepsilon_x \varepsilon_w .$$

The previous result can be conveniently generalized to arbitrary finite irreducible non–branching Coxeter graphs.

Outline

1. Preliminaries
   - Coxeter Groups
   - The Hecke Algebra
   - The Generalized Temperley–Lieb Algebra
   - Polynomials $D_{x,w}$
2. My Results
   - Combinatorial Properties of $D_{x,w}$
   - Combinatorial properties of $L_{x,w}$
   - Combinatorial properties of $a_{x,w}$

Non–recursive Formula for $L_{x,w}$

Theorem (A. Pesiri)

For all elements $x, w \in W_c(X)$ such that $x < w$ we have

$$L_{x,w}(q^{-\frac{1}{2}}) = q^{\frac{\ell(x)-\ell(w)}{2}} \sum \left((-1)^k \prod_{i=1}^{k+1} P_{x_{i-1},x_i}(q) \right),$$

where the sum runs over all the chains $x = x_0 < x_1 < \cdots < x_{k+1} = w$ such that $x_i \notin W_c(X)$ if $1 \leq i \leq k$, and $0 \leq k \leq \ell(x,w) - 1$.

Corollaries

Corollary (A. Pesiri)

Let $x, w \in W_c(X)$ be such that $x \leq w$. Then

- $L_{x,w}(q^{-\frac{1}{2}}) = L_{x^{-1},w^{-1}}(q^{-\frac{1}{2}})$;
- $L_{x,w}(q^{-\frac{1}{2}}) = L_{w_0 x w_0, w_0 w w_0}(q^{-\frac{1}{2}})$.

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Corollary (A. Pesiri)

Let $v \in W_c(X)$ and define

$$F_v(q) \stackrel{\text{def}}{=} \sum_{\substack{u \in W_c(X) \\ u \leq v}} \varepsilon_u q^{-\frac{\ell(u)}{2}} L_{u,v}(q^{-\frac{1}{2}}).$$

Then $F_v(q) = F_v(q^{-1}) = \delta_{e,v}$.

Outline

Non–recursive Formula for $a_{x,w}$

Proposition (A. Pesiri)

Let $x, w \in W_c(X)$ be such that $x \leq w$. Then

$$a_{x,w}(q) = \varepsilon_x \varepsilon_w R_{x,w}(q) + \\ + \sum_{\substack{y \notin W_c(X) \\ x<y<w}} \varepsilon_y \varepsilon_w R_{y,w}(q) \left(\sum (-1)^k \prod_{i=1}^{k} P_{x_{i-1},x_i}(q) \right),$$

where the second sum runs over all the chains $x = x_0 < \cdots < x_k = y$ such that $x_i \notin W_c(X)$ if $i > 0$.

Corollaries

Corollary (A. Pesiri)

For all $x, w \in W_c(X)$ *such that* $x < w$ *we have*

(i) $a_{x,w}(1) = 0$;

(ii) $a_{x,w}(0) = \sum(-1)^k$,

where the sum is taken over all the chains $x = x_0 < x_1 < \cdots < x_{k+1} = w$ *such that* $x_i \notin W_c(X)$ *if* $1 \leq i \leq k$, *and* $0 \leq k \leq \ell(x,w) - 1$.

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Corollary (A. Pesiri)

Let $x, w \in W_c(X)$. Then we have that

(i) $a_{x,w}(q) = a_{x^{-1},w^{-1}}(q)$;

(ii) $a_{x,w}(q) = a_{w_0 x w_0, w_0 w w_0}(q)$.

More Corollaries

Corollary (A. Pesiri)

Let $w \in W_c(X)$. Then

$$\sum_{\substack{x \in W_c(X) \\ x \leq w}} \varepsilon_x \varepsilon_w a_{x,w}(q) = q^{\ell(w)}.$$

More Corollaries

Corollary (A. Pesiri)

Let $w \in W_c(X)$. Then

$$\sum_{\substack{x \in W_c(X) \\ x \leq w}} \varepsilon_x \varepsilon_w a_{x,w}(q) = q^{\ell(w)}.$$

Lastly, we are able to compute the degree of $a_{x,w}$.

Corollary (A. Pesiri)

Let $x, w \in W_c(X)$ and $x \leq w$. Then $deg(a_{x,w}(q)) = \ell(w) - \ell(x)$.

For Further Reading I

Green, R. M. ; Losonczy, J.
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Math. Res. Lett., **6** (1999), no. 2, 213–222.

Green, R. M. ; Losonczy, J.
A projection property for Kazhdan-Lusztig bases.
Internat. Math. Res. Notices, 2000, no. 1, 23–34.

Losonczy, Jozsef.
The Kazhdan-Lusztig basis and the Temperley-Lieb quotient in type D.
J. Algebra, **233** (2000), no. 1, 1–15.

Rules

Consider the symmetric group $S_4 \cong W(A_3)$. In $TL(A_3)$ the following relations hold:

$$t_{s_i s_{i+1} s_i} + t_{s_i s_{i+1}} + t_{s_{i+1} s_i} + t_{s_i} + t_{s_{i+1}} + t_e = 0, \text{ for all } i \in \{1, 2\}.$$

By the expression untying the braid, we mean performing the substitution

$$t_{s_i s_{i+1} s_i} = -t_{s_i s_{i+1}} - t_{s_{i+1} s_i} - t_{s_i} - t_{s_{i+1}} - t_e.$$

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Recall that

$$t_w t_s = \begin{cases} t_{ws} & \text{if } \ell(ws) > \ell(w), \\ q t_{ws} + (q-1) t_w & \text{if } \ell(ws) < \ell(w). \end{cases}$$

Worked Example

Let $w = s_1 s_2 s_3 s_2 s_1 = [1, 2, 3, 2, 1] \in W(A_3)$. To compute $D_{x,w}(q)$ we have to untie the braids.

Worked Example

Let $w = s_1 s_2 s_3 s_2 s_1 = [1,2,3,2,1] \in W(A_3)$. To compute $D_{x,w}(q)$ we have to untie the braids.

$$t_{1,2,3,2,1} = t_1 \cdot t_{2,3,2} \cdot t_1$$

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$$\begin{aligned} t_{1,2,3,2,1} &= t_1 \cdot t_{2,3,2} \cdot t_1 \\ &= t_1 \cdot (-t_{2,3} - t_{3,2} - t_2 - t_3 - t_e) \cdot t_1 \end{aligned}$$

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&= -t_{1,2,3,1} - t_{1,3,2,1} - t_{1,2,1} - t_{1,3} \cdot t_1 - t_1 \cdot t_1 \\
&= -t_{1,2,1,3} - t_{3,1,2,1} - t_{1,2,1} - (qt_3 + (q-1)t_{1,3}) + \\
&\quad - (qt_e + (q-1)t_1)
\end{aligned}$$

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Let $w = s_1 s_2 s_3 s_2 s_1 = [1,2,3,2,1] \in W(A_3)$. To compute $D_{x,w}(q)$ we have to untie the braids.

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t_{1,2,3,2,1} &= t_1 \cdot t_{2,3,2} \cdot t_1 \\
&= t_1 \cdot (-t_{2,3} - t_{3,2} - t_2 - t_3 - t_e) \cdot t_1 \\
&= -t_{1,2,3,1} - t_{1,3,2,1} - t_{1,2,1} - t_{1,3} \cdot t_1 - t_1 \cdot t_1 \\
&= -t_{1,2,1,3} - t_{3,1,2,1} - t_{1,2,1} - (qt_3 + (q-1)t_{1,3}) + \\
&\quad - (qt_e + (q-1)t_1) \\
&= \cdots \cdots \cdots \\
&= (1-q)t_e + (2-q)t_1 + t_2 + (2-q)t_3 + (3-q)t_{1,3} + \\
&\quad + t_{1,2} + t_{2,1} + t_{2,3} + t_{3,2} + t_{1,2,3} + t_{3,2,1} + t_{1,3,2} + t_{2,1,3}.
\end{aligned}$$

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&= -t_{1,2,3,1} - t_{1,3,2,1} - t_{1,2,1} - t_{1,3} \cdot t_1 - t_1 \cdot t_1 \\
&= -t_{1,2,1,3} - t_{3,1,2,1} - t_{1,2,1} - (qt_3 + (q-1)t_{1,3}) + \\
&\quad - (qt_e + (q-1)t_1) \\
&= \cdots\cdots\cdots \\
&= (1-q)t_e + (2-q)t_1 + t_2 + (2-q)t_3 + (3-q)t_{1,3} + \\
&\quad + t_{1,2} + t_{2,1} + t_{2,3} + t_{3,2} + t_{1,2,3} + t_{3,2,1} + t_{1,3,2} + t_{2,1,3}.
\end{aligned}$$

Therefore we get $D_{s_1,w}(q) = D_{s_3,w}(q) = 2-q$, $D_{s_1 s_3,w} = 3-q$ and $D_{x,w}(q) = 1$ for the rest of the elements $x \leq w$.

A Key Observation

One may wonder whether the map $\sigma : \mathcal{H}(X) \to \mathcal{H}(X)/J(X)$ satisfies

$$\sigma(C'_w) = \begin{cases} c_w & \text{if } w \in W_c(X), \\ 0 & \text{if } w \notin W_c(X). \end{cases}$$

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Proposition

The answer is affirmative for non–branching graphs, that is for types A, B, $I_2(m)$, F_4, H_3 and H_4, and negative for branching graphs, that is for types D, E_6, E_7 and E_8.

Sketch of the Proof. I

$$\sigma(C'_w) = q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x,w}(q)\sigma(T_x)$$

$$= q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x,w}(q) \left(\sum_{\substack{y \in W_c(X) \\ y \leq x}} D_{y,x}(q) t_y \right)$$

$$= q^{-\frac{\ell(w)}{2}} \sum_{\substack{y \in W_c(X) \\ y \leq w}} \left(\sum_{y \leq x \leq w} D_{y,x}(q) P_{x,w}(q) \right) t_y$$

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$$= q^{-\frac{\ell(w)}{2}} \sum_{\substack{y \in W_c(X) \\ y \leq w}} \left(\sum_{y \leq x \leq w} D_{y,x}(q) P_{x,w}(q) \right) t_y$$

On the other hand, when $w \notin W_c(X)$ we get $\sigma(C'_w) = 0$. Therefore the expression highlighted in red is equal to 0.

Sketch of the Proof. II

Keep in mind that

$$\sum_{y \le x \le w} D_{y,x}(q) P_{x,w}(q) = 0, \text{ for all } y \in W_c(X)$$

and proceed by induction on $\ell(x, w) \stackrel{\text{def}}{=} \ell(w) - \ell(x)$.

Sketch of the Proof. II

Keep in mind that

$$\sum_{y \le x \le w} D_{y,x}(q) P_{x,w}(q) = 0, \text{ for all } y \in W_c(X)$$

and proceed by induction on $\ell(x,w) \stackrel{\text{def}}{=} \ell(w) - \ell(x)$.

If $\ell(x,w) = 1$, then we get $D_{x,w}(q) = -P_{x,w}(q)$. If $\ell(x,w) > 1$, then

$$D_{x,w}(q) = -P_{x,w}(q) - \sum_{\substack{t \notin W_c(X) \\ x < t < w}} D_{x,t}(q) P_{t,w}(q)$$

and the statement follows by applying the induction hypothesis on $D_{x,t}(q)$.