Combinatorics and Symmetrized Tensors

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(Joint work with Maria M. Torres)

Outline

- Terminology on symmetrized tensors

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- ☐ Statement of the problem

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- □ Root systems of symmetrized tensors

Some terminology on symmetrized tensors

$V = \mathbb{C}^n$ and $(e_1, \ldots, e_n)$ o.n. basis of V.

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$$\Gamma_{m,n} = \Big\{\text{words of lenght } m \text{ on the alphabet } [n] = \{1, \ldots, n\}\Big\},$$

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The λ-**symmetry class of tensors** V_λ is the linear span of the set of **decomposable symmetrized tensors**

$$\left\{ e_\alpha^{*\lambda} := \frac{\lambda(id)}{m!} \sum_{\sigma \in S_m} \lambda(\sigma) e_{\alpha\sigma^{-1}(1)} \otimes \ldots \otimes e_{\alpha\sigma^{-1}(m)} \mid \alpha \in \Gamma_{m,n} \right\}.$$

It is well known that,

$$V_\lambda = \bigoplus_{\alpha \in G_{m,n}} V_\alpha^\lambda,$$

where $G_{m,n}$ is the set of weakly increasing words of lenght m on the alphabet $\{1, \ldots, n\}$, and

$$V_\alpha^\lambda = \langle E_\alpha^\lambda \rangle,$$

is the linear span of the **orbital set** associated to $\lambda \vdash m$ and $\alpha \in G_{m,n}$,

$$E_\alpha^\lambda = \{e_{\alpha\sigma}^{*\lambda} \ : \ \sigma \in S_m\}.$$

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We say that the orbital subspace V_α^λ is **critical** if λ is the multiplicity partition of α.

In that case, $\dim V_\alpha^\lambda = \lambda(id)$ and we set $E_\alpha = E_\alpha^\lambda$, $V_\alpha = V_\alpha^\lambda$.

Inner product of tensors

The inner product of two decomposable symmetrized tensors $e_{\alpha}^{*\lambda}$, $e_{\beta}^{*\lambda}$ in the same critical orbital set of multiplicity partition $\lambda \vdash m$ is given by

$$(e_{\alpha}^{*\lambda}, e_{\beta}^{*\lambda}) = \frac{\lambda(id)}{m!} \sum_{\tau \in S_\alpha} \lambda(\sigma^{-1}\tau),$$

where S_α is the stabilizer subgroup of α and $\beta = \alpha\sigma$.

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Results on the existence of orthogonal basis for V_α consisting of decomposable symmetrized tensors by Wang *et al* (1991) and Pournaki (2001).

Orthogonal dimension problem [J Dias da Silva, MM Torres]:

What is the orthogonal dimension of E_α, i.e., the maximum cardinality of an orthogonal subset of a critical orbital set E_α?

This dimension only depends on the multiplicity partition λ of α and shall be denoted $\dim^\perp \lambda$.

Bessenrodt *et al* (2003) and Dias da Silva and Torres (2005) proved that $\dim^\perp(2, 1^{m-2}) = 2$.

Dias da Silva and Torres approach relies on a combinatorial necessary and sufficient condition for the orthogonality in critical orbital sets of symmetry classes of tensors.

λ-regular bipartite graphs

We say that a bipartite graph $G = (X, Y, E)$ with $|X| = |Y|$

is λ-**regular** for some partition $\lambda \vdash |E|$

if we can enumerate the sets of vertices

$$X = \{x_1, \ldots, x_r\} \text{ and } Y = \{y_1, \ldots, y_r\}$$

so that

$$\lambda = (\deg(x_1), \ldots, \deg(x_r)) = (\deg(y_1), \ldots, \deg(y_r)).$$

From now on we will only consider λ-*regular bipartite graphs*.

Full edge colorings

A **full edge coloring** of a λ-regular graph $G = (X, Y, E)$ is an ordered set partition $\mathcal{L} = (U_1, \ldots, U_{\lambda_1})$ of the edge family E, such that each U_j, $j = 1, \ldots, \lambda_1$, is a matching and $\lambda^* := (|U_1|, |U_2|, \ldots, |U_{\lambda_1}|)$ is the conjugate partition of λ.

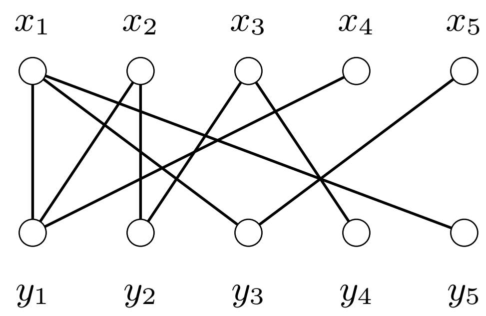
In particular, U_1 is a complete matching of G.

The **sign** of a full edge coloring $\mathcal{L} = (U_1, \ldots, U_{\lambda_1})$ is defined as

$$\operatorname{sign}(\mathcal{L}) = \prod_{i=1}^{\lambda_1} \operatorname{sign}(U_i),$$

where $\operatorname{sign}(U_i)$ is the sign of the permutation of the indices of the vertices of X and Y, induced by the complete matching U_i.

Let $G = (X, Y, E)$ be the λ-regular graph, with $\lambda = (3, 2^2, 1^2)$ and $\lambda^* = (5, 3, 1)$.



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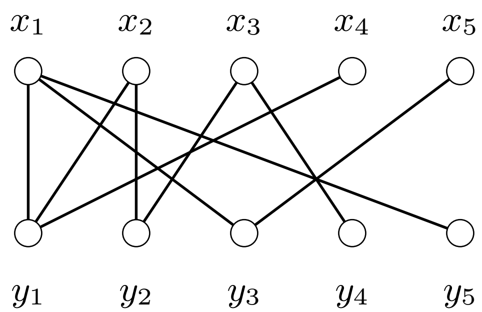

The edge set E is the disjoint union of the sets U_1, U_2, U_3,

$$U_1 = \{\{x_1, y_5\}, \{x_2, y_2\}, \{x_3, y_4\}, \{x_4, y_1\}, \{x_5, y_3\}\},$$
$$U_2 = \{\{x_1, y_3\}, \{x_2, y_1\}, \{x_3, y_2\}\},$$
$$U_3 = \{\{x_1, y_1\}\}.$$

Then $\mathcal{L} = (U_1, U_2, U_3)$ is a full edge coloring of G and

$$\operatorname{sign}(\mathcal{L}) = \operatorname{sign}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix} \cdot \operatorname{sign}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \cdot \operatorname{sign}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1.$$

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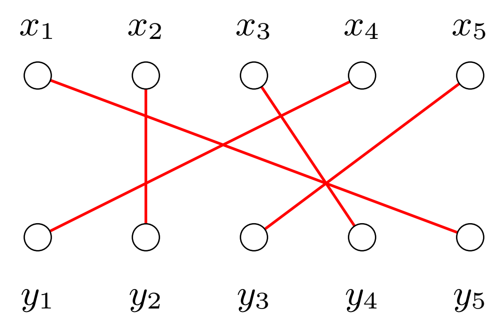

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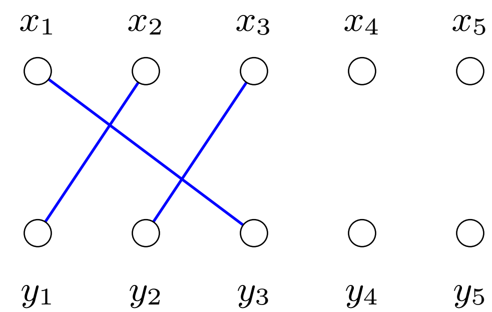

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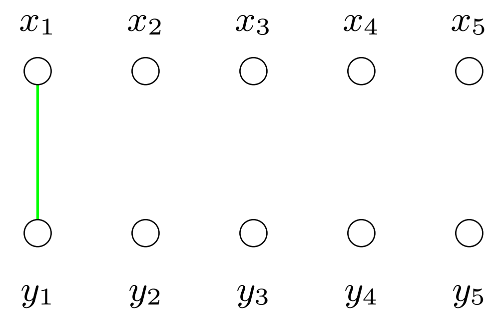

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Strong sign uniform partitions

We call a partition λ **strong sign uniform** if for every λ-regular bipartite graph $G = (X, Y, E)$ we have

- All full edge colorings of G have the same sign, which we denote by $\operatorname{sign}(G)$ (**sign uniform partition**)
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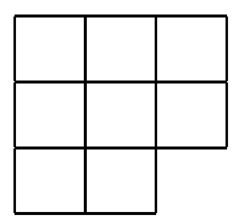
Theorem [JA Dias da Silva, MM Torres]
A partition is sign uniform if and only if its Ferrers diagram does not contain the diagram below.



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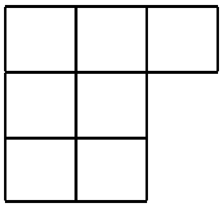

The class of strong sign uniform partitions corresponds to the class of the partitions of the form $(\ell, k, 1^r)$ or $(2^s, 1^t)$.

Let $\alpha \in \Gamma_{m,n}$ be a **normal** word of multiplicity λ i.e., $\alpha(i) < \alpha(j) \Rightarrow \text{mult}(\alpha(i)) \geq \text{mult}(\alpha(j))$.

Let β be a rearrangement of α.

We denote by $G_{\alpha,\beta} = (X, Y, E)$ the λ-regular graph s.t.

- $X = \{x_1, \ldots, x_{\lambda_1^*}\}$ and $Y = \{y_1, \ldots, y_{\lambda_1^*}\}$.
- E is a multiset of multi-edges $\{x_{\alpha(i)}, y_{\beta(i)}\}$, in 1-1 correspondence with the pairs $(\alpha(i), \beta(i))$.

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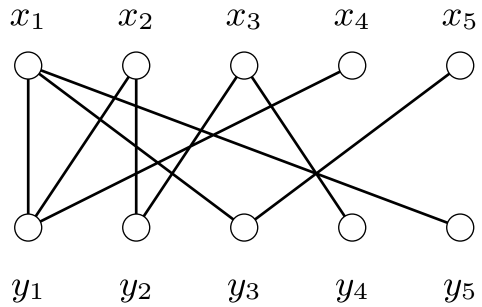
Ex: $\alpha = (1, 1, 1, 2, 2, 3, 3, 4, 5), \beta = (1, 3, 5, 1, 2, 2, 4, 1, 3) \in \Gamma_{7,5}$,

then $G_{\alpha,\beta}$ is the λ-regular bipartite graph with $\lambda = (3, 2^2, 1^2)$,



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Let $\mathcal{C}(G_{\alpha,\beta})$ be the set (possibly empty) of full colorings of $G_{\alpha,\beta}$.

Theorem [JA Dias da Silva, MM Torres]

$$(e_\alpha^{*\lambda}, e_\beta^{*\lambda}) = \begin{cases} 0, & |\mathcal{C}(G_{\alpha,\beta})| = 0, \\ \frac{\lambda(id)}{m!}\mathrm{sign}(G_{\alpha,\beta})|\mathcal{C}(G_{\alpha,\beta})|, & \textit{otherwise.} \end{cases}$$

A combinatorial criterion for orthogonality of tensors

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Using this theorem and calculating the number of full colorings we compute explicitly the inner product of two symmetrized tensors $e_\alpha^{*\lambda}$ and $e_\beta^{*\lambda}$ assuming λ **strong** sign uniform.

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a) If $\lambda = (\ell, 1^t)$,

$$(e^{*\lambda}_{\alpha}, e^{*\lambda}_{\beta}) = \begin{cases} 0, & \text{if } \mu_{1,1} \leq \ell - 2, \\ \\ \frac{\lambda(id)}{m!}\,\mathrm{sign}(G_{\alpha,\beta})\,\mu_{1,1}!, & \text{otherwise.} \end{cases}$$

In particular, $e^{*\lambda}_{\alpha}$ and $e^{*\lambda}_{\beta}$ are orthogonal iff $\mu_{1,1} \leq \ell - 2$.

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b) If $\lambda = (\ell, k)$, $(e^{*\lambda}_{\alpha}, e^{*\lambda}_{\beta}) = \frac{\lambda(id)}{m!}(-1)^{\mu_{1,2}}\,\mu_{1,1}!\,\mu_{1,2}!(\mu_{2,1} + \mu_{2,2})!$.

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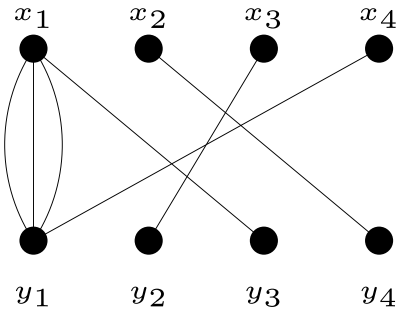
d) If $\lambda = (2^s, 1^t)$, $(e_\alpha^{*\lambda}, e_\beta^{*\lambda}) = \frac{\lambda(id)}{m!} \operatorname{sign}(G_{\alpha,\beta})\mathrm{per}(A_{G_{\alpha,\beta}})$.

Example

If $\alpha = (1, 1, 1, 1, 2, 3, 4)$ and $\beta = (1, 1, 3, 1, 4, 2, 1)$, $G_{\alpha,\beta}$ is the $(4, 1^3)$-regular graph depicted below.



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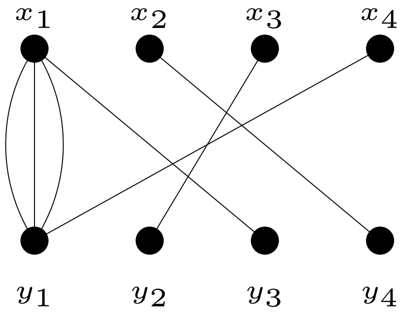

Thus $\mu_{1,1} = 3 > \ell - 2 = 1$, $\mathcal{C}(G_{\alpha,\beta}) = 1$, $\mathrm{sign}(\mathcal{L}) = -1$ and

$$(e_\alpha^{*\lambda}, e_\beta^{*\lambda}) = \frac{\lambda(id)}{7!}\,\mathrm{sign}(G_{\alpha,\beta})\,\mu_{1,1}! = \frac{20}{7!}(-3) = -\frac{1}{84}.$$

In particular, $e_\alpha^{*\lambda}$ and $e_\beta^{*\lambda}$ are not orthogonal.

Connections with coding theory

Denote by $A(n, d, w)$ the maximum number of binary sequences of length n with w *positions equal to 1* and *pairwise Hamming distance greater than or equal to* d.

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3. $A(m,6,\ell+k) \leq \dim^{\perp}(\ell,k,1^{m-\ell-k}) \leq A(m,4,\ell+k)$.
4. $A(m,2s+2,2s) \leq \dim^{\perp}(2^s,1^{m-2s}) \leq A(m,4,2s)$.

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4. $A(m, 2s + 2, 2s) \leq \dim^{\perp}(2^s, 1^{m-2s}) \leq A(m, 4, 2s)$.

The computation of $A(n, w, d)$ is an important open problem in coding theory known as the **error-correcting code problem** which is the discrete analogue of sphere packing problems.

Consequences

- $\dim^{\perp}(2, 1^{m-2}) = \lfloor \frac{m}{2} \rfloor$
  [C Bessenrodt *et al* (2003); JA Dias da Silva and MM Torres (2005)]
- $\dim^{\perp}(w, 1^{n-w}) = \dim^{\perp}(n-w, 1^{w})$
- $\dim^{\perp}(w, 1^{n-w}) \leq \lfloor \frac{n}{w} \dim^{\perp}(w-1, 1^{n-w}) \rfloor$
- $\dim^{\perp}(w, 1^{n-w}) \leq \left\lfloor \frac{n}{n-w} \dim^{\perp}(w, 1^{n-w-1}) \right\rfloor$
- $\dim^{\perp}(w, 1^{n-w}) \leq \dim^{\perp}(w-1, 1^{n-w}) + \dim^{\perp}(w, 1^{n-w-1})$
- $\dim^{\perp}(3, 1^{n-3}) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{m-1}{2} \right\rfloor \right\rfloor, & n \not\equiv 5 \pmod 6 \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1, & n \equiv 5 \pmod 6 \end{cases}$
- $\dim^{\perp}(4, 1^{n-4}) \begin{cases} \frac{n(n-1)(n-2)}{24}, & n \equiv 2 \text{ or } 4 \pmod 6 \\ \frac{n(n-1)(n-3)}{24}, & n \equiv 3 \text{ or } 5 \pmod 6 \\ \frac{n(n^2-3n-6)}{24}, & n \equiv 0 \pmod 6 \end{cases}$
- Etc...

Consider $\lambda = (2, 1^{m-2})$ and $\alpha = (\mathbf{1}, \mathbf{1}, 2, \ldots, m-1)$.

For $1 \leq i < j \leq m$ let $\alpha[i, j]$ be the rearrangement of α,

$$(2, 3, \ldots, i, \mathbf{1}, i+1, i+2, \ldots, j-1, \mathbf{1}, j, j+1, \ldots, m-1),$$

and set

$$\Pi_\alpha = \left\{ e^{*\lambda}_{\alpha[i,i+1]} \, : \, i = 1, \ldots, m-1 \right\}.$$

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Theorem *The set Π_α is a basis for $V_\alpha = \langle E_\alpha \rangle$ s.t.*

$$e^{*\lambda}_\beta = \operatorname{sign}(G_{\alpha[i,j],\beta})(-1)^{j-i+1} \sum_{s=i}^{j-1} e^{*\lambda}_{\alpha[s,s+1]}, \quad \forall\, e^{*\lambda}_\beta \in E_\alpha.$$

where $i < j$ are the positions of β that are equal to one.

Example

Consider $\alpha = (1,1,2,3,4)$ and $\beta = (2,4,1,3,1)$. Then

$$\Pi_\alpha = \left\{ e^{*\lambda}_{(1,1,2,3,4)}, e^{*\lambda}_{(2,1,1,3,4)}, e^{*\lambda}_{(2,3,1,1,4)}, e^{*\lambda}_{(2,3,4,1,1)} \right\},$$

and we get

$$e^{*\lambda}_{(2,4,1,3,1)} = (-1)\times(-1)^{5-3+1} \sum_{s=3}^{4} e^{*\lambda}_{\alpha[s,s+1]} = e^{*\lambda}_{(2,3,1,1,4)} + e^{*\lambda}_{(2,3,4,1,1)}.$$

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Let $V_\alpha^{\mathbb{R}} = \bigoplus_{s=1}^{m-1} \mathbb{R}\, e^{*\lambda}_{\alpha[s,s+1]} \subset V_\alpha$ be the real span of Π_α. By the previous theorem $E_\alpha \subset V_\alpha^{\mathbb{R}}$. Moreover, $V_\alpha^{\mathbb{R}}$ is endowed with the induced inner product.

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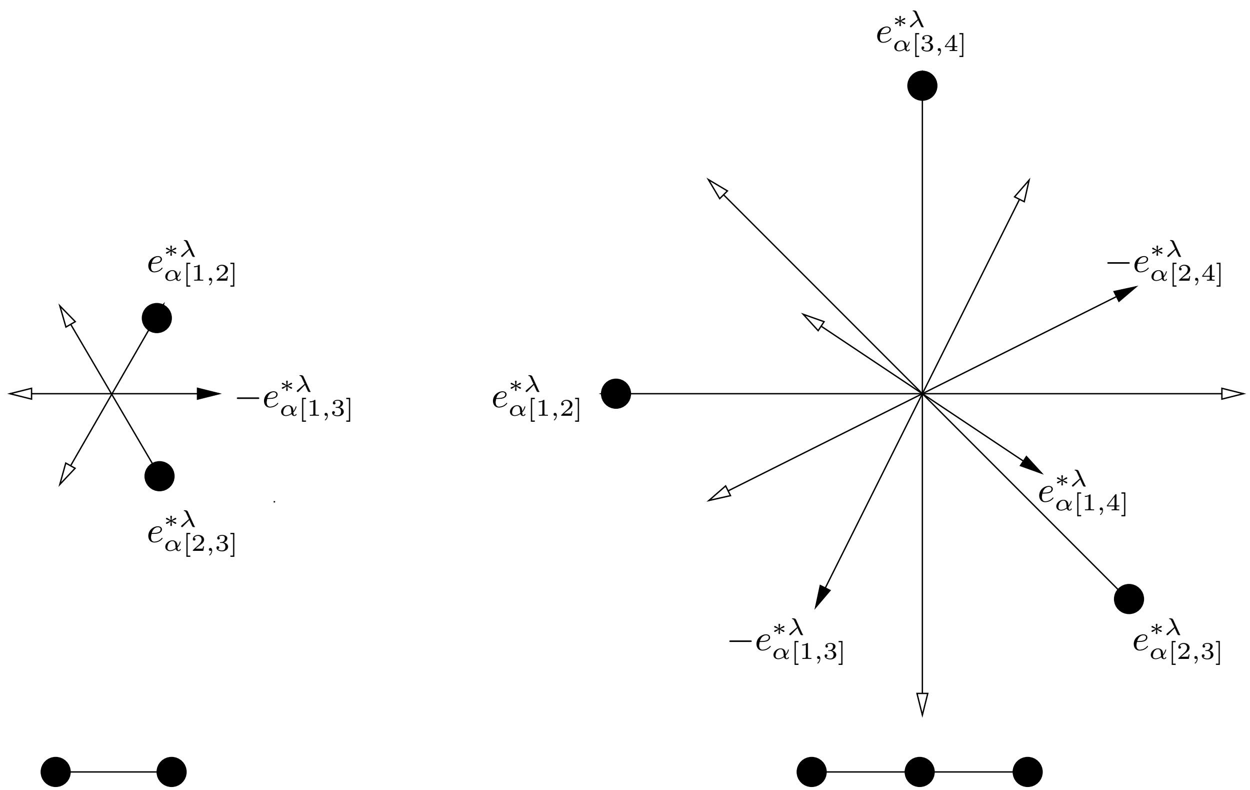
Theorem [MM Torres, –] *Let $\alpha = (1, 1, 2, \ldots, m-1)$. The following hold.*

1. *$E_\alpha = E_\alpha^+ \cup -E_\alpha^+$ is a regular crystallographic root system of rank $m-1$, set of positive roots E_α^+, simple system Π_α and Dynkin diagram A_{m-1}.*

2. *The critical orbital sets E_α with multiplicity $(2, 1^{m-2})$, are the only crystallographic root systems consisting entirely of critical decomposable symmetrized tensors associated to a single hook partition.*

Next figure depicts the root systems E_α for $m = 3, 4$ and the corresponding Dynkin diagrams



The simple roots were marked with *filled dots*, the remaining positive roots by *filled arrows* and the negative roots by *white arrows*.

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Corollary *The orthogonal dimension of the critical orbital sets associated to the hook partition $\lambda = (2, 1^{m-2})$ can be interpreted as the independence number of the Dynkin diagram of the root system E_α, i.e.,*

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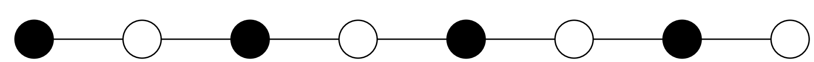
A_8: 

Symmetrized tensors
Statement of the problem
The combinatorial approach
Connections with coding theory
▷ Root systems of tensors

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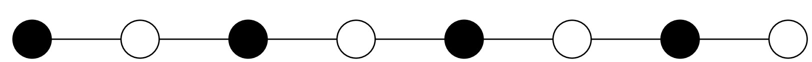
A_8: 

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Thank you for your attention!