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Noncommutative free cumulants

Matthieu Josuat-Vergès[1], Frédéric Menous[2], Jean-Christophe Novelli[1] and Jean-Yves Thibon[1]

[1]*Laboratoire d'Informatique Gaspard Monge, Université Paris-Est Marne-la-Vallée*
[2]*Département de Mathématiques, Université d'Orsay*

Abstract. The relation between moments and free cumulants in free probability is essentially a compositional inversion. We lift it at the level of the noncommutative Faà di Bruno algebra, and of an operad of Schröder trees. We get a new formula for free cumulants in terms of trees, and we recover an interpretation of the relation in terms of characters due to Ebrahimi-Fard and Patras.

Résumé. La relation entre moments et cumulants libres en probabilités libres est essentiellement une inversion compositionnelle. Nous la relevons au niveau de l'agèbre de Faà di Bruno non-commutative et d'une opérade d'arbres de Schröder. Nous obtenons une formule nouvelle pour les cumulants libres en termes d'arbres, et retrouvons une interprétation de la relation en termes de caractères du à Ebrahimi-Fard et Patras.

Keywords: free cumulants, Hopf algebras, operads, noncrossing partitions

1 Introduction

Recent works on certain functional equations involving reversion of formal power series have revealed that the appropriate setting for their combinatorial understanding involved a series of noncommutative generalizations, ending up as an equation in the group of an operad. Roughly speaking, this amounts to first interpreting the equation in the Faà di Bruno Hopf algebra, lifting it to its noncommutative version, and get series expanded on trees. We refer to [10, 11, 9].

Free probability provides other examples of functional equations with a combinatorial solution. The relation between the moments and the free cumulants of a single random variable is a functional inversion, which can be treated combinatorially by the formalism of [9]. However, the case of several random variables is classically formulated as a triangular system of equations which is solved by Möbius inversion over the lattice of noncrossing partitions [12]. We shall see that this system can be encoded by a single equation in the group of an operad. The solution arises as a sum over reduced plane trees which reduces to Speicher's solution in the scalar case. Also, our functional equation gives back that of Ebrahimi-Fard and Patras [5], which interpret the map from free cumulants to moments as a dendriform exponential sending an infinitesimal character to a character.

2 Free cumulants as symmetric functions

The free cumulants k_n of a probability measure μ on $\mathbb{R}$ are defined (see *e.g.*, [12]) by means of the generating series of its moments m_n

$$M_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = z^{-1} + \sum_{n\geq 1} m_n z^{-n-1} \tag{2.1}$$

as the coefficients of its compositional inverse

$$K_\mu(z) := M_\mu(z)^{\langle -1\rangle} = z^{-1} + \sum_{n\geq 1} k_n z^{n-1}\,. \tag{2.2}$$

We can interpret the coefficients of these formal power series as specializations of symmetric functions. In the present context, we set $m_n = \phi(h_n) = h_n(X)$ (in the notation of [8]). Indeed, the process of functional inversion (Lagrange inversion) admits a simple expression within this formalism (see [8], ex. 24 p. 35). If the symmetric functions h_n^* are defined by the equations

$$u = tH(t) \iff t = uH^*(u) \tag{2.3}$$

where $H(t) := \sum_{n\geq 0} h_n t^n$, $H^*(u) := \sum_{n\geq 0} h_n^* u^n$, then,

$$h_n^*(X) = \frac{1}{n+1}[t^n]E(-t)^{n+1} \tag{2.4}$$

where $E(t)$ is defined by $E(t)H(t) = 1$. This defines an involution $f \mapsto f^*$ of the ring of symmetric functions. Now, if one sets $m_n = h_n(X)$ as above, then $M_\mu(z) = z^{-1}H(z^{-1}) = u$, so that

$$z = K_\mu(u) = \frac{1}{u}E^*(-u) = u^{-1} + \sum_{n\geq 1}(-1)^n e_n^* u^{n-1}\,, \tag{2.5}$$

and finally

$$k_n = (-1)^n e_n^*(X)\,. \tag{2.6}$$

An explicit formula for e_n^* is given in [8, p. 35]. Moreover, $-e_n^*$, twisted by the sign character, becomes the Frobenius characteristic of the action of $\mathfrak{S}_n$ on prime parking functions (see [10]).

A *noncommutative probability space* is a pair (A, ϕ) where A is a unital algebra over $\mathbb{C}$ and ϕ a linear form on A such that $\phi(1) = 1$ (see, *e.g.*, [12]). The *free moments* are the functions m_n defined by

$$m_n[a_1, \ldots, a_n] = \phi(a_1 \cdots a_n). \tag{2.7}$$

The *free cumulants* κ_n are defined by the implicit equations

$$\phi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}_n} \kappa_\pi[a_1, \dots, a_n] \tag{2.8}$$

where NC_n is the set of noncrossing partitions of $[n]$ and

$$k_\pi[a_1, \dots, a_n] = \prod_{B \in \pi} \kappa[B] \text{ and for } B = \{b_1 < \dots < b_p\},\ \kappa[B] = \kappa[b_1, \dots, b_p]. \tag{2.9}$$

By Möbius inversion over the lattice of noncrossing partitions, this yields

$$\kappa_\pi[a_1, \dots, a_n] = \sum_{\sigma \le \pi} \mu(\sigma, \pi) \phi_\sigma[a_1, \dots, a_n]. \tag{2.10}$$

where ϕ_π is defined similarly [12].

3 The noncommutative Faà di Bruno Hopf algebra

The construction of the Faà di Bruno algebra can be done starting from the algebra *Sym* of symmetric functions [6]. Its algebra structure is the same as in *Sym*, but instead of the usual coproduct $\Delta_0 h_n = \sum_{i=0}^n h_i \otimes h_{n-i}$ (with $h_0 = 1$), we have a new one:

$$\Delta_1 h_n = \sum_{i=0}^{n} h_i \otimes h_{n-i}((i+1)X) \quad (h_0 = 1). \tag{3.1}$$

It comes from the interpretation of h_n as coordinates on the group of formal power series $\sum_{n \ge 0} a_n z^{n+1}$ with $(a_0 = 1)$ under composition. In particular, the antipode is essentially given by Lagrange inversion.

This can be repeated literally with the algebra **Sym** of noncommutative symmetric functions. It is a free associative (and noncommutative) graded algebra with one generator S_n in each degree. The coproduct

$$\Delta_1 S_n = \sum_{i=0}^{n} S_i \otimes S_{n-i}((i+1)A) \quad (S_0 = 1) \tag{3.2}$$

remains coassociative, and **Sym** endowed with this coproduct is a Hopf algebra, known as Noncommutative Formal Diffeomorphims [1, 11], or as the noncommutative Faà di Bruno algebra [3].

Let $\mathcal{H}_{\mathrm{ncdif}}$ denote this Hopf algebra, and let γ denote its antipode. The image $h = \gamma(\sigma_1)$ of the formal sum of its generators $\sigma_1 = \sum_{n \ge 0} S_n$ is characterized by the functional equation

$$h^{-1} = \sum_{n \ge 0} S_n h^n. \tag{3.3}$$

The *noncommutative Lagrange series* g is defined by the functional equation

$$g = \sum_{n\geq 0} S_n g^n. \tag{3.4}$$

Recall that for $f \in \mathbf{Sym}$, $f(-A)$ is the image of f by the automorphism $S_n \mapsto (-1)^n \Lambda_n$. It is proved in [11] that $h(A) = g(-A)$, and that

$$g_n = \sum_{\pi \in \mathrm{NDPF}_n} S^{\mathrm{ev}(\pi)} \tag{3.5}$$

where NDPF_n is the set of nondecreasing parking functions of length n, *i.e.*, nondecreasing words over the positive integers such that $\pi_i \leq i$, and $\mathrm{ev}(\pi) = (|\pi|_i)_{i=1..n}$. There is a simple bijection between NDPF_n and NC_n, and g_n can as well be written as a sum over noncrossing partitions.

4 Noncommutative free cumulants

In the case of a single random variable, the free cumulants κ_n are the images of the noncommutative symmetric functions K_n defined by the functional equation

$$\sigma_1 = \sum_{n\geq 0} K_n \sigma_1^n \tag{4.1}$$

by the character χ of $\mathcal{H}_{\mathrm{ncdif}}$ such that $\chi(S_n) = m_n = \phi(a^n)$, where a is some element of a noncommutative probability space (A, ϕ).

This equation is formally similar to (3.4), so that we can write down immediately an expression of S_n in terms of the basis $K^I := K_{i_1} \cdots K_{i_r}$, by replacing S^I by K^I in the expression of g_n given in (3.5). These expressions are sums over Catalan sets

$$S_n = \sum_{\pi \in \mathrm{NDPF}_n} K^{\mathrm{ev}(\pi)} \tag{4.2}$$

in the guise of nondecreasing parking functions, instead of noncrossing partitions.

Solving recursively for K_n, it appears that $(-1)^{n-1}K_n$ is given by the following rule. Let Ω be the linear operator defined in [10] by $\Omega S^{i_1,\ldots,ir} = S^{i_1+1,i_2,\ldots,i_r}$ and $\Omega(1) = S_1$, then $K_n = -(\Omega g_{n-1})(-A)$. It is proved in [10] that $g^{-1} = 1 - \Omega g$, and we have indeed

Theorem 4.1.

$$K := 1 + \sum_{n\geq 1} K_n = g^{-1}(-A). \tag{4.3}$$

Proof. Essentially, it follows from the properties of the antipode of $\mathcal{H}_{\mathrm{ncdif}}$. $\square$

5 Free cumulants in the Schröder operad

To obtain a combinatorial expression for K_n, one can work in the group of the Schröder operad as in [9, Section 10.2]. This will cover the case of several random variables. Let PT_n be the set of reduced plane trees, *i.e.*, plane trees for which any internal node has at least two descendants. The *Schröder operad* [9] is the $\mathbb{C}$-vector space

$$\mathcal{S} = \bigoplus_{n\geq 1} \mathcal{S}_n, \quad \text{where } \mathcal{S}_n = \mathbb{C}\,\mathrm{PT}_n \tag{5.1}$$

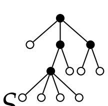
endowed with the composition operations

$$\mathcal{S}_n \otimes \mathcal{S}_{k_1} \otimes \ldots \otimes \mathcal{S}_{k_n} \longrightarrow \mathcal{S}_{k_1+\ldots+k_n} \ (n \geqslant 1, k_i \geqslant 1) \tag{5.2}$$

which map the tensor product of trees $t_0 \otimes t_1 \otimes \ldots \otimes t_n$ to the tree $t_0 \circ (t_1, \ldots, t_n)$ obtained by replacing the leaves of t_0, from left to right, by the trees $t_1, \ldots, t_n$.

The number of leaves of a tree t will be called its degree $d(t)$, and we define the weight $\mathrm{wt}(t)$ of a tree as its degree minus 1.

We can represent trees by noncommutative monomials in indeterminates S_n $(n \geq 0)$, by interpreting a node of arity k as a k-ary operator denoted by S_{k-1}, and writing the resulting expression in Polish notation. For example,

S 

$$= S_2S_0S_1S_3S_0S_0S_0S_0S_0S_1S_0S_0 = S^{201300000100} \tag{5.3}$$

is of degree 8 and weight $7 = 2+1+3+1$. The sum of the components of I is therefore the weight of the associated tree. We shall indifferently use the notations S^I or S^t. Let $\hat{\mathcal{S}}$ be the completion of the vector space $\mathcal{S}$ with respect to the weight $\mathrm{wt}(t) = d(t) - 1$. The group of the operad $\mathcal{S}$ is defined as [2]

$$G_{\mathcal{S}} = \left\{ \circ + \sum_{n\geqslant 2} p_n, \quad p_n \in \mathcal{S}_n \right\} \subset \hat{\mathcal{S}} \tag{5.4}$$

endowed with the composition product

$$p \circ q = q + \sum_{n\geqslant 2} p_n \circ \big(\underbrace{q, \ldots, q}_{n}\big) \in G_{\mathcal{S}} \tag{5.5}$$

for $p = \circ + \sum_{n\geqslant 2} p_n$ and $q \in G_{\mathcal{S}}$. Elements of $G_{\mathcal{S}}$ can be described by their coordinates

$$p = \sum_{t\in\mathrm{PT}} p_t t \quad \text{and} \quad q = \sum_{t\in\mathrm{PT}} q_t t \quad (\mathrm{PT} = \cup_{n\geq 1}\mathrm{PT}_n) \tag{5.6}$$

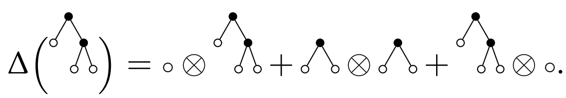
(with $q_\circ = p_\circ = 1$) so that the coordinates of $r = p \circ q$ are given by

$$r_t = \sum_{t = t_0 \circ (t_1, \ldots, t_n)} p_{t_0} g_{t_1} \cdots g_{t_n}. \tag{5.7}$$

This allows to consider the group $G_{\mathcal{S}}$ as the group of characters of a graded Hopf algebra $H_{\mathcal{S}}$. It is the noncommutative polynomial algebra over reduced plane trees PT (with unit $\circ$), endowed with the coproduct given by *admissible cuts* (see [9]): an admissible cut of a tree T is a possibly empty subset of internal vertices $c = \{i_1, \ldots, i_k\}$ such that along any path from the root to a leaf, there is at most one internal vertex in c. For any such cut, one defines

- $P^c(T) = T_{i_1} \ldots T_{i_k}$ as the product of the subtrees of T having their root in c, ordered as in T from the top and from left to right.
- $R^c(T)$ as the trunk which remains after removing these trees.

The coproduct of $H_{\mathcal{S}}$ is then $\Delta(T) = \sum_c R^c(T) \otimes P^c(T)$ so that $H_{\mathcal{S}}$ is a graded Hopf algebra. For instance



(5.8)

The bijection between $G_{\mathcal{S}}$ and the group of characters on $H_{\mathcal{S}}$ is obvious: since $H_{\mathcal{S}}$ is a polynomial algebra, a character χ is entirely determined by its restriction to trees of positive weight, in other words, by its *residue* in the sense of [4] (which can be considered as an infinitesimal character $\mathrm{Res}(\chi)$) and the values of this residue are given by the coordinates in $G_{\mathcal{S}}$.

Consider the series of corollas

$$f_c = S_0 + \sum_{n \geqslant 1} S_n S_0^{n+1}. \tag{5.9}$$

The inverse of f_c in $G_{\mathcal{S}}$ is, in terms of trees,

$$g_c = \sum_{t \in \mathrm{PT}} (-1)^{i(t)} S^t \tag{5.10}$$

where $i(t)$ denotes the number of internal nodes of t. Indeed, denoting by $\wedge(t_1, \ldots, t_n)$ the tree whose subtrees of the root are $t_1, \ldots, t_n$,

$$g_c = S_0 + \sum_{n \geqslant 1} \sum_{\substack{t = \wedge(t_1 \cdot \ldots \cdot t_{n+1}) \\ t_i \in \mathrm{PT}}} (-1)^{i(t)} S^{\wedge(t_1 \cdot \ldots \cdot t_{n+1})} \tag{5.11}$$

$$= S_0 + \sum_{n \geqslant 1} \sum_{\substack{t = \wedge(t_1 \cdot \ldots \cdot t_{n+1}) \\ t_i \in \mathrm{PT}}} (-1)^{1 + i(t_1) + \ldots + i(t_{n+1})} S_n S^{t_1} \cdots S^{t_{n+1}} = S_0 - \sum_{n \geqslant 1} S_n g_c^{n+1} \tag{5.12}$$

so that

$$S_0 = g_c + \sum_{n\geqslant 1} S_n g_c^{n+1} = f_c \circ g_c. \tag{5.13}$$

Definition 5.1. *A Schröder tree is prime if its rightmost subtree is a leaf. We denote by* PST_n *the set of prime Schröder trees of weight* n.

Prime Schröder trees are enumerated by the large Schröder numbers.

The series g_c, introduced in [9, Eq. (158)], projects onto the antipode $g(-A)$ of $\mathcal{H}_{\mathrm{ncdif}}$ by the map $S_0 \mapsto 1$. Imitating the interpretation of the Lagrange series given in [9, Eq. (164)] and exchanging the roles of g_n, K_n and S_n as above, we obtain the following result.

Theorem 5.2. *Define* η *and* κ *by*

$$g_c = \eta \cdot S_0, \qquad \kappa := \eta^{-1} \cdot S_0 \tag{5.14}$$

where the exponent -1 *denotes here the multiplicative inverse. Then, the image of* κ *by the algebra morphism* $S_0 \mapsto 1$ *is the series* K *of* **Sym**. *In terms of trees,*

$$\kappa_n = \sum_{t\in\mathrm{PST}_n} (-1)^{i(t)-1} S^t. \tag{5.15}$$

Proof. For $f \in G_S$, write $f = \tilde{f} S_0$, and for $f, g \in G_{\mathcal{S}}$, define

$$f \dashv g = (\tilde{f} \circ g) S_0 = S_0 + ((\tilde{f} - 1) \circ g) S_0 \tag{5.16}$$

This is a partial composition: if $g = \circ + \sum_{n\geqslant 2} g_n$ and $f = \circ + \sum_{n\geqslant 2} f_n$, then

$$f \dashv g = \circ + \sum_{n\geq 2} f_n(\underbrace{g, \dots, g}_{n-1}, \circ). \tag{5.17}$$

From (5.14) and $g_c^{-1} = f_c$, we have $\tilde{\kappa} g_c = S_0$ and $(\tilde{\kappa} \circ f_c) S_0 = f_c$, so that

$$f_c = \kappa \dashv f_c \tag{5.18}$$

which implies (5.15). Indeed, plugging f_c in this expression, we get an alternating sum of trees obtained by grafting corollas to leaves of prime Schröder trees t except to the rightmost one. The sign of the resulting tree t' is $(-1)^{i(t)-1} = (-1)^{i(t')-k-1}$ if k is the number of grafted corollas. Hence, each t' which is not a corolla has coefficient $(1-1)^n$ where n is the number of its internal nodes whose all descendants are leaves. □

Our formula is multiplicity-free, and the number of terms is given by the large Schröder numbers. This expression is finer than Speicher's formula (2.10), for example if $n = 3$ the term $2\phi(a_1)\phi(a_2)\phi(a_3)$ is now separated into two binary trees.

Equation (5.18) seems to have the same structure as that of [4], which involves dendriform (or shuffle) and codendriform (or unshuffle) algebras.

In terms of the Hopf algebra structure, instead of the convolution of characters, which on trees T (of positive weight) reads as

$$(f * g)(T) = \pi \circ (f \otimes g) \circ \Delta(T) = \sum_{c=\{i_1,\dots,i_k\}} f(R^c(T))g(P^c(T)) \tag{5.19}$$

we get

$$(f \dashv g)(T) = \pi \circ (f \otimes g) \circ \Delta^+_{\prec}(T) = \sum_{c=\{i_1,\dots,i_k\}}^{\prec} f(R^c(T))g(P^c(T)) \tag{5.20}$$

where the sum is restricted to admissible cuts such that the rightmost leaf of T remains in $R^c(T)$ or, equivalently, such that the rightmost subtree in $P^c(T)$ does not contain the rightmost leaf of T.

Extending to Schröder trees the constructions of [4, 5], we can now state:

Theorem 5.3. *For any tree T of positive weight, let*

$$\Delta^+_{\prec}(T) = \sum_{c=\{i_1,\dots,i_k\}}^{\prec} R^c(T) \otimes P^c(T) \text{ and } \Delta^+_{\succ}(T) = \Delta(T) - \Delta^+_{\prec}(T). \tag{5.21}$$

These maps can be extended to the augmentation ideal $H^+_{\mathcal{S}}$ of $H_{\mathcal{S}}$ by the rule:

$$\Delta^+_{\prec}(T_1T_2\dots T_s) = \Delta^+_{\prec}(T_1).\Delta(T_2\dots T_s) \tag{5.22}$$

$$\Delta^+_{\succ}(T_1T_2\dots T_s) = \Delta^+_{\succ}(T_1).\Delta(T_2\dots T_s) \tag{5.23}$$

so that $H_{\mathcal{S}}$ is a codendriform bialgebra.

The last statement means that on $H^+_{\mathcal{S}}$, if

$$\Delta(a) = \bar{\Delta}(a) + 1 \otimes a + a \otimes 1 \tag{5.24}$$

$$\Delta^+_{\prec}(a) = \Delta_{\prec}(a) + a \otimes 1 \tag{5.25}$$

$$\Delta^+_{\succ}(a) = \Delta_{\succ}(a) + 1 \otimes a \tag{5.26}$$

then

$$(\Delta_{\prec} \otimes I) \circ \Delta_{\prec} = (I \otimes \bar{\Delta}) \circ \Delta_{\prec} \tag{5.27}$$

$$(\Delta_{\succ} \otimes I) \circ \Delta_{\prec} = (I \otimes \Delta_{\prec}) \circ \Delta_{\succ} \tag{5.28}$$

$$(\bar{\Delta} \otimes I) \circ \Delta_{\succ} = (I \otimes \Delta_{\succ}) \circ \Delta_{\succ} \tag{5.29}$$

As in [4], the space $\mathrm{Lin}(H_{\mathcal{S}}, k)$, which is a $\mathbb{K}$-algebra for the convolution product

$$(f * g) = \pi \circ (f \otimes g) \circ \Delta \tag{5.30}$$

is also a dendrifrom algebra for left and right half-convolutions

$$(f \prec g) = \pi \circ (f \otimes g) \circ \Delta_{\prec} \tag{5.31}$$
$$(f \succ g) = \pi \circ (f \otimes g) \circ \Delta_{\succ}. \tag{5.32}$$

In terms of characters in $G_{\mathcal{S}}$, the operation $\dashv$ (see (5.20)) coincides on trees with $\prec$, and the relation between g_c and κ in Theorem 5.2 translates now into the character equation

$$f_c = \epsilon + (\mathrm{Res}(\kappa)) \prec f_c \tag{5.33}$$

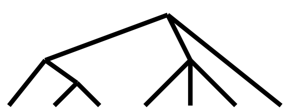
where ϵ is the unit of the group, corresponding to S_0.

As we shall see, this equation implies [5, Th. 13], and thus also Speicher's formula for the free cumulants. We shall first explain in the forthcoming section how to derive the latter from (5.15) by a direct combinatorial argument.

6 Speicher's formula

Equation (5.15) is a formula for the free cumulants in terms of moments, involving prime Schröder trees instead of noncrossing partitions as in Equation (2.10) (which was originally the definition of free cumulants). We show that our formula implies Speicher's. Let us first rewrite (5.15).

We label the sectors of a prime Schröder tree from left to right by $1, 2, \ldots, n$:



1 2 3 4 5 6

and denote $v\measuredangle i$ if the internal vertex v has a clear view to the ith sector between the ith and $(i+1)$st leaves. Then the formula for κ_n is:

$$\kappa_n[a_1, \ldots, a_n] = \sum_{t \in \mathrm{PST}_n} (-1)^{i(t)-1} \prod_{v \in \mathrm{int}(T)} \phi\left(\prod_{v\measuredangle i} a_i\right) \tag{6.1}$$

Note that if we gather the sectors i and j viewed from a same $v \in \mathrm{int}(t)$, we get a noncrossing partition $\eta(t)$. For example, the above tree gives $1|2|36|45$, and this corresponds to the term $\phi(a_1)\phi(a_2)\phi(a_3a_6)\phi(a_4a_5)$.

Definition 6.1. *A noncrossing arrangement of binary trees is a set of binary trees, whose leaves are labeled with integers from* 1 *to* n*, in such a way that the canonical drawing of the trees does not create any crossing. Let* $\mathcal{A}_n$ *denote the set of such objects.*

Figure 1: The bijection of Proposition 6.2.

Proposition 6.2. *There is a bijection between* PST_n *and* $\mathcal{A}_n$*, such that the image of* $t \in \mathrm{PST}_n$ *is obtained by: removing each middle edge (i.e. an edge which is not leftmost or rightmost among all edges below some internal vertex), then removing the root and all edges below.*

See Figure 1 for an example. If we forget the tree structure of the arrangement and only keep blocks of elements connected to each other, we get a noncrossing partition. The example in Figure 1 gives $1456|23|78A|9$. Going through the bijection of the previous proposition, this gives another noncrossing partition associated to $t \in \mathrm{PST}_n$, denoted by $\nu(t)$.

Lemma 6.3. *The noncrossing partition* $\nu(t)$ *is the Kreweras complement of* $\eta(t)$*.*

We refer to [7] for the definition of Kreweras complement, that we denote $\pi \mapsto \pi^c$. Rewriting (6.1), we get:

$$\kappa_n[a_1, \dots, a_n] = \sum_{t \in \mathrm{PST}_n} (-1)^{i(t)-1} \phi_{\nu(t)}[a_1, \dots, a_n]$$

In terms of noncrossing partitions and using the previous lemma, we get:

$$\kappa_n[a_1, \dots, a_n] = \sum_{\pi \in \mathrm{NC}_n} (-1)^{n-\#\pi} \#\{t \in \mathrm{PST}_n : \eta(t) = \pi\} \phi_{\pi^c}[a_1, \dots, a_n].$$

The number $\#\{t \in \mathrm{PST}_n : \eta(t) = \pi\}$ can be obtained as the number of noncrossing arrangement of binary trees projected to π, so it is a product of Catalan numbers:

$$\#\{t \in \mathrm{PST}_n : \eta(t) = \pi\} = \prod_{B \in \pi} C_{\#B-1}.$$

But this number is also the value of the Möbius function $\mu(\hat{0}, \pi)$ (see [7]), so

$$\kappa_n[a_1, \dots, a_n] = \sum_{\pi \in \mathrm{NC}_n} (-1)^{n-\#\pi} \mu(\hat{0}, \pi) \phi_{\pi^c}[a_1, \dots, a_n].$$

We also have $\mu(\hat{0}, \pi) = \mu(\pi^c, \hat{1})$ by self duality properties of NC_n. Then we recover Speicher's formula by replacing π^c with π.

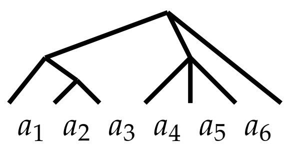
7 The Hopf algebra of decorated Schröder trees

Let A be any set (decorations), and $T(A) = \mathbb{K} \otimes (\otimes_{n\geq 1} T_n(A))$ the free associative $\mathbb{K}$-algebra over A, regarded as the tensor algebra of the linear span of A.

Using the grading of $H_{\mathcal{S}}$, we can define a decorated version of the algebra $H_{\mathcal{S}}$

$$H_{\mathcal{S}}(A) = \mathbb{K} \oplus \bigoplus_{n\geq 1} (H_{\mathcal{S},n} \otimes T_n(A)). \tag{7.1}$$

This space has an obvious algebra structure, and it is also easy to extend the Hopf algebra structure of $H_{\mathcal{S}}$. Consider a tree $T \in H_{\mathcal{S},n}$ and $w = a_1 \ldots a_n$ in $T_n(A)$. Since T has $n+1$ leaves, one can label its sectors from left to right with $a_1, \ldots, a_n$ and identify $T \otimes w$ with this decorated tree. For instance



(7.2)

In an admissible cut c for such a tree, $P^c(T)$ obviously inherits the letters a_i associated with the subtrees in $P^c(T)$, and $R^c(T)$ keeps the letters which can be viewed from the internal vertices of T that are still in $R^c(T)$. It is clear that $H_{\mathcal{S}}(A)$ is a Hopf algebra, and a straightforward adaptation of the proof of Theorem 5.3 shows that

Theorem 7.1. *$H_{\mathcal{S}}(A)$ is a codendriform bialgebra.*

The decorated analog of Theorem 5.2 reads on characters

Theorem 7.2. *Let ϕ be a linear form on $T(A)$. Extend it to a map $\phi: H_{\mathcal{S}}(A) \to \mathbb{C}$ sending the decorated corollas to $\phi(w)$ where w is the decorating word and the other trees to 0 (regarded as an infinitesimal character of $\mathcal{H}_{\mathcal{S}}(A)$), and let Φ be its extension to a character of $\mathcal{H}_{\mathcal{S}}(A)$. Then,*

$$\Phi = \epsilon + \kappa \prec \Phi \tag{7.3}$$

where κ is the infinitesimal character on $H_{\mathcal{S}}(A)$ defined by

$$\kappa(T \otimes a_1 \ldots a_n) = \begin{cases} (-1)^{i(t)-1} \prod_{v \in \mathrm{int}(T)} \phi\left(\prod_{v \measuredangle i} a_i\right) & \textit{if } T \in \mathrm{PST} \\ 0 & \textit{otherwise} \end{cases} \tag{7.4}$$

where $v \measuredangle i$ means that the internal vertex v has a clear view to the ith sector between the ith and $(i+1)$th leaves.

In [4] and [5], free cumulants appear as the solution of a dendriform equation for characters of $T(T_{\geq 1}(A))$ (the double tensor algebra).

Theorem 7.3. *Let ι be the algebra morphism $T(T_{\geq 1}(A)) \to H_{\mathcal{S}}(A)$ sending a word $w = a_1 \cdots a_n$ to the sum of all trees with n sectors decorated from left to right by $a_1, \ldots, a_n$. Then,*
(i) ι is a coalgebra morphism;
(ii) ι is a codendriform morphism: $(\iota \otimes \iota) \circ \Delta_{\prec}(w) = \Delta_{\prec} \circ \iota(w)$.

This result implies the formula for free cumulants, as ι send the maps Φ and κ of Theorem 7.2 to the free moments $\tilde{\Phi} = \phi \circ \iota$ and cumulants $\tilde{\kappa} = \kappa \circ \iota$ of [4].

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References

[1] C. Brouder, A. Frabetti, and C. Krattenthaler. "Non-commutative Hopf algebra of formal diffeomorphisms". *Adv. Math.* **200** (2006), pp. 479–524. DOI.

[2] F. Chapoton. "Rooted trees and an exponential-like series". 2002. arXiv:math/0209104.

[3] K. Ebrahimi-Fard, A. Lundervold, and D. Manchon. "Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras". *Internat. J. Alg. Comput.* **24** (2014), pp. 671–705. DOI.

[4] K. Ebrahimi-Fard and F. Patras. "Cumulants, free cumulants and half-shuffles". *Proc. Royal Soc. London A* **471** (2015), p. 20140843.

[5] K. Ebrahimi-Fard and F. Patras. "The splitting process in free probability theory". *Internat. Math. Res. Notices* **2016** (2016), pp. 2647–2676. DOI.

[6] S. A. Joni and G.-C. Rota. "Coalgebras and bialgebras in combinatorics". *Contemp. Math.* **6** (1982), pp. 1–47. DOI.

[7] G. Kreweras. "Sur les partitions non croisées d'un cycle". *Discrete Math.* **1** (1972), pp. 333–350. DOI.

[8] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 1995.

[9] F. Menous, J.-C. Novelli, and J.-Y. Thibon. "Combinatorics of Poincaré's and Schröder's equations". 2015. arXiv:1506.08107.

[10] J.-C. Novelli and J.-Y. Thibon. "Hopf algebras and dendriform structures arising from parking functions". *Fund. Math.* **193** (2007), pp. 189–241. DOI.

[11] J.-C. Novelli and J.-Y. Thibon. "Noncommutative symmetric functions and Lagrange inversion". *Adv. Appl. Math.* **40** (2008), pp. 8–35. DOI.

[12] R. Speicher. "Multiplicative functions on the lattice of non-crossing partitions and free convolution". *Math. Ann.* **298** (1994), pp. 141–159. DOI.