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Shifted Hecke insertion and K-theory of $OG(n, 2n+1)$

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Abstract. Patrias and Pylyavskyy introduced shifted Hecke insertion as an application of their theory of dual filtered graphs. We show that shifted Hecke insertion has a natural place in the combinatorial study of the K-theory of the maximal orthogonal Grassmannian. In particular, we relate it to the K-theoretic jeu de taquin of Clifford-Thomas-Yong and use it to create new symmetric functions, which we use to derive a Littlewood-Richardson rule for the K-theory of the orthogonal Grassmannian equivalent to the rules of Clifford-Thomas-Yong and Buch-Samuel.

Résumé. Patrias and Pylyavskyy ont introduit l'insertion de Hecke décalée comme une application de leur théorie des graphes filtrés en dualité. Nous montrons que l'insertion de Hecke décalée a une place naturelle dans l'étude combinatoire de la K-théorie de $OG(n, 2n+1)$. En particulier, nous la relions au jeu de taquin K-théorique de Clifford-Thomas-Yong et nous l'utilisons pour créer de nouvelles fonctions symétriques. Nous utilisons ces fonctions symétriques pour dériver une règle de Littlewood-Richardson pour la K-théorie de $OG(n, 2n+1)$ qui est équivalente aux règles de Clifford-Thomas-Yong et de Buch-Samuel.

Keywords: symmetric functions, K-theory, insertion algorithms

1 Introduction

In [9], Patrias and Pylyavskyy introduce shifted Hecke insertion as an application of their theory of dual filtered graphs. It is a bijection between finite words in the positive

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integers and pairs (P_{SK}, Q_{SK}) of shifted tableaux of the same shape, where P_{SK} is increasing and Q_{SK} is set-valued. We show that shifted Hecke insertion has a natural place in the combinatorial study of the K-theory for the orthogonal Grassmannian $OG(n, 2n+1)$. In particular, shifted Hecke insertion plays a role in K-theory of $OG(n, 2n+1)$ similar to the role of (unshifted) Hecke insertion in the K-theory of the Grassmannian. Additionally, we highlight the role of shifted Hecke insertion as a tool for studying certain symmetric functions, similar to the relationship between RSK and Schur functions. We next describe the setting and results in more detail.

The K-theory of the orthogonal Grassmannian $OG(n, 2n+1)$ is well understood. It has as a basis of Schubert structure sheaves indexed by shifted shapes, $\{\mathcal{O}_\lambda\}$. The product structure is determined by a combinatorial Littlewood-Richardson rule

$$\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c^\nu_{\lambda,\mu} \mathcal{O}_\nu$$

first proven by Clifford, Thomas, and Yong [4] using shifted K-theoretic jeu de taquin and a geometric Pieri rule of Buch and Ravikumar [1]. An analogous rule was presented by Buch and Samuel as the special case of a more general theory [2]. After identifying the equivariant K-theory of $OG(n, 2n+1)$ with a ring consisting of polynomials, Ikeda and Naruse show the factorial Schur P-functions form a complete set of representatives [6]. These can be specialized to the *shifted stable Grothendieck polynomials* GP_λ, which are the image of each $\mathcal{O}_\lambda$ under this identification.

The Littlewood-Richardson rule of Clifford, Thomas, and Yong is formulated in terms of *shifted K-jeu de taquin*, a variant of jeu de taquin for *increasing shifted tableaux*. They use shifted K-jeu de taquin slides to define a map between increasing shifted skew tableaux and increasing shifted tableaux that we call *shifted superstandard K-rectification*. Identifying a word w with the skew tableau T_w whose entries lie on an anti-diagonal, we prove the following relationship.

Theorem 1. *Let w be a word and P be the shifted superstandard K-rectification of T_w. Then $P_{SK}(w) = P$ where $P_{SK}(w)$ is the shifted Hecke insertion tableau.*

This is the shifted analogue of Theorem 4.2 of [14]. As a consequence of Theorem 1, we can easily rephrase the Littlewood-Richardson rule of Clifford-Thomas-Yong in terms of shifted Hecke insertion as follows.

Theorem 2. *The product structure for the K-theory of $OG(n, 2n+1)$ is described by*

$$\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c^\nu_{\lambda,\mu} \mathcal{O}_\nu,$$

where $c^\nu_{\lambda,\mu}$ is equal to the number of increasing shifted skew tableaux R of shape ν/λ for which $P_{SK}(\mathfrak{row}(R))$ is the minimal increasing tableau of shape μ.

Another important consequence of Theorem 1 is that shifted Hecke insertion respects the *weak K-Knuth equivalence* of Buch and Samuel [2]. More precisely, every word u is weakly K-Knuth equivalent to the reading word of its shifted Hecke insertion tableaux $P_{SK}(u)$ (see Corollary 10), so if $P_{SK}(u) = P_{SK}(v)$ then u is weak K-Knuth equivalent to v. As with (unshifted) Hecke insertion, the converse is not true: one weak K-Knuth equivalence class may correspond to more than one insertion tableau. In Sections 3 and 4, we present an independent proof of Theorem 2 by using shifted Hecke insertion to study GP_λ directly. Here, we are using shifted Hecke insertion as a tool for the combinatorial study of symmetric functions, independent of the geometry.

For each shifted shape λ, we define the *weak shifted stable Grothendieck polynomial* K_λ as a generating function over multiset-valued shifted tableaux. We then show how to expand K_λ in terms of the *fundamental quasisymmetric functions* f_α, which form a basis for the ring of quasisymmetric functions QSym (see Section 3 for precise definitions).

Theorem 3. *For any fixed increasing shifted tableau T of shape λ,*

$$K_\lambda = \sum_{P_{SK}(w)=T} f_{\mathcal{D}(w)},$$

where $\mathcal{D}(w)$ is the descent set of the word w.

We show that the K_λ are symmetric (Proposition 19) and that they differ from Ikeda and Naruse's GP_λ by a sign and change of variables (Proposition 17). As a consequence, we see the product structure of K_λ, GP_λ, and $\mathcal{O}_\lambda$ are identical, up to a predictable sign.

To understand this product structure at the level of symmetric functions, we define the shifted K-theoretic Poirier-Reutenauer algebra called $SKPR$ using weak K-Knuth equivalence. For any word h, we define elements $[[h]] = \sum_{w \triangleq h} w$, where $\triangleq$ denotes weak K-Knuth equivalence. Multiplication is defined as the shuffle product. This generalizes the shifted Poirier-Reutenauer bialgebra of Jing and Li [7], which is a shifted analogue of the Poirier-Reutenauer Hopf algebra [11]. Our approach closely follows work of Patrias and Pylyavskyy on a K-theoretic Poirier-Reutenauer bialgebra [10]. Surprisingly, the shifted K-theoretic Poirier-Reutenauer algebra does not have the coalgebra structure found in the shifted Poirier-Reutenauer and K-theoretic Poirier Reutenauer bialgebras.

We then construct an algebra homomorphism from $SKPR$ to QSym that we prove sends $[[h]]$ to a sum of K_λ for any word h.

Theorem 4. *Letting $\lambda(T)$ denote the shape of T, we have*

$$\phi([[h]]) = \sum_{\mathfrak{row}(T) \triangleq h} K_{\lambda(T)}.$$

Using Theorem 4, we present a Littlewood-Richardson rule for K_λ. In light of our previous comments, this rule also applies to $GP_\lambda \cdot GP_\mu$ and $\mathcal{O}_\lambda \cdot \mathcal{O}_\mu$ and thus provides an independent proof of Theorem 2. Our proof does not rely on Buch and Ravikumar's Pieri rule [1].

Proofs have been omitted for brevity but may be found in [5].

2 Shifted Hecke Insertion and Weak K-Knuth Equivalence

We outline our argument that the shifted Hecke insertion given in [9] respects the weak K-Knuth equivalence given in [2]. Before presenting our argument, we review previous work on increasing shifted tableaux, shifted Hecke insertion, and shifted K-jeu de taquin.

2.1 Increasing shifted tableaux and shifted Hecke insertion

To each strict partition $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_k)$ we associate the *shifted shape*, which is an array of boxes where the ith row has λ_i boxes and is indented $i - 1$ units. A *shifted tableau* is a filling of the shifted shape with positive integers. A shifted tableau is *increasing* if the labels are strictly increasing from left to right along rows and top to bottom down columns. The *reading word* of an increasing shifted tableau T, denoted $\mathfrak{row}(T)$, is the word obtained by reading the entries left to right from the bottom row to the top row. The increasing shifted tableau below has reading word 8471367.

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}$$

Lemma 1. *There are finitely many increasing shifted tableaux filled with a given finite alphabet.*

We now briefly recall the rules for shifted Hecke insertion and refer the reader to [9] for further reading. It is simultaneously a shifted analogue of Buch, Kresch, Shimozono, Tamvakis and Yong's Hecke insertion [3] and a K-theoretic analogue of Sagan-Worley insertion, due independently to Sagan and Worley [12, 15]. From this point on, "insertion" will always refer to shifted Hecke insertion unless stated otherwise.

First, we describe how to insert a positive integer x into a given shifted increasing tableau T. We start by inserting x into the first row of T. For each insertion, we assign a box to record where the insertion terminates. This data will be used when we define the recording tableau.

The rules for inserting x into a row or column of T are as follows:

(1) If x is weakly larger than all integers in the row (respectively column) and adjoining x to the end of the row (respectively column) results in an increasing tableau T', then T' is the resulting tableau. We say the insertion terminates at the new box.

(2) If x is weakly larger than all integers in the row (respectively column) and adjoining x to the end of the row (respectively column) does not result in an increasing

tableau, then $T' = T$. If x is row inserted into a nonempty row, we say the insertion terminated at the box at the bottom of the column containing the rightmost box of this row. If x is row inserted into an empty row, we say that the insertion terminated at the rightmost box of the previous row. If x is column inserted into a nonempty column, we say the insertion terminated at the rightmost box of the row containing the bottom box of the column x could not be added to.

For the next two rules, suppose the row (respectively column) contains a box with label strictly larger than x, and let y be the smallest such label.

(3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer for this step and is to be inserted in the next step. If x was inserted into a column or if y was on the main diagonal, proceed to insert each future output integer into the next column to the right its column of origin. If x was inserted into a row and y was not on the main diagonal, then insert y into the row below.

(4) If replacing y with x does not result in an increasing tableau, then do not change the entries of the row (respectively column). In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert each future output integer into the next column to the right its column of origin. If x was inserted into a row, then insert y into the row below.

For any given word $w = w_1 w_2 \cdots w_n$, we define the *insertion tableau* $P_{SK}(w)$ of w to be $(\cdots((\varnothing \overset{SK}{\longleftarrow} w_1) \overset{SK}{\longleftarrow} w_2) \cdots \overset{SK}{\longleftarrow} w_n)$, where $\varnothing$ denotes the empty shape and $T \overset{SK}{\longleftarrow} x$ denotes the insertion of the letter x into the tableau T. See Example 6.

In order to describe the recording tableau for shifted Hecke insertion of a word w, we need the following definition.

Definition 5. [6] A *set-valued shifted tableau* is defined to be a filling of the boxes of a shifted shape with finite, nonempty subsets of primed and unprimed positive integers with ordering $1' < 1 < 2' < 2 < \ldots$ such that:

1. The smallest number in each box is greater than or equal to the largest number in the box directly to the left of it, if that box exists.
2. The smallest number in each box is greater than or equal to the largest number in the box directly above it, if that box exists.
3. There are no primed entries on the main diagonal.
4. Each unprimed integer appears in at most one box in each column, and each primed integer appears in at most one box in each row.

A set-valued shifted tableau is called *standard* if the set of labels is exactly $[n]$ for some n, each appearing either primed or unprimed exactly once. The *recording tableau* of a word $w = w_1w_2 \dots w_n$, denoted $Q_{SK}(w)$, is a standard set-valued shifted tableau that records where the insertion of each letter of w terminates. We define it inductively.

Start with $Q_{SK}(\emptyset) = \emptyset$. If the insertion of w_k added a new box to $P_{SK}(w_1w_2 \dots w_{k-1})$, then add the same box with label k (k' if this box was added by column insertion) to $Q_{SK}(w_1w_2 \dots w_{k-1})$. If w_k did not change the shape of $P_{SK}(w_1w_2 \dots w_{k-1})$, we obtain $Q_{SK}(w_1w_2 \dots w_k)$ from $Q_{SK}(w_1 \dots w_{k-1})$ by adding the label k (k' if it ended with column insertion) to the box where the insertion terminated. If insertion terminated when a letter failed to insert into an empty row, label the box where the insertion terminated k'.

Example 6. Let $w = 451132$. We insert w letter by letter, writing the insertion tableau at each step in the top row and the recording tableau at each step in the bottom row.

$$
\begin{array}{|c|}\hline 4 \\ \hline\end{array}
\quad
\begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}
= P_{SK}(w)
$$

$$
\begin{array}{|c|}\hline 1 \\ \hline\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 2 & 3' \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3'4' & 6' \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}
= Q_{SK}(w)
$$

Theorem 7. [9, Theorem 5.19] *The map $w \mapsto (P_{SK}(w), Q_{SK}(w))$ is a bijection between words of positive integers and pairs of shifted tableaux (P, Q) of the same shape where P is an increasing shifted tableau and Q is a standard set-valued shifted tableau.*

2.2 Weak K-Knuth equivalence and shifted jeu de taquin

The Knuth relations determine which words have the same Robinson-Schensted-Knuth insertion tableau [13, Theorem A1.1.4]. We present Buch and Samuel's shifted K-theoretic analogue, called weak K-Knuth equivalence [2]. As we will see in Corollary 10 and Remark 11, weak K-Knuth equivalence is a necessary but not sufficient condition for two words to have the same shifted Hecke insertion tableau.

Definition 8. [2, Definition 7.6] Define the *weak K-Knuth equivalence relation* on the alphabet $\{1,2,3,\cdots\}$, denoted by $\triangleq$, as the symmetric transitive closure of the following relations, where $\mathbf{u}$ and $\mathbf{v}$ are (possibly empty) words of positive integers, and $a < b < c$ are distinct positive integers:

1. $(\mathbf{u}, a, a, \mathbf{v}) \triangleq (\mathbf{u}, a, \mathbf{v})$
2. $(\mathbf{u}, a, b, a, \mathbf{v}) \triangleq (\mathbf{u}, b, a, b, \mathbf{v})$
3. $(\mathbf{u}, b, a, c, \mathbf{v}) \triangleq (\mathbf{u}, b, c, a, \mathbf{v})$
4. $(\mathbf{u}, a, c, b, \mathbf{v}) \triangleq (\mathbf{u}, c, a, b, \mathbf{v})$
5. $(a, b, \mathbf{u}) \triangleq (b, a, \mathbf{u})$

Two words w and w' are *weak K-Knuth equivalent*, denoted $w \mathrel{\hat{=}} w'$, if w' can be obtained from w by a finite sequence of weak K-Knuth equivalence relations. Two shifted increasing tableaux T and T' are *weak K-Knuth equivalent* if $\mathfrak{row}(T) \mathrel{\hat{=}} \mathfrak{row}(T')$.

By removing relation (5), we obtain the K-Knuth relations of Buch and Samuel [2]. As in the K-Knuth equivalence but in contrast to Knuth equivalence, each weak K-Knuth equivalence class has infinitely many elements and contains words of arbitrary length.

In [4], Clifford, Thomas, and Yong create a shifted K-theoretic jeu de taquin algorithm for increasing shifted tableaux that is a natural analogue of the Thomas-Yong K-theoretic jeu de taquin [14] and of Schützenberger's jeu de taquin. We refer the reader to [4] for details. One important difference between Schützenberger's jeu de taquin and these K-theoretic analogues is that the order in which one performs the slides in the K-theoretic analogues may change the outcome of the procedure. In particular, a given skew tableau T may rectify to more than one straight-shaped increasing shifted tableau depending on the *rectification order*. We fix one such rectification order, which we call *shifted superstandard rectification* for Theorem 1 based on the *superstandard tableaux* defined in Section 2.3.

For any word $w = w_1 \dots w_n$, let T_w denote the shifted skew tableau consisting of n boxes on the antidiagonal with reading word w. Theorem 1 explains the relationship between shifted Hecke insertion and shifted K-jdt, affirming that shifted Hecke insertion is a K-theoretic analogue of Sagan-Worley insertion. Our proof of Theorem 1 relies on showing how a single insertion step replicates a sequence of K-jdt moves. This requires a quite involved combinatorial argument, which we omit.

Using a result of Buch and Samuel [2], we can now relate shifted Hecke insertion to the weak K-Knuth relations. Tableaux T and T' are called *jeu de taquin equivalent* if one can be obtained from another using shifted K-jdt. Their result says that weak K-Knuth equivalence and jeu de taquin equivalence are the same for increasing shifted tableaux. From this point on, we refer to both as "equivalence."

Theorem 9. *[2, Theorem 7.8] Let T and T' be increasing shifted tableaux. Then $\mathfrak{row}(T) \mathrel{\hat{=}} \mathfrak{row}(T')$ if and only if T and T' are jeu de taquin equivalent.*

Corollary 10. *We have $u \mathrel{\hat{=}} \mathfrak{row}(P_{SK}(u))$. As a consequence, if $P_{SK}(u) = P_{SK}(v)$, then $u \mathrel{\hat{=}} v$.*

Remark 11. The converse of the second part of Corollary 10 does not hold. Consider the words 12453 and 124533, which are easily seen to be weakly K-Knuth equivalent. We compute that shifted Hecke insertion gives the following distinct tableaux.

$$P_{SK}(12453) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \qquad P_{SK}(124533) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

2.3 Unique Rectification Targets

As we have seen in Remark 11, weak K-Knuth equivalence classes may have several corresponding insertion tableaux. This is a key difference between weak K-Knuth equivalence and the classical Knuth equivalence. Of particular importance in our setting are the classes of words with only one tableau.

Definition 12. [2, Definition 3.5] An increasing shifted tableau T is a *unique rectification target*, or a URT, if it is the only tableau in its weak K-Knuth equivalence class. Equivalently, T is a URT if for every $w \triangleq \mathfrak{row}(T)$ we have $P_{SK}(w) = T$. If $P_{SK}(w)$ is a URT, we call the equivalence class of w a *unique rectification class*.

We refer the reader to [2, 4] for a more detailed discussion of URTs for shifted tableaux and straight shape tableaux. The tableaux given in Remark 11 are equivalent to each other, and hence neither is a URT.

The *minimal increasing shifted tableau* M_λ of a shifted shape λ is the tableau obtained by filling the boxes of λ with the smallest values allowed in an increasing tableau. The *superstandard shifted tableau* S_λ of shifted shape λ is obtained by filling the boxes in row λ_i with $\lambda_1 + \ldots + \lambda_{i-1} + 1$ through $\lambda_1 + \ldots + \lambda_{i-1} + \lambda_i$.

$$S_{(4,2)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array} \quad M_{(5,2,1)} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$$

In [2, Corollary 7.2], Buch and Samuel proved that minimal increasing shifted tableaux are URTs, and in [4, Theorem 1.1], Clifford, Thomas, and Yong show that superstandard tableaux are URTs. As a consequence, we see there are URTs for every shifted shape. Moreover, we can reformulate the Buch-Samuel and Clifford-Thomas-Yong rules using Hecke insertion.

Corollary (Theorem 2)**.** Let T be a URT of shape λ. The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ for K-theory of $OG(n, 2n+1)$ enumerates increasing shifted skew tableaux R of shape ν/μ with $P_{SK}(\mathfrak{row}(R)) = T_\lambda$ (up to sign).

3 Weak shifted stable Grothendieck polynomials

We define the weak shifted stable Grothendieck polynomial K_λ as a weighted generating function over weak set-valued shifted tableaux.

Definition 13. A *weak set-valued shifted tableau* is a filling of the boxes of a shifted shape with finite, nonempty multisets of primed and unprimed positive integers with ordering $1' < 1 < 2' < 2 < \cdots$ such that the conditions of Definition 5 are satisfied.

Note that the difference between set-valued shifted tableaux and weak set-valued shifted tableaux is that we allow multisets in the boxes of the latter. For example, the first and third tableaux in Example 16 are weak set-valued but not set-valued shifted tableaux.

Given any weak set-valued shifted tableau T, we define x^T to be the monomial $\prod_{i \geq 1} x_i^{a_i}$, where a_i is the number of occurrences of i and i' in T.

Definition 14. The *weak shifted stable Grothendieck polynomial* is $K_\lambda = \sum_T x^T$, where the sum is over the set of weak set-valued tableaux T of shape λ.

Remark 15. Stable Grothendieck polynomials and their analogues typically have a sign $(-1)^{|T|-|\lambda|}$ for each monomial, where $|T|$ is the degree of x^T. We suppress this sign for our definition K_λ as others have done e.g. [8]. It is easily reintroduced when necessary.

Example 16. We have $K_{(2,1)} = x_1^2 x_2 + 2x_1x_2x_3 + 3x_1^2x_2^2 + 5x_1^2x_2x_3 + 5x_1x_2^2x_3 + \cdots$, where the coefficient of $x_1^2 x_2^2$ is 3 because of the tableaux shown below.

$$\begin{array}{|c|c|}\hline 11 & 2' \\ \hline \end{array}\qquad \begin{array}{|c|c|}\hline 1 & 12' \\ \hline \end{array}\qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$
$$\begin{array}{c|c|}\cline{2-2} & 2 \\ \cline{2-2} \end{array}\qquad \begin{array}{c|c|}\cline{2-2} & 2 \\ \cline{2-2} \end{array}\qquad \begin{array}{c|c|}\cline{2-2} & 22 \\ \cline{2-2} \end{array}$$

Note that the lowest degree terms of K_λ are a sum over shifted semistandard Young tableaux, so they form the Schur P-function P_λ.

The weak shifted stable Grothendieck polynomial K_λ is closely related to the shifted stable Grothendieck polynomial GP_λ defined by Ikeda and Naruse in [6]. Here, $GP_\lambda = \sum_T (-1)^{|T|-|\lambda|} x^T$, where we sum over all set-valued shifted tableaux T of shape λ (see Definition 5), $|T|$ is the degree of x^T, and $|\lambda|$ is the number of boxes in λ. Given a set-valued tableau T, we may convert it to a multiset valued tableau by replacing each instance of i or i' with (potentially) multiple copies of that entry. We now interpret this observation at the level of symmetric functions.

Proposition 17. *We have* $K_\lambda(x_1, x_2, \ldots) = (-1)^{|\lambda|} GP_\lambda \left(\frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \ldots \right)$.

3.1 The symmetry of K_λ

From Proposition 17 and [6, Theorem 9.1], we can conclude that the weak shifted stable Grothendieck polynomial K_λ is symmetric. However, using shifted Hecke insertion, we provide a direct proof. This also shows that K_λ is a sum of stable Grothendieck polynomials. Moreover, by Proposition 17, we obtain a new proof of symmetry for the GP_λ. To our mind, this proof of symmetry is easier than the Ikeda-Naruse proof, as might be expected since their result follows as a consequence of symmetry for the more general factorial K-theoretic Schur P-functions.

We first write the K_λ as a sum of quasisymmetric functions. To any $\mathcal{D} \subset [n-1]$, we associate the *fundamental quasisymmetric function* $f_\mathcal{D} = \sum x_{i_1} \ldots x_{i_n}$, where $i_1 \leq i_2 \leq \cdots \leq i_n$ and $i_j < i_{j+1}$ if $j \in \mathcal{D}$. The fundamental quasisymmetric functions form a basis for the *ring of quasisymmetric functions* QSym, which is comprised of formal power series with bounded degree that are *quasisymmetric*, i.e., the coefficients of $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $x_{i_1}^{\alpha_1}\dots x_{i_n}^{\alpha_n}$ are the same for any $i_1 < \cdots < i_n$.

The *descent set* of a word $w = w_1w_2\dots w_n$ is $\mathcal{D}(w) = \{i : w_i > w_{i+1}\}$. For example, 354211 has descent set $\{2,3,4\}$. Similarly, the *descent set* of a standard set-valued shifted tableau T is

$$\mathcal{D}(T) = \left\{ i \;:\; \begin{array}{l} \text{both } i \text{ and } (i+1)' \text{ appear} \\ \qquad\textsc{or} \\ i \text{ is strictly above } i+1 \\ \qquad\textsc{or} \\ i' \text{ is weakly below } (i+1)' \text{ but not in the same box} \end{array} \right\}.$$

The next result says that shifted Hecke insertion respects descent sets. Analogues of this result hold for many insertion algorithms, including RSK and Hecke insertion.

Proposition 18. *For any word $w = w_1w_2\dots w_n$, $\mathcal{D}(w) = \mathcal{D}(Q_{SK}(w))$.*

By associating fundamental quasisymmetric functions to descent sets of words and applying Proposition 18, we can prove Theorem 3.

A *multiset-valued tableau T* of ordinary (unshifted) shape satisfies conditions (1) and (2) of Definition 13 and has no primed entries. In [8], Lam and Pylyavskyy define the *weak stable Grothendieck polynomial* $J_\mu = \sum_T x^T$ where the sum is now over multiset-valued tableaux. Note that our definition differs from theirs by taking the transpose. Weak stable Grothendieck polynomials are symmetric functions. Using Hecke insertion, each J_μ can be expressed as a sum over words instead of tableaux. These words are equivalent under a subset of the weak K-Knuth relations, allowing us to express K_λ in terms of weak stable Grothendieck polynomials, which gives the proposition below. From Proposition 17, it also follows that GP_λ is symmetric.

Proposition 19. *For any shifted shape λ, K_λ is symmetric.*

4 Shifted K-Poirier-Reutenauer Algebra and a Littlewood-Richardson rule

In [11], Poirier and Reutenauer define a Hopf algebra spanned by the set of standard Young tableaux. Jing and Li developed a shifted version [7], and Patrias and Pylyavskyy developed a K-theoretic analogue [10]. In this section, we combine these approaches to introduce a shifted K-theoretic analogue.

We say a word h is *initial* if the letters appearing in it are exactly the numbers in $[k]$ for some positive integer k. For example, the words 54321 and 211345 are initial, but 2344 is not. For an initial word h, define $[[h]]$ to be the formal sum of words in the weak K-Knuth equivalence class of h: $[[h]] = \sum_{h \,\hat{=}\, w} w$. This is an infinite sum; however, the number of terms in $[[h]]$ of each length is finite. For example, $[[213]] = 213 + 231 + 123 + 321 + 3221 + 3321 + 3211 + 32111 + \cdots$. As a consequence of Lemma 1, the set of shifted Hecke insertion tableaux obtained by inserting the terms in $[[h]]$ is finite for any h. We define $SKPR$ to be the vector space over $\mathbb{R}$ spanned by these elements.

Let $\sqcup\!\sqcup$ denote the usual shuffle product of words and $w[n]$ be the word obtained from w by increasing each letter by $[n]$. For example, if $w = 312$, $w[4] = 756$. Given words h and h' in alphabets $[n]$ and $[m]$, respectively, we define the product of their classes to be

$$[[h]] \cdot [[h']] = \sum_{w \hat{=} h, w' \hat{=} h'} w \sqcup\!\sqcup w'[n].$$

For example $[[12]] \cdot [[1]] = [[123]] + [[312]] + [[3123]]$. In general, $[[h]] \cdot [[h']]$ can be expressed as a sum of classes.

Proposition 20. *For any two initial words h and h', the product of their classes can be written as*

$$[[h]] \cdot [[h']] = \sum_{h''} [[h'']],$$

where the sum is over some finite set of initial words h''.

It turns out that classes corresponding to URTs have particularly simple products, which we can express as an explicit sum over sets of tableaux as follows.

Proposition 21. *Let T_1 and T_2 be two URTs. Then*

$$\left(\sum_{P_{SK}(u)=T_1} u \right) \cdot \left(\sum_{P_{SK}(v)=T_2} v \right) = \sum_{T \in \mathcal{T}(T_1 \sqcup\!\sqcup T_2)} \sum_{P_{SK}(w)=T} w,$$

where $\mathcal{T}(T_1 \sqcup\!\sqcup T_2)$ is the finite set of shifted tableaux T such that $T|_{[n]} = T_1$ and $P_{SK}(\mathfrak{row}(T)|_{[n+1,n+m]}) = T_2$.

Define ϕ : SKPR$\rightarrow$ QSym by setting $\phi([[h]]) = \sum_{w \hat{=} h} f_{\mathcal{D}(w)}$. The proof that this map is a homomorphism is straight-forward. Together with Theorem 3, this implies Theorem 4. Combined with Proposition 21, we can then prove a Littlewood-Richardson rule for K_λ independent of Buch and Ravikumar's Pieri rule.

Theorem 22. *Let T be a URT of shape λ. Then we have $K_\lambda K_\mu = \sum_\nu c^\nu_{\lambda,\mu} K_\nu$, where $c^\nu_{\lambda,\mu}$ is given by the number of increasing shifted skew tableaux R of shape ν/μ such that $P_{SK}(\mathfrak{row}(R)) = T$.*

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