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Realizing Simion's type B associahedron as a pulling triangulation of the Legendre polytope

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Abstract. We show that Simion's type B associahedron is combinatorially equivalent to a pulling triangulation of a type B root polytope called the Legendre polytope. Furthermore, we show that every pulling triangulation of the Legendre polytope yields a flag complex. Our triangulation refines a decomposition of the Legendre polytope given by Cho. We extend Cho's cyclic group action to the triangulation in such a way that it corresponds to rotating centrally symmetric triangulations of a regular $(2n+2)$-gon.

Résumé. Nous montrons que l'associaèdre du type B de Simion est combinatoriellement èquivalent à une triangulation obtenue en tirant les sommets d'un polytope des racines du type B, appelé le polytope de Legendre. De plus, nous montrons que toute triangulation obtenue en tirant les sommets de ce polytope est un complex de drapeau. Notre triangulation raffine une décomposition du polytope de Legendre donné par Cho. Nous éteignons l'action du groupe cyclique de Cho à la triangulation tel qu'elle correspond à la rotation des triangulations centrallement symétriques d'un $(2n+2)$-gone régulier.

Keywords: root polytope, associahedron, type B

1 Introduction

Root polytopes arising as convex hulls of roots in a root system have become the subject of intensive interest in recent years [1, 10, 13, 16, 21, 22]. Another important area where geometry meets combinatorics is the study of noncrossing partitions, associahedra and their generalizations. In this context Simion [24] constructed a type B associahedron whose facets correspond to centrally symmetric triangulations of a regular $(2n+2)$-gon. Burgiel and Reiner [5] described Simion's construction as providing "the first motivating example for an equivariant generalization of fiber polytopes, that is, polytopal subdivisions which are invariant under symmetry groups". It was recently observed by Cori

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and Hetyei [11] that the face numbers in this type B associahedron are the same as the face numbers in any pulling triangulation of the boundary of a type B root polytope, called the Legendre polytope in [16].

In this presentation we explain that the equality of these face numbers is not a mere coincidence: the type B associahedron is combinatorially equivalent to a pulling triangulation of the Legendre polytope P_n. The convex hull of the positive roots among the vertices of the Legendre polytope and of the origin is a type A root polytope P_n^+. Cho [9] has shown that the Legendre polytope P_n may be decomposed into copies of P_n^+ that meet only on their boundaries and that there is a $\mathbb{Z}_{n+1}$-action on this decomposition. Our triangulation representing the type B associahedron as a triangulation of the Legendre polytope refines Cho's decomposition in such a way that extends the $\mathbb{Z}_{n+1}$-action to the triangulation. The effect of this $\mathbb{Z}_{n+1}$-action on the centrally symmetric triangulations of the $(2n+2)$-gon is rotation.

2 Preliminaries

2.1 Simion's type B associahedron

Simion [24] introduced a simplicial complex denoted by Γ_n^B on $n(n+1)$ vertices as follows. Consider a centrally symmetric convex $(2n+2)$-gon, and label its vertices in the clockwise order with $1, 2, \ldots, n, n+1, \overline{1}, \overline{2}, \ldots, \overline{n}, \overline{n+1}$. The vertices of Γ_n^B are the *B-diagonals*, which are one of the two following kinds: diagonals joining antipodal pairs of points, and antipodal pairs of noncrossing diagonals. The diagonals joining antipodal points are all pairs of the form $\{i, \overline{i}\}$ satisfying $1 \leq i \leq n+1$, and they are called *diameters*. The B-diagonals that are antipodal pairs of noncrossing diagonals are either of the form $\{\{i,j\},\{\overline{i},\overline{j}\}\}$ satisfying $1 \leq i < i+1 < j \leq n+1$ or of the form $\{\{i,\overline{j}\},\{\overline{i},j\}\}$ satisfying $1 \leq j < i \leq n+1$.

The simplicial complex Γ_n^B is the family of sets of pairwise noncrossing B-diagonals. Simion showed the simplicial complex Γ_n^B is the boundary complex of an n-dimensional convex polytope. The dual of this polytope is also known as the Bott–Taubes polytope [4] and the cyclohedron [20]. Associahedra are usually defined as simple polytopes. However, Simion followed Perles' convention by calling this simplicial polytope the associahedron. Since the only associahedron we will consider is the one constructed by Simion, we prefer to use her nomenclature.

Simion also computed the face numbers and h-vector. These turn out to be identical with the face numbers and h-vector of any pulling triangulation of the Legendre polytope. We will discuss this polytope in the next subsection. We end with a fact that is implicit in the work of Simion [24, Section 3.3].

Lemma 2.1 (Simion)**.** *Each facet Γ_n^B of Simion's type B associahedron contains exactly one B-diagonal of the form $\{i, \bar{i}\}$ connecting an antipodal pair of points.*

2.2 The Legendre polytope or "full" type A root polytope

Consider an $(n+1)$-dimensional Euclidean space with orthonormal basis $\{e_1, \ldots, e_{n+1}\}$. The convex hull of the vertices $\pm 2e_1, \ldots, \pm 2e_{n+1}$ is an $(n+1)$-dimensional cross-polytope. The intersection of this cross-polytope with the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$ is an n-dimensional centrally symmetric polytope P_n first studied by Cho [9]. It is called the *Legendre polytope* in the work of Hetyei [16], since the polynomial $\sum_{j=0}^{n} f_{j-1} \cdot ((x-1)/2)^j$ is the nth Legendre polynomial, where f_i is the number of i-dimensional faces in any pulling triangulation of the boundary of P_n. See Lemma 2.7 below. Furthermore, it is called the "full" type A root polytope in the work of Ardila–Beck–Hoşten–Pfeifle–Seashore [1]. It has $n(n+1)$ vertices consisting of all points of the form $e_i - e_j$ where $i \neq j$.

We use the shorthand notation (i, j) for the vertex $e_j - e_i$ of the Legendre polytope P_n. We may think of these vertices as the set of all directed nonloop edges on the vertex set $\{1, 2, \ldots, n+1\}$. A subset of these edges is contained in some face of P_n exactly when there is no $i \in \{1, 2, \ldots, n+1\}$ that is both the head and the tail of a directed edge. Equivalently, the faces are described as follows.

Lemma 2.2. *The faces of the Legendre polytope P_n are of the form* $\mathrm{conv}(I \times J) = \mathrm{conv}(\{(i, j) : i \in I, j \in J\})$ *where I and J are two non-empty disjoint subsets of the set $\{1, 2, \ldots, n+1\}$. The dimension of a face is given by $|I| + |J| - 2$. A face is a facet if and only if the union of I and J is the set $\{1, 2, \ldots, n+1\}$.*

Especially, when the two sets I and J both have cardinality two, the associated face is a square. Furthermore, the other two-dimensional faces are equilateral triangles.

Affine independent subsets of vertices of faces of the Legendre polytope are easy to describe. A set $S = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ is a $(k-1)$-dimensional simplex if and only if, disregarding the orientation of the directed edges, the set S contains no cycle, that is, it is a forest [16, Lemma 2.4].

The Legendre polytope P_n contains the polytope P_n^+, defined as the convex hull of the origin and the set of points $e_i - e_j$, where $i < j$. The polytope P_n^+ was first studied by Gelfand, Graev and Postnikov [13] and later by Postnikov [23]. Some of the results on P_n^+ may be easily generalized to P_n.

2.3 Pulling triangulations

The notion of pulling triangulations is originally due to Hudson [17, Lemma 1.4]. For more modern formulations, see [25, Lemma 1.1] and [2, End of Section 2]. We refer to [16, Section 2.3] for the version presented here.

For a polytopal complex $\mathcal{P}$ and a vertex v of $\mathcal{P}$, let $\mathcal{P} - v$ be the complex consisting of all faces of $\mathcal{P}$ not containing the vertex v. Also for a facet F let $\mathcal{P}(F)$ be the complex of all faces of $\mathcal{P}$ contained in F.

Definition 2.3 (Hudson)**.** *Let $\mathcal{P}$ be a polytopal complex and let $<$ be a linear order on the set V of its vertices. The pulling triangulation $\triangle(\mathcal{P})$ with respect to $<$ is defined recursively as follows. We set $\triangle(\mathcal{P}) = \mathcal{P}$ if $\mathcal{P}$ consists of a single vertex. Otherwise let v be the least element of V with respect to $<$ and set*

$$\triangle(\mathcal{P}) = \triangle(\mathcal{P} - v) \cup \bigcup_{F} \{\mathrm{conv}(\{v\} \cup G) \,:\, G \in \triangle(\mathcal{P}(F))\},$$

where the union runs over the facets F not containing v of the maximal faces of $\mathcal{P}$ which contain v. The triangulations $\triangle(\mathcal{P} - v)$ and $\triangle(\mathcal{P}(F))$ are with respect to the order $<$ restricted to their respective vertex sets.

Theorem 2.4 (Hudson)**.** *The pulling triangulation $\triangle(\mathcal{P})$ is a triangulation of the polytopal complex $\mathcal{P}$ without introducing any new vertices.*

In particular, any pulling triangulation of the boundary of P_n is compressed as defined by Stanley [25], and has the same face numbers [16, Corollary 4.11]. This important fact and the analogous statement for P_n^+ is a direct consequence of the following two fundamental results [14, 15, 25].

Proposition 2.5 (Stanley)**.** *Suppose that one of the vertices of a polytope P is the origin and that the matrix whose rows are the coordinates of the vertices of P is totally unimodular. Let $<$ be any ordering on the vertex set of P such that the origin is the least vertex with respect to $<$. Then the pulling order $<$ is compressed, that is, all of the facets in the induced triangulation have the same relative volume.*

Theorem 2.6 (Heller)**.** *The incidence matrix of a directed graph is totally unimodular.*

2.4 Face vectors of pulling triangulations of the Legendre polytope

Among all triangulations of the boundary of the Legendre polytope P_n obtained by pulling the vertices, counting faces is most easily performed for the *lexicographic triangulation* in which we pull (i, j) before (i', j') exactly when $i < i'$ or when $i = i'$ and $j < j'$. Counting faces in this triangulation amounts to counting lattice paths; see [16, Lemma 5.1] and [1, Proposition 17]. From this we obtain the following expression for the face numbers [16, Theorem 5.2].

Lemma 2.7 (Hetyei)**.** *For any pulling triangulation of the boundary of P_n, the number f_{j-1} of $(j-1)$-dimensional faces is*

$$f_{j-1} = \binom{n+j}{j}\binom{n}{j} \quad \text{for } 0 \le j \le n. \tag{2.1}$$

3 The flag property

Recall that a simplicial complex is a *flag complex* if every minimal nonface has two elements. The main result of this section is the following.

Theorem 3.1. *Every pulling triangulation of the boundary of the Legendre polytope P_n is a flag simplicial complex.*

A key ingredient for proving this theorem and Theorem 6.1 is the following observation.

Lemma 3.2. *Let $\{x_1, x_2, y_1, y_2\}$ be a four element subset of the set $\{1, 2, \ldots, n+1\}$. Then the set $\{x_1, x_2\} \times \{y_1, y_2\} = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_1)\}$ is the vertex set of a square face of the Legendre polytope P_n, and the sets $\{(x_1, y_1), (x_2, y_2)\}$ and $\{(x_1, y_2), (x_2, y_1)\}$ are the diagonals of this square. For any pulling triangulation the diagonal containing the vertex that was pulled first is an edge of the triangulation and the other diagonal is not an edge.*

Since the Cartesian product of an m-dimensional simplex and an n-dimensional simplex is a face of the Legendre polytope of dimension $m+n+1$, we obtain the following corollary.

Corollary 3.3. *Every pulling triangulation of the Cartesian product of two simplices is a flag complex.*

4 The arc representation of Γ_n^B

Consider a regular $(2n+2)$-gon whose vertices are labeled $1, 2, \ldots, n+1, \overline{1}, \overline{2}, \ldots, \overline{n+1}$ in the clockwise order. Identify each vertex $\overline{i}$ with $n+1+i$ for $i = 1, 2, \ldots, n+1$. Subject to this identification, each B-diagonal, may be represented as an unordered pair of diagonals of the form $\{\{u, v\}, \{u+n+1, v+n+1\}\}$ for some 2-element subset $\{u, v\}$ of $\{1, 2, \ldots, 2n+2\}$, where addition is modulo $2n+2$. For B-diagonals $\{k, \overline{k}\}$ joining antipodal points, the unordered pair $\{\{k, k+n+1\}, \{k+n+1, k+2n+2\}\}$ contains two copies of the same two-element set.

For any two points x and y on the circle $\mathbb{R}/(2n+2)\mathbb{Z}$ which are not antipodal, let $[x, y]$ denote the shortest arc from x to y.

Definition 4.1. *We define the* arc-representation *on the vertices of Γ_n^B as follows. Subject to the above identifications, represent the B-diagonal $\{\{u, v\}, \{u+n+1, v+n+1\}\}$ with the centrally symmetric pair of arcs $\{[u, v-1], [u+n+1, v+n]\}$ on the circle $\mathbb{R}/(2n+2)\mathbb{Z}$.*

Note that for B-diagonals of the form $\{\{k, n+1+k\}, \{k, n+1+k\}\}$ corresponding to antipodal pairs of points, the union of the arcs $[k, k+n]$ and $[k+n+1, k-1]$ is not the full circle.

See Figure 1 for an example where $n = 7$ with the B-diagonal $\{\{2, 5\}, \{\overline{2}, \overline{5}\}\}$.

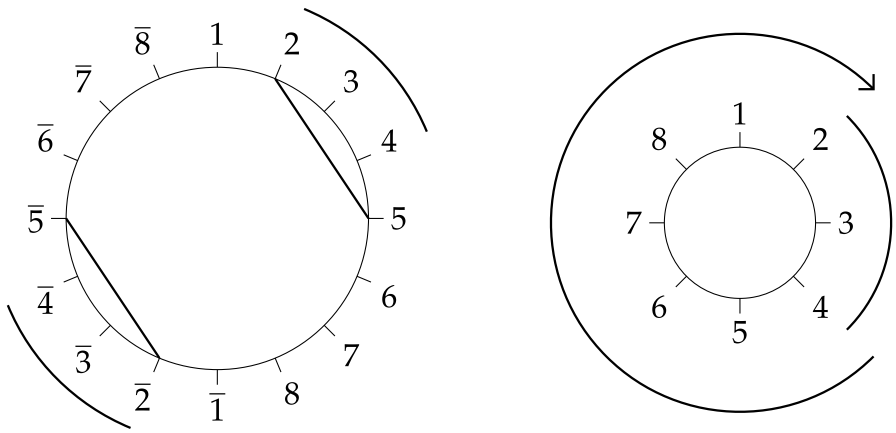


Figure 1: The arc representation of the B-diagonal consisting of the two diagonals $\{2,5\}$ and $\{\overline{2},\overline{5}\} = \{10,13\}$ is the two arcs $[2,4]$ and $[\overline{2},\overline{4}] = [10,12]$. By considering the arcs modulo $n+1 = 8$ (see the second circle) we obtain that this B-diagonal is represented by the arrow $(4,2)$.

Lemma 4.2. *The arc-representation of the vertices of Γ_n^B is one-to-one: distinct B-diagonals are mapped to distinct centrally symmetric pairs of arcs.*

The following theorem plays an important role in connecting the type B associahedron with the Legendre polytope.

Theorem 4.3. *The B-diagonal represented by the pair of arcs $\{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and the B-diagonal represented by the pair of arcs $\{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$ are noncrossing if and only if for either arc $I \in \{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and for either arc $J \in \{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$, the arcs I and J are either nested or disjoint.*

Corollary 4.4. *The B-diagonal represented by the pair of arcs $\{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and the B-diagonal represented by the pair of arcs $\{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$ are noncrossing if and only the set $[u_1, v_1 - 1] \cup [u_1 + n + 1, v_1 + n]$ and the set $[u_2, v_2 - 1] \cup [u_2 + n + 1, v_2 + n]$ are nested or disjoint.*

5 Embedding Γ_n^B as a family of simplices on ∂P_n

We begin by defining a bijection between the vertex set of Γ_n^B and that of P_n. Recall that we use the shorthand notation (i, j) for the vertex $e_j - e_i$ of P_n. We refer to (i, j) as *the arrow from i to j*. Using the term "arrow" as opposed to "directed edge" will eliminate the confusion that $e_j - e_i$ is a vertex of P_n.

Definition 5.1. *Let $\{[i, j], [\overline{i}, \overline{j}]\}$ be the arc representation of a B-diagonal in Γ_n^B, where $1 \leq i \leq n+1$ and $i < j$. Define* the arrow representation *of this B-diagonal in P_n to be the arrow (j, i).*

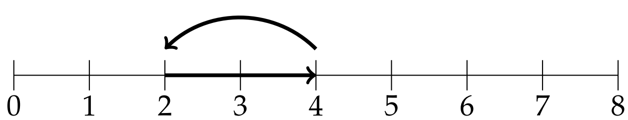
In other words, the arrow encodes the complement of the image of the arcs in the circle $\mathbb{R}/(n+1)\mathbb{Z}$. We refer to the second circle in Figure 1 for the continuation of the example of the B-diagonal $\{\{2,5\},\{\overline{2},\overline{5}\}\}$.

Definition 5.2. *Define the map* $\pi : \mathbb{R}/(2n+2)\mathbb{Z} \longrightarrow \mathbb{R}/(n+1)\mathbb{Z}$ *to be the modulo* $n+1$ *map. Furthermore, identify the circle* $\mathbb{R}/(n+1)\mathbb{Z}$ *with the half-open interval* $(0, n+1]$. *Thus the map* π *sends each* $x \in (0, n+1]$ *to* x *and each* $x \in (n+1, 2n+2]$ *to* $x - n - 1$.

Although the map π depends on n, we suppress this dependency in our notation. Also observe that the map π is a two-to-one mapping: for each $y \in \mathbb{R}/(n+1)\mathbb{Z}$ we have $|\pi^{-1}(y)| = 2$.

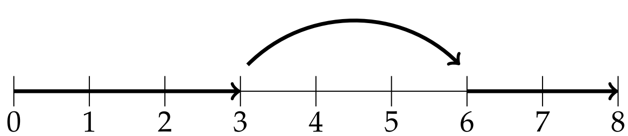
Remark 5.3. For any pair of arcs $\{[u, v-1], [u+n+1, v+n]\}$ there is a unique way to select u to be an element of the set $\{1, 2, \ldots, n+1\}$, that is, $\pi(u) = u$. We may distinguish two cases depending upon whether the arc $[u, v-1]$ is a subset of the arc $[u, n+1]$ or not.

(i) If $[u, v-1] \subseteq [u, n+1]$ then visualize the set $\pi([u, v-1]) = \pi([u+n+1, v+n])$ as the subinterval $[u, v-1]$ of $(0, n+1]$. The direction of both arcs $[u, v-1]$ and $[u+n+1, v+n]$ corresponds to parsing the interval $[u, v-1]$ in increasing order. Adding the associated backward arrow $(v-1, u)$ closes a directed cycle with this directed interval.



As an example, when $n = 7$ then the B-diagonal $\{\{2,5\},\{\overline{2},\overline{5}\}\}$ is represented by the backward arrow $(4,2)$ as drawn above on the interval $(0,8]$.

(ii) If the arc $[u, v-1]$ is not contained in the arc $[u, n+1]$ then $n+1 < v-1 < u+n+1$. The integer $\pi(v-1) = v-1-(n+1)$ is congruent to $v-1$ modulo $(n+1)$ and satisfies $1 \leq \pi(v-1) < u$. The image of the arc $[u, v-1]$ under π, that is, $\pi([u, v-1]) = \pi([u+n+1, v+n])$ is then the subset $(0, \pi(v)-1] \cup [u, n+1]$ of the interval $(0, n+1]$. We may consider $(0, \pi(v)-1] \cup [u, n+1]$ as a "wraparound interval" modulo $n+1$ from u to $\pi(v-1)$. The direction of both pieces corresponds to listing the elements of this "wraparound interval" in increasing order modulo $n+1$. Adding the associated forward arrow $(\pi(v-1), u)$ closes a directed cycle with the directed wraparound interval.



For instance, when $n = 7$ the B-diagonal $\{\{4, \overline{6}\}, \{\overline{4}, 6\}\}$ yields the forward arrow $(3, 6)$.

Proposition 5.4. *The B-diagonal represented by the arrow $(\pi(v_1 - 1), \pi(u_1))$ and the B-diagonal represented by the arrow $(\pi(v_2 - 1), \pi(u_2))$ are noncrossing if and only if the images $\pi([u_1, v_1 - 1])$ and $\pi([u_2, v_2 - 1])$ are disjoint or contain each other.*

Proposition 5.5. *Suppose a pair of B-diagonals is represented by a pair of arrows as defined in Definition 5.1. These B-diagonals cross if and only if one of the following conditions is satisfied:*

(1) Both arrows are backward and they cross.
(2) Both arrows are forward and they do not nest.
(3) One arrow is forward, the other one is backward, and the backward arrow nests or crosses the forward arrow.
(4) The head of one arrow is the tail of the other arrow.

Corollary 5.6. *Noncrossing sets of B-diagonals are represented by subsets of vertices contained in a facet of the Legendre polytope P_n.*

6 The type B associahedron represented as a pulling triangulation

Theorem 6.1. *Let $<$ be any linear order on the vertex set of P_n subject to the following conditions:*

1. $(x_1, y_1) < (x_2, y_2)$ *whenever* $x_1 - y_1 > 0 > x_2 - y_2$.
2. *On the subset of vertices (x, y) satisfying $x < y$, we have $(x_1, y_1) < (x_2, y_2)$ whenever the interval $[x_1, y_1] = \{x_1, x_1 + 1, \ldots, y_1\}$ is contained in the interval $[x_2, y_2]$.*
3. *On the subset of vertices (x, y) satisfying $x > y$, we have $(x_1, y_1) < (x_2, y_2)$ whenever the interval $[y_1, x_1]$ is contained in the interval $[y_2, x_2]$.*

Then the arc representation of Γ_n^B given in Definition 5.1 is a pulling triangulation of the boundary of the Legendre polytope P_n with respect to $<$.

The main idea of the proof is the following. Recall that Γ_n^B is a flag complex and its minimal nonfaces are the pairs of crossing B-diagonals. By Theorem 3.1 the pulling triangulation we defined is also a flag complex. It suffices to show that the minimal nonfaces are in bijection. Equivalently, for any pair of arrows $\{(x_1, y_1), (x_2, y_2)\}$ that form an edge in the pulling triangulation of P_n, these arrows correspond to a pair of noncrossing B-diagonals in Γ_n^B. By Proposition 5.5 this amounts to showing the following: backward arrows cannot cross, forward arrows must nest, and for a pair of arrows of opposite direction the backward arrow cannot cross or nest the forward arrow. The rest of the proof is an easy verification of these statements for the six possible relative positions of a pair of arrows with distinct endpoints.

As a corollary we obtain Simion's polytopal result.

Corollary 6.2 (Simion)**.** *The Simion type B associahedron Γ_n^B is the boundary complex of a simplicial polytope.*

Since the associahedron of type A is the link of a B-diagonal of the form $\{i, \bar{i}\}$, we obtain the following classical result; see the work of Haiman, Lee [18] and Stasheff. For a brief history, see the introduction of [7].

Corollary 6.3. *The associahedron is the boundary complex of a simplicial polytope.*

We end this section by describing the structure of all facets of the Simion's type B associahedron in terms of arrows.

Theorem 6.4. *A set of arrows $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ represents a facet of Simion's type B associahedron Γ_n^B if and only if the following conditions are satisfied:*

1. *There is exactly one integer k satisfying $1 \leq k \leq n+1$ such that $(k-1, k)$ (or $(n+1, 1)$ if $k = 1$) belongs the set S. We call this k the* type *of the facet.*
2. *Backward arrows do not nest any forward arrow, in particular, they cannot nest $(k-1, k)$ if $k > 1$.*
3. *If $k = 1$ then there is no forward arrow in the set S.*
4. *Forward arrows must nest. In particular, if $k > 1$ then for each each forward arrow $(x, y) \in S$ must satisfy $x \leq k-1$ and $y \geq k$. (Forward arrows must nest $(k-1, k)$.)*
5. *No head of an arrow in the set S is also the tail of another arrow in S.*
6. *No two arrows cross.*

7 Triangulating Cho's decomposition

The type A root polytope P_n^+ is the convex hull of the origin and the set of points $\{e_i - e_j : 1 \leq i < j \leq n+1\}$. Cho [9] gave a decomposition of the Legendre polytope P_n into $n+1$ copies of P_n^+ as follows. The symmetric group $\mathfrak{S}_{n+1}$ acts on the Euclidean space $\mathbb{R}^{n+1}$ by permuting the coordinates, that is, the permutation $\sigma \in \mathfrak{S}_{n+1}$ sends the basis vector e_i into $e_{\sigma(i)}$. Hence the permutation σ acts on the Legendre polytope P_n by sending each $e_i - e_j$ into $e_{\sigma(i)} - e_{\sigma(j)}$. Cho's main result [9, Theorem 16] is the following decomposition.

Theorem 7.1 (Cho)**.** *The Legendre polytope P_n has the decomposition*

$$P_n = \bigcup_{k=0}^{n} \zeta^k(P_n^+)$$

where ζ is the cycle $(1, 2, \ldots, n+1)$. Furthermore, for $0 \leq k < r \leq n$ the polytopes $\zeta^k(P_n^+)$ and $\zeta^r(P_n^+)$ have disjoint interiors.

The following theorem implies that each copy $\zeta^k(P_n^+)$ of P_n^+ is the union of simplices of the triangulation given in Definition 5.1, representing the boundary complex Γ_n^B of Simion's type B associahedron.

Theorem 7.2. *Every facet F of the arc representation of Γ_n^B given in Definition 5.1 is contained in $\zeta^{k-1}(P_n^+)$ where k is the unique arrow of the form $(k-1,k)$ in F or $(n+1,1)$ if $k=1$. Equivalently, the facet F is contained in $\zeta^k(P_n^+)$ exactly when it represents a facet of Γ_n^B that contains the diagonal $\{k,\overline{k}\}$.*

8 Concluding Remarks

For complete proofs, we refer the reader to the full length version of this extended abstract [12].

Simion observed algebraically that the number of k-dimensional faces of the type B associahedron is given by the number of *balanced Delannoy paths* between $(0,0)$ and $(2n,0)$ taking k up steps $(1,1)$, k down steps $(1,-1)$, and $n-k$ horizontal steps $(2,0)$. We have found a combinatorial proof by providing a non-recursive bijection between the faces of the type B associahedron and Delannoy paths [12, Section 8].

In a recent paper, Cellini and Marietti [8] used abelian ideals to produce a triangulation for various root polytopes. In the case of type A, their construction yields once again a lexicographic triangulation of each face. Restricting to the positive roots yields Gelfand, Graev and Postnikov's anti-standard tree bases for the type A positive root polytope. Is there an ideal corresponding to the reverse lexicographic triangulation?

The h-vector of Simion's type B associahedron may be computed from the f-vector using elementary operations on binomial coefficients; see [24, Corollary 1].

Lemma 8.1 (Simion)**.** *The h-vector $(h_0, h_1, \ldots, h_n)$ of Simion's type B associahedron Γ_n^B satisfies*

$$h_i = \binom{n}{i}^2 \quad \text{for } 0 \le i \le n.$$

One of the referees pointed out the recent work of Ceballos, Padrol and Sarmiento, in which they recover our pulling triangulation of the bounday of the Legendre polytope [6, Theorem 8.5]. Much earlier Billera, Cushman and Sanders describe an explicit shelling using lattice paths and recover the values h_i as counting those lattice paths having i corners [3, Section 3]. The fact that the number of such lattice paths is given by $\binom{n}{i}^2$ is a classical result of MacMahon from 1915 [19, Vol. I, Article 89, pp. 119-120].

Finally, are there other interesting simplicial polytopes that can be better understood as pulling triangulations of less complicated polytopes?

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