Séminaire Lotharingien de Combinatoire **78B** (2017)
Article #23, 12 pp.

Proceedings of the 29th Conference on Formal Power Series and Algebraic Combinatorics (London)

Filters in the partition lattice

Richard Ehrenborg and Dustin Hedmark

Department of Mathematics, University of Kentucky, Lexington, KY 40506

Abstract. Given a filter Δ in the poset of compositions of n, we form the filter Π^*_Δ in the partition lattice. We determine all the reduced homology groups of the order complex of Π^*_Δ as $\mathfrak{S}_{n-1}$-modules in terms of the reduced homology groups of the simplicial complex Δ and in terms of Specht modules of border shapes. We also obtain the homotopy type of this order complex. These results generalize work of Calderbank–Hanlon–Robinson and Wachs on the d-divisible partition lattice. Our main theorem applies to a plethora of examples, including filters associated to integer knapsack partitions and filters generated by all partitions having block sizes a or b. We also obtain the reduced homology groups of the filter generated by all partitions having block sizes belonging to the arithmetic progression $a, a+d, \ldots, a+(a-1)\cdot d$, extending work of Browdy.

Résumé. Étant donné un filtre Δ dans l'ensemble ordonné des compositions de n, nous formons le filtre Π^*_Δ dans le treillis de partitions. Nous déterminons tous les groupes d'homologie réduits du complex des chaînes de Π^*_Δ comme $\mathfrak{S}_{n-1}$-modules en termes des groupes d'homologie réduits du complexe simplicial Δ et des modules de Specht des bandes frontalières. Nous arrivons aussi à déterminer le type d'homotopie de ce complex des chaînes. Ces résultats généralisent les travaux de Calderbank–Hanlon–Robinson et de Wachs sur le treillis des partitions d-divisibles. Notre théor'eme principal s'applique à une pléthore d'exemples, y compris les filtres associés aux partitions entier sac-à-dos et les filtres engendrés par toutes les partitions dont la taille des blocs est a ou b. En généralisant le travail de Browdy, nous obtenons aussi les groupes d'homologie réduits du filtre engendré par toutes les patitions dont les tailles de bloc appartiennent à la suite arithmétique $a, a+d, \ldots, a+(a-1)\cdot d$.

Keywords: Partition lattice, Composition lattice, Specht module, Equivariant Quillen's Fiber Lemma, Frobenius complex

1 Introduction

In his physics dissertation Sylvester [13] considered the even partition lattice, that is, the poset of all set partitions where the blocks have even size. He computed the Möbius function of this lattice and showed that it equals, up to a sign, the tangent number. Stanley then introduced the d-divisible partition lattice. This is the collection of all set partitions with blocks having size divisible by d, denoted by Π^d_n. He showed that the Möbius function is, up to a sign, the number of permutations in the symmetric group $\mathfrak{S}_{n-1}$ with descent set $\{d, 2d, \ldots, n-d\}$; see [10].

Calderbank, Hanlon and Robinson [5] continued this work by studying the top homology group of the order complex $\triangle(\Pi_n^d - \{\hat{1}\})$ and gave an explicit description of the $\mathfrak{S}_{n-1}$-action on this homology group in terms of a Specht module. However, they were unable to obtain the other homology groups and asked Wachs if it was possible that the complex $\triangle(\Pi_n^d - \{\hat{1}\})$ was shellable, which would imply that the other homology groups are trivial. Wachs [14] proved that this was indeed the case by showing that the poset $\Pi_n^d \cup \{\hat{0}\}$ is EL-shellable, and thus the homotopy type of the complex $\triangle(\Pi_n^d - \{\hat{1}\})$ is a wedge of spheres of the same dimension. Additionally, Wachs gave a different proof for the $\mathfrak{S}_{n-1}$-action on the top homology of Π_n^d, as well as matrices for the action of $\mathfrak{S}_n$ on this homology.

Ehrenborg and Jung [7] further generalized the d-divisible partition lattice by defining a subposet $\Pi_{\vec{c}}^*$ of the partition lattice for a composition $\vec{c}$ of n. The subposet reduces to the d-divisible partition lattice when the composition $\vec{c}$ is given by $\vec{c} = (d, d, \ldots, d)$. Their work consists of three main results. First, they showed that the Möbius function of $\Pi_{\vec{c}}^* \cup \{\hat{0}\}$ equals, up to a given sign, the number of permutations in $\mathfrak{S}_n$ ending with the element n having descent composition $\vec{c}$. Second, they showed that the order complex $\triangle(\Pi_{\vec{c}}^* - \{\hat{1}\})$ is homotopy equivalent to a wedge of spheres of the same dimension. Lastly, they proved that the action of $\mathfrak{S}_{n-1}$ on the top homology group of $\triangle(\Pi_{\vec{c}}^* - \{\hat{1}\})$ is given by the Specht module corresponding to the composition $\vec{c} - 1$.

In the current paper we continue this research program by considering a more general class of filters in the partition lattice. Let Δ be a filter in the poset of compositions. Since the poset of compositions is isomorphic to a Boolean algebra, the filter Δ under the reverse order is a lower order ideal and hence can be viewed as the face poset of a simplicial complex. We define the associated filter Π_Δ^* in the partition lattice. This extends the definition of $\Pi_{\vec{c}}^*$. In fact, when Δ is a simplex generated by the composition $\vec{c}$ the two definitions agree.

Our main result is that we can determine all the reduced homology groups of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ in terms of the reduced homology groups of links in Δ and of Specht modules of border shapes; see Theorem 7.5. The proof proceeds by showing that if the result holds for the two complexes Δ, Γ and also for their intersection $\Delta \cap \Gamma$, then it holds for their union $\Delta \cup \Gamma$. Furthermore, the proof relies on Mayer–Vietoris sequences to construct the isomorphism of Theorem 6.3. As our main tool, we use Quillen's fiber lemma to translate topological data from the filter Q_Δ^* to the filter Π_Δ^*.

We also present a second proof of our main result, Theorem 6.3, using an equivariant poset fiber theorem of Björner, Wachs and Welker [3]. Even though this approach is concise, it does not yield an explicit construction of the isomorphism of Theorem 6.3. In particular, our hands on approach using Mayer–Vietoris sequences reveals how the homology groups of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ are changing as the complex Δ is built up. Once again, the Ehrenborg–Jung result on $Q^*_{\vec{c}}$ is needed to apply the poset fiber theorem.

Our main result yields explicit expressions for the reduced homology groups of the complex $\triangle(\Pi^*_\Delta - \{\hat{1}\})$, most notably when Δ is homeomorphic to a ball or to a sphere. The same holds when Δ is a shellable complex. We are able to describe the homotopy type of the order complex $\triangle(\Pi^*_\Delta - \{\hat{1}\})$ using the homotopy fiber theorem of [3]. Again, when Δ is homeomorphic to a ball or to a sphere, we obtain that Π^*_Δ is a wedge of spheres. We are also able to lift discrete Morse matchings from Δ and its links to form a discrete Morse matching on the filter of ordered set partitions Q^*_Δ.

In Section 8 we study the case when Λ is a semigroup of positive integers and we consider the filter of partitions whose block sizes belong to the semigroup Λ. When Λ is generated by the arithmetic progression $a, a+d, a+2d, \ldots$ we are able to describe the reduced homology groups of the associated filter in the partition lattice. The particular case when d divides a was studied by Browdy [4], where the filter Λ consists of partitions whose block sizes are divisible by d and are greater than or equal to a. Finally, in Section 9 we study the filter corresponding to the semigroup generated by two relative prime integers. Here we are able to give explicit results for the top and bottom reduced homology groups.

Other previous work in this area is due to Björner and Wachs [2]. Additionally, Sundaram studied the subposet of the partition lattice defined by a set of forbidden block sizes using plethysm and the Hopf trace formula; see [11, 12].

We end the paper by posing questions for further study.

2 Integer and set partitions

We define an integer partition λ to be a finite multiset of positive integers. Thus the multiset $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Let I_n be the set of all integer partitions of n. We form a poset on these integer partitions where the cover relation is given by adding two parts. In terms of multisets the cover relation is $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k\} \prec \{\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_k\}$. Note that the partition $\{1, 1, \ldots, 1\}$ is the minimal element and $\{n\}$ is the maximal element in the partial order.

Let Π_n denote the poset of all set partitions of $[n] = \{1, 2, \ldots, n\}$ where the partial order is given by merging blocks, that is, $\{B_1, B_2, B_3, \ldots, B_k\} \prec \{B_1 \cup B_2, B_3, \ldots, B_k\}$. The poset Π_n is in fact a lattice, called the partition lattice. Let $|\pi|$ denote the number of blocks of the partition π. Furthermore, for a set partition $\pi = \{B_1, B_2, \ldots, B_k\}$ define its type to be the integer partition of n given by the multiset $\mathrm{type}(\pi) = \{|B_1|, |B_2|, \ldots, |B_k|\}$.

The symmetric group $\mathfrak{S}_n$ acts on subsets of $[n]$ by relabeling the elements. Similarly, the symmetric group $\mathfrak{S}_n$ acts on the partition lattice by relabeling the elements of the blocks. For $\pi = \{B_1, B_2, \ldots, B_k\}$ a set partition the action is given by $\alpha \cdot \pi = \{\alpha(B_1), \alpha(B_2), \ldots, \alpha(B_k)\}$. Finally, when we speak about the action of the symmetric group $\mathfrak{S}_{n-1}$, we view the group $\mathfrak{S}_{n-1}$ as the subgroup $\{\alpha \in \mathfrak{S}_n : \alpha_n = n\}$ of the symmetric group $\mathfrak{S}_n$.

3 Compositions and ordered set partitions

A composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n is an ordered list of positive integers such that $c_1 + c_2 + \cdots + c_k = n$. Let $\text{Comp}(n)$ be the set of all compositions of n. We make $\text{Comp}(n)$ into a poset by introducing the cover relation given by adding adjacent entries, that is, $(c_1, \ldots, c_i, c_{i+1}, \ldots, c_k) \prec (c_1, \ldots, c_i + c_{i+1}, \ldots, c_k)$. The poset $\text{Comp}(n)$ is isomorphic to the Boolean algebra on $n-1$ elements. Note that $(1, 1, \ldots, 1)$ and (n) are the minimal and maximal elements of $\text{Comp}(n)$, respectively. Define the type of a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ to be the integer partition $\text{type}(\vec{c}) = \{c_1, c_2, \ldots, c_k\}$ of n. Furthermore, let $|\vec{c}|$ denote the number of parts of the composition $\vec{c}$.

For $\alpha \in \mathfrak{S}_n$, let the descent set of α, denoted by $\text{Des}(\alpha)$, be the subset of $[n-1]$ given by $\text{Des}(\alpha) = \{i \in [n-1] : \alpha(i) > \alpha(i+1)\}$. Throughout this paper it will be more convenient to consider $\text{Des}(\alpha)$ as a composition of n, namely, if $\text{Des}(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$, then we consider $\text{Des}(\alpha)$ as a composition of n given by $\text{Des}(\alpha) = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$. Note that the identity permutation $(1, 2, \ldots, n)$ has descent composition (n).

Let $\beta_n(\vec{c})$ be the number of permutations α in $\mathfrak{S}_n$ such that $\text{Des}(\alpha) = \vec{c}$. Likewise, define $\beta_n^*(\vec{c})$ to be the number of permutations α in $\mathfrak{S}_n$ with descent composition $\vec{c}$ and $\alpha(n) = n$.

An *ordered set partition* $\sigma = (C_1, C_2, \ldots, C_p)$ of $[n]$ is a list of non-empty blocks such that the set $\{C_1, C_2, \ldots, C_p\}$ is a partition of the set $[n]$, where the order of the blocks now matters. Let $|\sigma|$ denote the number of blocks in the ordered set partition σ.

Let Q_n be the set of all ordered set partitions on the set $[n]$. Introduce a partial order on Q_n where the cover relation is joining adjacent blocks, that is, $(C_1, \ldots, C_i, C_{i+1}, \ldots, C_p) \prec (C_1, \ldots, C_i \cup C_{i+1}, \ldots, C_p)$. Observe that the poset Q_n has the maximal element $([n])$, along with $n!$ minimal elements, namely the ordered set partitions $(\{\alpha_1\}, \{\alpha_2\}, \ldots, \{\alpha_n\})$, one for each permutation $\alpha_1 \alpha_2 \cdots \alpha_n \in \mathfrak{S}_n$. Moreover, every interval in Q_n is isomorphic to a Boolean algebra.

Define the *type* of an ordered set partition $\sigma = (C_1, C_2, \ldots, C_k)$ to be the composition of n given by $\text{type}(\sigma) = (|C_1|, |C_2|, \ldots, |C_k|)$. Finally, the symmetric group $\mathfrak{S}_n$ acts on ordered set partitions by relabeling, that is $\alpha \cdot (C_1, C_2, \ldots, C_k) = (\alpha(C_1), \alpha(C_2), \ldots, \alpha(C_1))$.

4 Topological considerations

Let P be a poset. Recall the order complex of P, denoted $\triangle(P)$, is the simplicial complex whose i-dimensional faces are the chains in P with $i+1$ elements. If P has a minimal element $\hat{0}$ or a maximal element $\hat{1}$, then $\triangle(P)$ is a contractible complex. Thus we will be removing these elements to ensure interesting topology.

Recall a simplicial complex Δ is a finite collection of sets such that the empty set belongs to Δ and Δ is closed under inclusion. We will find it easier to view a simplicial complex as a partially ordered set Δ such that (i) Δ has a unique minimal element $\hat{0}$ and (ii) every interval $[\hat{0}, x]$ for $x \in \Delta$ is isomorphic to a Boolean algebra. A poset P satisfying these conditions is called a *simplicial poset*. Notice that a poset P is simplicial if P is the face poset of a simplicial complex. Furthermore, note that the second condition in the definition of a simplicial poset makes the poset Δ ranked since every saturated chain between the minimal element $\hat{0}$ and an element x has the same length. Thus the *dimension* of an element x is defined by its rank minus one, that is, $\dim(x) = \rho(x) - 1$.

A filter in a poset P is an upper order ideal. Hence if F is a filter in P, then the dual filter F^* in the dual poset P^* is now a lower order ideal. In particular, if $\Delta \subseteq \mathrm{Comp}(n)$ is a filter, since upper order ideals in $\mathrm{Comp}(n)$ are isomorphic to Boolean algebras, the dual of Δ is a simplicial poset in the dual space $\mathrm{Comp}(n)^*$, which has cover relation given by splitting rather than merging. To emphasize that we have dualized, we use $\leq^*$ to denote the order relation in the dualized $\mathrm{Comp}(n)$.

Lastly, the *link* of a face F in a simplicial complex Δ is given by $\mathrm{lk}_F(\Delta) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$. However, working with the poset definition of a simplicial complex, we have the following equivalent definition of the link. The link is the principle filter generated by the face x, that is, $\mathrm{lk}_x(\Delta) = \{y \in \Delta : x \leq y\}$. One advantage of this definition is that we do not have to relabel the faces when considering the link.

From now on our simplicial complex Δ will be a filter in the composition lattice, $\mathrm{Comp}(n)$, with the dual order $\leq^*$.

Finally, for simplicial complexes Δ and Γ in $\mathrm{Comp}(n)$ and $\mathrm{Comp}(m)$ respectively, their *join* is defined to be poset $\Delta * \Gamma = \{\vec{c} \circ \vec{d} : \vec{c} \in \Delta, \vec{d} \in \Gamma\}$, where $\circ$ denote the concatenation of compositions.

5 Border strips and Specht modules

A border strip B is a connected skew-shape which does not contain a two by two square. For each composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ there is a unique border strip such that the number of boxes in the ith row is given by c_i and every two adjacent rows overlap in one position. Denote this border strip by $B(\vec{c})$.

We now define two operations on compositions. The motivation comes from the associated Specht and permutation modules. For a composition $\vec{c} = (c_1, \ldots, c_{k-1}, c_k)$ let $\vec{c} - 1$ denote the composition $(c_1, \ldots, c_{k-1}, c_k - 1)$ if $c_k \geq 2$, and otherwise let $\vec{c} - 1$ denote the empty composition. Similarly, let $\vec{c}/1$ denote the composition $(c_1, \ldots, c_{k-1}, c_k - 1)$ if $c_k \geq 2$, and otherwise let $\vec{c}/1$ denote the composition $(c_1, \ldots, c_{k-1})$. Note that if $\vec{c}$ is a composition of n then $\vec{c}/1$ is always a composition of $n-1$.

For a composition $\vec{c}$ of n let $B^*(\vec{c}\,)$ denote the border strip $B(\vec{c}-1)$. All our results of this paper are stated in terms of the Specht modules $S^{B^*(\vec{c}\,)}$ where the group action is by $\mathfrak{S}_{n-1}$. We think of this Specht module as a submodule of $S^{B(\vec{c}\,)}$ spanned by all standard Young tableaux where the northeastern-most box is filled with n. Note that when the composition ends with the entry 1, there are no such standard Young tableaux, and hence $S^{B^*(\vec{c}\,)}$ is the zero module.

6 The ordered partition filter Q^*_Δ

We now introduce the ordered partition filter Q^*_Δ. This filter will serve us as an important stepping stone to understanding the topology of general filters in the partition lattice. The transition from Q^*_Δ to the partition lattice uses Quillen's Fiber Lemma; see Section 7. Note that by considering the reverse orders in $\text{Comp}(n)$ and in Q_n we obtain two simplicial posets. Hence for Δ a non-empty filter in $\text{Comp}(n)$, we view Δ as a simplicial complex under the reverse order $\leq^*$. See the discussion in Section 4.

Definition 6.1. *Let Δ be a filter in $\text{Comp}(n)$, that is, Δ is a simplicial complex consisting of compositions of n. Define the ordered partition filter Q^*_Δ to be all ordered set partitions whose type is in the complex Δ and whose last block contains the element n, that is,*

$$Q^*_\Delta = \{\sigma = (C_1, C_2, \ldots, C_k) \in Q_n \,:\, \text{type}(\sigma) \in \Delta,\, n \in C_k\}.$$

Note that we view Q^*_Δ as a simplicial complex. Our purpose is to study the reduced homology groups of this complex.

Recall that the link of a composition $\vec{c}$ in Δ is the filter $\text{lk}_{\vec{c}}(\Delta) = \{\vec{d} \in \Delta \,:\, \vec{d} \leq^* \vec{c}\,\}$, where $\leq^*$ is the reverse of the partial order of $\text{Comp}(n)$. Since $\text{lk}_{\vec{c}}(\Delta)$ is now a simplicial poset with minimal element $\vec{c}$, we have a dimension shift from Δ to $\text{lk}_{\vec{c}}(\Delta)$ given by

$$\dim_{\text{lk}_{\vec{c}}(\Delta)}(\vec{d}\,) = \dim_\Delta(\vec{d}\,) - |\vec{c}\,| + 1 \tag{6.1}$$

for $\vec{d} \in \text{lk}_{\vec{c}}(\Delta)$. A special case of Q^*_Δ is when the simplicial complex Δ is a simplex, that is, Δ is generated by one composition $\vec{c}$. This case was studied by Ehrenborg and Jung in [7]. Their results are given below.

Theorem 6.2 (Ehrenborg–Jung)**.** *Let $\vec{c}$ be a composition of n into k parts. Then the complex $Q^*_{\vec{c}}$ is a wedge of $\beta^*_n(\vec{c}\,)$ spheres of dimension $k-2$. Furthermore, the top homology group $\widetilde{H}_{k-2}(Q^*_{\vec{c}})$ is isomorphic to the Specht module $S^{B^*(\vec{c}\,)}$ as an $\mathfrak{S}_{n-1}$-module.*

Note that Ehrenborg and Jung formulated their result in terms of pointed set partitions. That is, our notation $Q^*_{\vec{c}}$ is $\Delta_{\vec{d}}$ in their notation, where $\vec{d} = (c_1, \ldots, c_{k-1}, c_k - 1)$.

They allow the last entry of a composition to be zero and similarly the last entry of an ordered set partition to be empty. Moreover, our notation $\Pi^*_{\vec{c}}$ is in their notation $\Pi^\bullet_{\vec{d}}$.

We can now state the main result of this section.

Theorem 6.3. *Let Δ be a simplicial complex of compositions of n. Then the ith reduced homology group of the simplicial complex Q^*_Δ is given by*

$$\widetilde{H}_i(Q^*_\Delta) \cong \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Furthermore, this isomorphism holds as $\mathfrak{S}_{n-1}$-modules.

7 Filters in the set partition lattice

In Theorem 6.3 we characterized each homology group of Q^*_Δ, a subspace of ordered set partitions. We will now translate the topological data we have gathered on Q^*_Δ into data on the usual partition lattice Π_n.

Recall that Q^*_Δ is the collection of ordered set partitions containing the element n in the last block, whose type is contained in the simplicial complex $\Delta \subseteq \mathrm{Comp}(n)$. Define the *forgetful map* $f : Q^*_\Delta \longrightarrow \Pi_n$ given by removing the order between blocks, that is, $f((C_1, C_2, \ldots, C_k)) = \{C_1, C_2, \ldots, C_k\}$.

Definition 7.1. *Let $\Pi^*_\Delta \subseteq \Pi_n$ be the image of Q^*_Δ under the forgetful map f.*

Lemma 7.2. *Suppose that F is a filter in the integer partition lattice. Let Δ_F be the filter of compositions given by $\{\vec{c} \in \mathrm{Comp}(n) : \mathrm{type}(\vec{c}) \in F\}$. Then the associated filter $\Pi^*_{\Delta_F}$ in the partition lattice is given by $\{\pi \in \Pi_n : \mathrm{type}(\pi) \in F\}$.*

Remark 7.3. In general, taking the image of a filter $\Delta \subseteq \mathrm{Comp}(n)$ under the map type does not define a filter in the integer partition lattice I_n. For example, consider the simplex Δ in $\mathrm{Comp}(6)$ generated by $(3, 2, 1)$. Note that $\mathrm{type}(\Delta)$ consists of the four partitions $\{\{3, 2, 1\}, \{3 + 2, 1\}, \{3, 2 + 1\}, \{3 + 2 + 1\}\} = \{\{3, 2, 1\}, \{5, 1\}, \{3, 3\}, \{6\}\}$. This is not a filter in I_6 since it does not contain the partition $\{4, 2\}$.

The $\mathfrak{S}_{n-1}$ action on Π^*_Δ extends to the chains in the order complex $\triangle(\Pi^*_\Delta - \{\hat{1}\})$. For the equivariant version of the Quillen Fiber Lemma, see [15, Theorem 5.2.2].

Proposition 7.4. *The forgetful map $f : Q^*_\Delta - \{\hat{1}\} \longrightarrow \Pi^*_\Delta - \{\hat{1}\} = P$ satisfies the condition of Quillen's Equivariant Fiber Lemma, that is, for a partition $\pi = \{B_1, B_2, \ldots, B_k\}$ in P, the order complex $\triangle(f^{-1}(P_{\geq \pi}))$ is the barycentric subdivision of a cone, and is therefore contractible and acyclic.*

Combining Proposition 7.4 with Theorem 6.3, we have the following result for the homology of the order complex $\triangle(\Pi^*_\Delta - \{\hat{1}\})$.

Theorem 7.5. *The ith reduced homology group of the order complex of* $\Pi^*_\Delta - \{\hat{1}\}$ *as an* $\mathfrak{S}_{n-1}$*-module is given by*

$$\widetilde{H}_i(\triangle(\Pi^*_\Delta - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c}\in\Delta} \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

8 The Frobenius complex

We now consider a class of examples stemming from [6]. Let Λ be a semigroup of positive integers, that is, a subset of the positive integers which is closed under addition. Let Δ_n be the collection of all compositions of n whose parts belong to Λ, that is,

$$\Delta_n = \{(c_1, \dots, c_k) \in \mathrm{Comp}(n) \ : \ c_1, \dots, c_k \in \Lambda\}.$$

Since Λ is closed under addition, we obtain that Δ_n is a filter in the poset of compositions $\mathrm{Comp}(n)$ and hence we view it as a simplicial complex. This complex is known as the Frobenius complex; see [6]. Moreover, since Λ is a semigroup, the collection of integer partitions of n with parts in Λ is a filter, therefore, using Lemma 7.2 the associated filter in the partition lattice is given by

$$\Pi^\Lambda_n = \{\{B_1, \dots, B_k\} \in \Pi_n \ : \ |B_1|, \dots, |B_k| \in \Lambda\}.$$

We continue by studying one concrete example. Let a and d be two positive integers. Let Λ be the semigroup generated by the arithmetic progression $\Lambda = \langle a, a+d, a+2d, \dots\rangle$. Since for $j \geq a$ we have that $a + j \cdot d = d \cdot a + a + (j-a) \cdot d$, the semigroup is generated by the finite arithmetic progression $\Lambda = \langle a, a+d, a+2d, \dots, a+(a-1)d\,\rangle$. Clark and Ehrenborg proved that the Frobenius complex Δ_n is a wedge of spheres of different dimensions; see [6, Theorem 5.1]. Observe that their result is formulated in terms of sets, instead of compositions. However, the two notions are equivalent via the natural bijection given by sending a composition $(c_1, c_2, \dots, c_k)$ of n to the subset $\{c_1, c_1 + c_2, \dots, c_1 + \cdots + c_{k-1}\}$ of the set $[n-1]$. To state their result, let A be the set $\{a+d, a+2d, \dots, a+(a-1)\cdot d\}$.

Proposition 8.1. *For n in the semigroup Λ, there is a discrete Morse matching on the Frobenius complex Δ_n such that the critical cells are compositions $\vec{c} = (c_1, \dots, c_k)$ characterized by (i) All but the last entry of the composition belongs to the set A, that is, $c_1, \dots, c_{k-1} \in A$. (ii) The last entry c_k belongs to $\{a\} \cup A$. Furthermore, all the critical cells are facets.*

Call the sum $c_1 + c_2 + \cdots + c_j$ an *initial sum* of a composition $(c_1, c_2, \dots, c_k)$ for $j \leq k$.

Definition 8.2. *For an interval $[\vec{d}, \vec{b}\,]$ in the lattice of compositions* $\mathrm{Comp}(n)$ *let $B^*(\vec{d}, \vec{b}\,)$ be the skew-shape where the row lengths are given by $d_1, d_2, \dots, d_{r-1}, d_r - 1$ and if the initial sum $d_1 + \cdots + d_j$ is equal to an initial sum of the composition $\vec{b} - 1$, then jth row and the $(j+1)$st row overlap in one column. All other rows of $B^*(\vec{d}, \vec{b}\,)$ are non–overlapping.*

Definition 8.3. *For a composition $\vec{d}$ of n with entries in the set $\{a\} \cup A$ let $\vec{b}(\vec{d})$ be the composition greater than or equal to $\vec{d}$ obtained by adding runs of entries of $\vec{d}$ together where each run ends with the entry a.*

Theorem 8.4. *Let a and d be two positive integers and let Π_n^Λ be the filter in the partition lattice Π_n where each partition π consists of blocks whose cardinalities belong to the semigroup Λ generated by the arithmetic progression $a, a+d, \ldots, a+(a-1)\cdot d$. Then the ith reduced homology group of the order complex $\triangle(\Pi_n^\Lambda - \{\hat{1}\})$ is given by the direct sum*

$$\widetilde{H}_i(\triangle(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d}} S^{B^*(\vec{d},\vec{b}(\vec{d}))},$$

where the sum is over all compositions $\vec{d}$ into $i+2$ parts such that every entry belongs to the set $\{a\} \cup A = \{a, a+d, a+2\cdot d, a+(a-1)\cdot d\}$.

Example 8.5. When the integer d divides the integer a, the homology groups of Π_n^Λ have been studied. In this case, the filter Π_n^Λ consists of all partitions where the block sizes are divisible by d and the block sizes are greater than or equal to a. This filter was studied by Browdy [4], and our Theorem 8.4 reduces to Browdy's result; see Corollary 5.3.3 in [4].

Example 8.6. The previous example is particularly nice when $d = 1$. The semigroup Λ is given by $\Lambda = \{n \in \mathbb{P} : n \geq a\}$ and the filter Π_n^Λ consists of all partitions where $1, 2, \ldots, a-1$ are forbidden block sizes. In this case it follows by Billera and Meyers [1] that Δ_n is non-pure shellable. Additionally, Björner and Wachs [2] gave an EL-labelling of $\Pi_n^\Lambda \cup \{\hat{0}\}$. This complex was also considered by Sundaram in Example 4.4 in [11].

9 The partition filter $\Pi_n^{\langle a,b\rangle}$

Let a and b be two relatively prime integers greater than 1. Let $\Pi_n^{\langle a,b\rangle}$ be the filter in Π_n generated by all partitions whose block sizes are all a or b. As an example, $\Pi_n^{\langle 2,3\rangle}$ consists of all partitions in Π_n with no singleton blocks. The corresponding complex Δ_n in $\mathrm{Comp}(n)$ consists of all compositions of n whose parts are contained in the set $\langle a, b\rangle = \{i \cdot a + j \cdot b : a, b \in \mathbb{N}\}$. When $a = 2$ and $b = 3$ the complex Δ_n is known as the complex of sparse sets; see [6, 8].

Following Theorem 4.1 in [6], we define the set $A = \{n \in \mathbb{P} : n \equiv 0, a, b \text{ or } a+b \bmod ab\}$ and the function $h : A \longrightarrow \mathbb{Z}_{\geq -1}$ as follows:

$$h(n) = \begin{cases} \frac{2n}{ab} - 2 & \text{if } n \equiv 0 \bmod ab, \\ \frac{2(n-a)}{ab} - 1 & \text{if } n \equiv a \bmod ab, \\ \frac{2(n-b)}{ab} - 1 & \text{if } n \equiv b \bmod ab, \\ \frac{2(n-a-b)}{ab} & \text{if } n \equiv a+b \bmod ab. \end{cases}$$

Then Theorem 4.1 in [6] states that Δ_n is either homotopy equivalent to a sphere or is contractible, according to

$$\Delta_n \simeq \begin{cases} S^{h(n)} & \text{if } n \in A, \\ \text{point} & \text{otherwise.} \end{cases}$$

For a composition $\vec{c} = (c_1, \ldots, c_k)$ of n with all of its parts in A, let $\dim(\vec{c})$ denote the dimension of the reduced homology of $\triangle(\Pi_n^{\langle a,b\rangle} - \{\hat{1}\})$ to which the composition $\vec{c}$ contributes. That is, $\dim(\vec{c})$ is given by $\dim(\vec{c}) = \sum_{i=1}^{k} h(c_i) + 2k - 2$.

Theorem 9.1. *Let $2 \leq a < b$ with $\gcd(a,b) = 1$. Then the ith reduced homology group of $\triangle(\Pi_n^{\langle a,b\rangle} - \{\hat{1}\})$ is given by the direct sum of Specht modules $\bigoplus_{\vec{c} \in F_i} S^{B^*(\vec{c})}$, where F_i is the collection of compositions $\vec{c}$ of n where all the parts are in the set A with $\dim(\vec{c}) = i$.*

We now describe the top and bottom reduced homology of the order complex $\Delta(\Pi_n^{\langle a,b\rangle} - \{\hat{1}\})$. We begin with the top homology.

Proposition 9.2. *Let $2 \leq a < b$ with $\gcd(a,b) = 1$. Let r be the unique integer such that $0 \leq r < a$ and $n \equiv rb \bmod a$. Then the top homology of $\triangle(\Pi_n^{\langle a,b\rangle} - \{\hat{1}\})$, which occurs in dimension $(n - r(b-a))/a - 2$, is given by the direct sum of Specht modules $\bigoplus_{\vec{c} \in R} S^{B^*(\vec{c})}$, where R is the collection of compositions $\vec{c}$ of n where exactly r of the parts are equal to b or $a+b$, and the remaining parts are all equal to a.*

We now turn our attention to the bottom reduced homology.

Proposition 9.3. *Let $3 \leq a < b$ with $\gcd(a,b) = 1$. Let r and s be the two unique integers such that*

$$n \equiv rb \bmod a, \quad 0 \leq r < a, \quad n \equiv sa \bmod b \quad \text{and} \quad 0 \leq s < b.$$

Then the bottom reduced homology of $\triangle(\Pi_n^{\langle a,b\rangle} - \{\hat{1}\})$ occurs in dimension $2 \cdot \frac{n-sa-rb}{ab} + r + s - 2$, and is given by the direct sum of Specht modules $S^{B^*(\vec{c})}$ over all compositions $\vec{c}$ such that the number of parts of $\vec{c}$ of the form $j \cdot ab + a$ and $j \cdot ab + a + b$ is s and the number of parts of the form $j \cdot ab + b$ and $j \cdot ab + a + b$ is r.

We end with a complete description in the case when $a = 2$.

Proposition 9.4. *Let b be odd and greater than or equal to 3. Then the ith reduced homology of $\triangle(\Pi_n^{\langle 2,b\rangle} - \{\hat{1}\})$ is given by the direct sum of Specht modules $S^{B^*(\vec{c})}$ over all compositions $\vec{c}$ with all parts congruent to 0 or 2 modulo b, where exactly $(b(i+2) - n)/(b-2)$ entries of $\vec{c}$ are congruent to 2 modulo b. The bottom reduced homology occurs in dimension $\lceil n/b \rceil - 2$.*

10 Concluding remarks

Using Theorem 7.5 we have been able to classify the action of $\mathfrak{S}_{n-1}$ on the top homology of $\triangle(\Pi^*_\Delta - \{\hat{1}\})$ for any complex $\Delta \subseteq \mathrm{Comp}(n)$. In the case when $\triangle(\Pi^*_\Delta - \{\hat{1}\})$ is shellable, is there an EL-labelling of $\Pi^*_\Delta \cup \{\hat{0}\}$ that realizes this shelling order?

Is there a way we can classify the $\mathfrak{S}_n$-action on the homology groups of $\triangle(\Pi^*_\Delta - \{\hat{1}\})$ rather than the $\mathfrak{S}_{n-1}$-action? Browdy described the matrices representing the action of $\mathfrak{S}_n$ on the cohomology groups of the filter with block sizes belonging to the arithmetic progression $k \cdot d, (k+1) \cdot d, \ldots$; see [4, Section 5.4].

The partition lattice is naturally associated with the symmetric group, that is, the Coxeter group of type A. Miller [9] has extended the results about the filter $\Pi^*_{\vec{c}}$ to other root systems. Hence it is natural to ask if our results for the filter Π^*_Δ can be extended to other root systems.

Lastly, all of our results are based upon Δ being a filter in the composition lattice $\mathrm{Comp}(n)$. What if we remove the filter constraint? That is, let Ω be an arbitrary collection of compositions of n not containing the extreme composition (n). Define Q^*_Ω to be all ordered set partitions $\sigma = (C_1, C_2, \ldots, C_k)$ such that $\mathrm{type}(\sigma) \in \Omega$ and containing n in the last block C_k. Let Π_Ω be the image of Q^*_Ω under the forgetful map f. What can be said about the homology groups and the homotopy type of the order complex $\triangle(\Pi_\Omega)$? We need to understand the topology of the links $\mathrm{lk}_{\vec{c}}(\Omega)$, even though these links are not themselves simplicial complexes.

Acknowledgements

The authors thank Bert Guillou and Kate Ponto for their homological guidance and expertise. They also thank Sheila Sundaram and Michelle Wachs for essential references. The authors thank Margaret Readdy for her comments on an earlier draft. Both authors were partially supported by National Security Agency grant H98230-13-1-0280. The first author wishes to thank the Princeton University Mathematics Department where this work began.

References

[1] L. J. Billera and A. N. Myers. "Shellability of interval orders". *Order* **15** (1998), pp. 113–117. DOI.

[2] A. Björner and M. L. Wachs. "Shellable nonpure complexes and posets. I". *Trans. Amer. Math. Soc.* **348** (1996), pp. 1299–1327. DOI.

[3] A. Björner, M. L. Wachs, and V. Welker. "Poset fiber theorems". *Trans. Amer. Math. Soc.* **357** (2005), pp. 1877–1899. DOI.

[4] A. Browdy. "The (Co)Homology of Lattices and Partitions with Restricted Block Size". PhD thesis. University of Miami, 1996.

[5] A. R. Calderbank, P. Hanlon, and R. W. Robinson. "Partitions into even and odd block size and some unusual characters of the symmetric groups". *Proc. London Math. Soc* **s3-53** (1986), pp. 288–320. DOI.

[6] E. Clark and R. Ehrenborg. "The Frobenius complex". *Ann. Combin.* **16** (2012), pp. 215–232. DOI.

[7] R. Ehrenborg and J. Jung. "The topology of restricted partition posets". *J. Algebraic Combin.* **37** (2012), pp. 643–666. DOI.

[8] D. N. Kozlov. "Complexes of directed trees". *J. Combin. Theory Ser. A* **88** (1999), pp. 112–122. DOI.

[9] A. R. Miller. "Reflection arrangements and ribbon representations". *European J. Combin.* **39** (2014), pp. 24–56. DOI.

[10] R. P. Stanley. "Exponential Structures". *Studies Appl. Math.* **59** (1978), pp. 73–82. DOI.

[11] S. Sundaram. "Applications of the Hopf trace formula to computing homology representations". *Jerusalem Combinatorics '93*. Contemp. Math., Vol. 178. Amer. Math. Soc., 1994, pp. 277–309. DOI.

[12] S. Sundaram. "On the topology of two partition posets with forbidden block sizes". *J. Pure. Appl. Algebra* **155** (2001), pp. 271–304. DOI.

[13] G. S. Sylvester. "Continuous-Spin Ising Ferromagnets". PhD thesis. Massachusetts Institute of Technology, 1976.

[14] M. L. Wachs. "A basis for the homology of the d-divisible partition lattice". *Adv. Math.* **117** (1996), pp. 294–318. DOI.

[15] M. L. Wachs. "Poset topology: Tools and applications". *Geometric Combinatorics*. IAS/Park City Math. Ser., Vol. 13. Amer. Math. Soc., 2007, pp. 497–615.