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Toric arrangements associated to graphs

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Abstract. We study certain toric arrangements associated to graphs. The arrangement depends on the choice of an integral lattice: we focus on the case of the (co)root lattice of type A, but also comment on the (simpler) case of the (co)weight lattice of type A. We obtain a combinatorial description for the intersection poset and derive several results on the characteristic polynomial and the arithmetic Tutte polynomial of the toric arrangement. The former counts proper colorings that satisfy an additional divisibility condition. By employing the Voronoi cell of the lattice, we show that the chambers of certain toric arrangements may be seen as equivalence classes for a canonical equivalence relation on the set of chambers of the corresponding linear arrangement. We study this relation in the graphic case.

Keywords: Toric arrangement, root system, graphic arrangement, Tutte polynomial, Voronoi cell, acyclic orientation

1 Introduction

We study arrangements of hypertori in a torus. Such an arrangement is associated to a set of integral vectors with respect to a lattice. We set up the relevant notions in Section 2. In Section 3 we discuss certain toric arrangements associated to a simple graph. The (co)root and (co)weight lattices of type A furnish the other necessary ingredient. The latter case exhibits a simpler behavior. We focus on the former for the most part. Section 4 deals with the intersection poset. Our first result provides a combinatorial description for this poset. These are followed by results on the characteristic and arithmetic Tutte polynomials in Section 5. They include an expression for the former in terms of certain proper colorings, the introduction of the d-divisible Tutte polynomial, an expression for this polynomial in terms of activities, and a formula relating it to the arithmetic Tutte polynomial. In Section 6 we introduce a canonical equivalence relation on the set of chambers of a linear arrangement. This relation depends on the lattice and is defined when the Voronoi cell is *confined* by the affine arrangement. We conclude by analyzing this relation in the case of graphic arrangements in Section 7.

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2 Toric arrangements

Let V be a real vector space of finite dimension d. Fix an inner product $\langle \cdot, \cdot \rangle$ in V. For each $\alpha \neq 0$ in V, and each $k \in \mathbb{R}$, let

$$H_\alpha := \{x \in V \mid \langle \alpha, x \rangle = 0\} \quad \text{and} \quad H_{\alpha,k} := \{x \in V \mid \langle \alpha, x \rangle = k\}.$$

Then H_α is a *linear hyperplane* in V, while $H_{\alpha,k}$ is an *affine hyperplane* in V. View V as an abelian group under addition. Each linear hyperplane H_α is then a subgroup, and each affine hyperplane $H_{\alpha,k}$ is a coset for this subgroup.

A *lattice* L in V is a subgroup of V generated by a basis of V. It is free abelian of rank d. The associated d-dimensional *torus* is the quotient group $T := V/L$. A *linear hypertorus* is a subgroup of V/L that is isomorphic to a $(d-1)$-dimensional torus, and an *affine hypertorus* is a coset in V/L of a linear hypertorus.

The *dual lattice* of L is $\widehat{L} := \{\alpha \in V \mid \langle \alpha, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in L\}$. It is a lattice in V.

A linear hyperplane H in V is *L-integral* if there is $\alpha \in \widehat{L}$ such that $H = H_\alpha$. Given a hyperplane H in V, or more generally a subset H of V, let $\overline{H}$ denote the image of H in V/L under the canonical projection. If H is an L-integral linear hyperplane in V, then $\overline{H}$ is a linear hypertorus in V/L. It follows that for any coset $H + x$ of H, its image $\overline{H} + \overline{x}$ is an affine hypertorus in V/L.

A vector α is *L-integral* if it belongs to $\widehat{L}$. Let Φ be a finite set of nonzero L-integral vectors. To this data we associate two hyperplane arrangements in V, one linear and the other affine:

$$\mathcal{A}(\Phi) := \{H_\alpha \mid \alpha \in \Phi\} \quad \text{and} \quad \widetilde{\mathcal{A}}(\Phi) := \{H_{\alpha,k} \mid \alpha \in \Phi,\, k \in \mathbb{Z}\}.$$

The associated *toric* arrangement in V/L is

$$\overline{\mathcal{A}}(\Phi, L) := \{\overline{H}_{\alpha,k} \mid \alpha \in \Phi,\, k \in \mathbb{Z}\}.$$

It consists of affine hypertori in V/L. Integrality guarantees that $\overline{\mathcal{A}}(\Phi, L)$ is finite. If Φ generates V over $\mathbb{R}$, the arrangement $\mathcal{A}(\Phi)$ is *essential*. In this case, the minimal flats of each arrangement are zero-dimensional.

Example 2.1. Let $V = \mathbb{R}^d$ with the standard inner product and let $L = \mathbb{Z}^d$. Any set of nonzero vectors $\Phi \subseteq \mathbb{Z}^d = \widehat{L}$ yields a toric arrangement in the *standard torus* V/L. △

Example 2.2. Let Φ be a crystallographic root system in V [11, Section 1.2]. The root lattice is $\mathbb{Z}\Phi$ and its dual $L = \widehat{\mathbb{Z}\Phi}$ is the coweight lattice of Φ. Then $\Phi \subseteq \widehat{L}$ and we obtain a toric arrangement $\overline{\mathcal{A}}(\Phi, L)$ in V/L. △

Example 2.3. Let Φ be as before, $\Phi^\vee$ the set of coroots, and $L = \mathbb{Z}\Phi^\vee$, the coroot lattice of Φ. Then $\widehat{L}$ is the weight lattice of Φ and the crystallographic assumption states that $\Phi \subseteq \widehat{L}$. Then $\overline{\mathcal{A}}(\Phi, L)$ is a toric arrangement in the *Steinberg torus* V/L of Φ. It is studied in [1, 7]. △

3 Toric arrangements associated to a graph

Let $\{e_i\}_{1\leq i\leq n}$ be the standard basis of $\mathbb{R}^n$. For $n \geq 2$, consider

$$A_{n-1} := \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \quad \text{and} \quad V_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}.$$

The former is a root system (of type A) and the latter is the subspace of $\mathbb{R}^n$ it spans. $\mathbb{R}^n$ is endowed with the standard inner product and V_n with the induced one. For root systems of type A, there is no distinction between roots and coroots, or between weights and coweights.

Given a simple graph G with with vertex set $V(G) = [n]$ and edge set $E(G)$, consider the subset Φ_G of A_{n-1} given by

$$\Phi_G := \{e_i - e_j \mid \{i, j\} \in E(G)\}.$$

There are three toric arrangements naturally associated to G, given by different choices for the space V and the lattice L. (In all three cases, $\Phi_G \subseteq \widehat{L}$).

(i) $V = \mathbb{R}^n$ and $L = \mathbb{Z}^n$. Then $\overline{\mathcal{A}}(\Phi_G, L)$ sits in the standard torus. This arrangement is studied in [6, 8, 14].

(ii) $V = V_n$ and $L = \widehat{\mathbb{Z}A_{n-1}} = \mathbb{Z}\{\frac{1}{n}(e_1 + \dots + e_n) - e_i \mid 1 \leq i \leq n\}$ the (co)weight lattice of A_{n-1}. The arrangement $\overline{\mathcal{A}}(\Phi_G, L)$ turns out to be the essentialization of the arrangement in (i). With the exception of a few remarks (notably in Theorem 6), we do not pursue its study in this note.

(iii) $V = V_n$ and $L = \mathbb{Z}A_{n-1}$ the (co)root lattice of A_{n-1}. The arrangement $\overline{\mathcal{A}}(\Phi_G, L)$ sits in the Steinberg torus of A_{n-1}. This note focuses on combinatorial aspects of this arrangement and the more general family of arrangements introduced next.

A *weighted graph* $\mathscr{G} = (G, \mathsf{wt})$ consists of a simple graph G with $V(G) = [n]$ and a *weight function*

$$\mathsf{wt} : V(G) \to \mathbb{Z}_{>0},$$

which associates a positive integer to each vertex of G. (These differ from the graphs in [3, Section 9], which carry weighted edges.) Given such $\mathscr{G}$, let

$$\begin{aligned} V_{\mathscr{G}} &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \mathsf{wt}(1)x_1 + \dots \mathsf{wt}(n)x_n = 0\}, \\ \langle x, y\rangle_{\mathscr{G}} &:= \mathsf{wt}(1)x_1y_1 + \dots + \mathsf{wt}(n)x_ny_n, \\ \Phi_{\mathscr{G}} &:= \left\{ \frac{e_i}{\mathsf{wt}(i)} - \frac{e_j}{\mathsf{wt}(j)} \;\middle|\; \{i, j\} \in E(\mathscr{G}) \right\}, \\ L_{\mathscr{G}} &:= \mathbb{Z}\left\{ \frac{\mathsf{wt}(j)e_i - \mathsf{wt}(i)e_j}{\gcd(\mathsf{wt}(i), \mathsf{wt}(j))} \;\middle|\; 1 \leq i \neq j \leq n \right\}. \end{aligned}$$

The resulting toric arrangement $\overline{\mathcal{A}}(\Phi_{\mathscr{G}}, L_{\mathscr{G}})$ is studied in Section 5. When the weight of every vertex is 1, this arrangement reduces to the one in (iii) above.

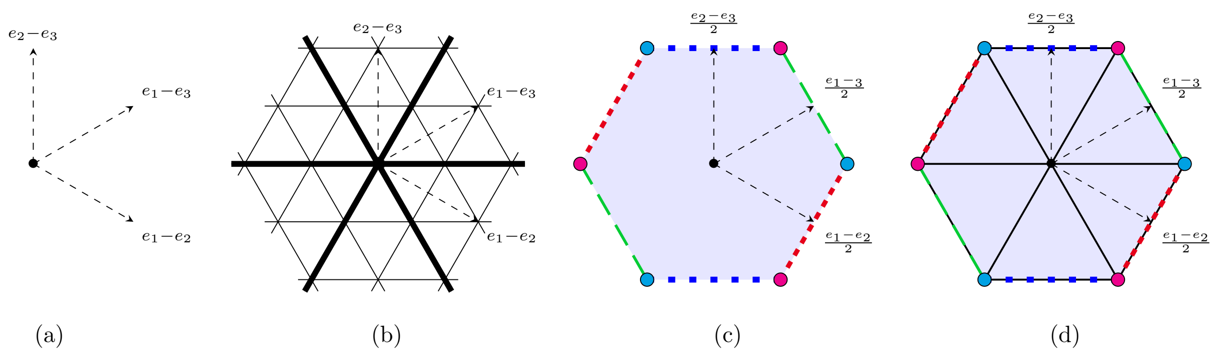

Figure 1: (a) The positive roots of A_2. (b) The affine arrangement of the complete graph K_3, with the linear arrangement in boldface. (c) The Steinberg torus of A_2. Note the identifications along the boundary. (d) The toric arrangement consists of 3 circles.

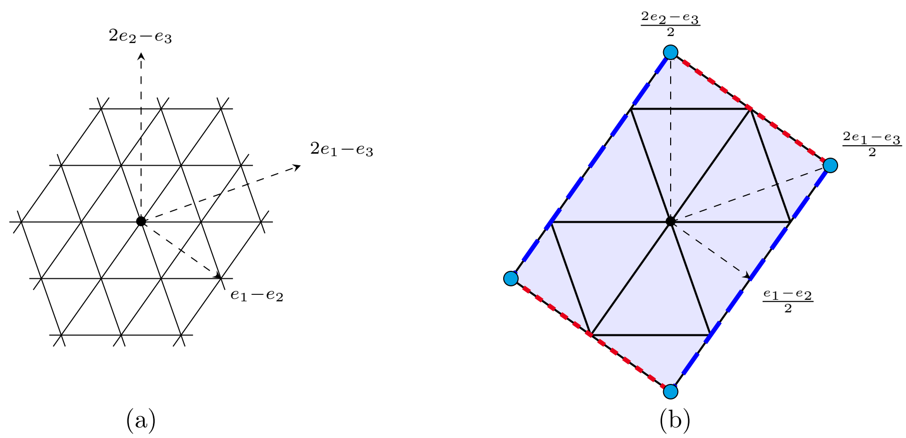

Figure 2: (a) The affine arrangement of the complete graph K_3 with weights $\mathrm{wt}(1) = \mathrm{wt}(2) = 1$, $\mathrm{wt}(3) = 2$. (b) The corresponding toric arrangement.

4 Intersection poset of toric graphic arrangements

Fix L and Φ as in Section 2. An *affine flat* is a nonempty intersection of affine hyperplanes in $\widetilde{\mathcal{A}}(\Phi, L)$. A *toric flat* is the image of an affine flat in the torus V/L. Equivalently, a toric flat is a connected component in a nonempty intersection of hypertori in $\overline{\mathcal{A}}(\Phi, L)$. The intersection of two toric flats (in particular, two hypertori) is in general not a toric flat, but instead a disjoint union of toric flats.

The *intersection poset* $\overline{\Pi}(\Phi, L)$ is the set of all toric flats of $\overline{\mathcal{A}}(\Phi, L)$, ordered by inclusion. The poset $\overline{\Pi}(\Phi, L)$ is graded. When the arrangement is essential, the rank of a flat is its topological dimension. There is a maximum element (the ambient torus $T := V/L$) and several minimal elements (points when the arrangement is essential).

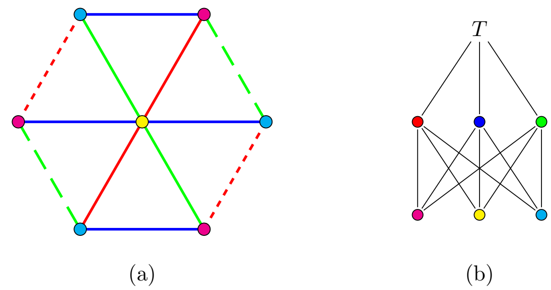

Figure 3: (a) The arrangement $\overline{\mathcal{A}}(\Phi_{K_3}, \mathbb{Z}A_2)$, with flats distinguished by color. Any two one-dimensional flats intersect in the same three points. (b) The intersection poset.

Let G be a simple graph. Consider the arrangement $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}A_{n-1})$ as in Section 3, item (iii). We turn to a combinatorial description of its intersection poset.

For any partition $\pi = \{B_1, \dots, B_k\}$ of $[n]$, let

$$\gcd(\pi) := \gcd\{|B_i| \mid 1 \leq i \leq k\}.$$

We order the partitions of $[n]$ by refinement: $\pi \leq \pi'$ if each block of π' is contained in a block of π. In this case, $\gcd(\pi')$ divides $\gcd(\pi)$.

A partition π is a *bond* of G if the graph induced by G on each block of π is connected. Let $\overline{\Pi}(G)$ be the set of pairs (π, t) where π is a bond of G and $t \in \mathbb{Z}_{\gcd(\pi)}$ is an integer modulo $\gcd(\pi)$. This set is partially ordered by

$$(\pi, t) \leq (\pi', t') \qquad \text{if } \pi \leq \pi' \text{ in } \Pi(G) \text{ and } t \equiv t' \bmod \gcd(\pi').$$

When π' refines π, $\gcd(\pi')$ divides $\gcd(\pi)$, and hence t modulo $\gcd(\pi')$ is well-defined.

Theorem 1. *The poset $\overline{\Pi}(G)$ is the intersection poset of the toric arrangement $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}A_{n-1})$.*

The correspondence is as follows. For any $(\pi, t) \in \overline{\Pi}(G)$, choose a block B of π, and choose $S_1, S_2 \subseteq B$ such that $|S_2| = t$ and $B = S_1 \sqcup S_2$. The pair (π, t) is identified with the image of the affine flat determined by the equalities:

$$\begin{aligned} x_i &= x_j && \text{if } i, j \text{ belong to the same block of } \pi \text{ and } i, j \notin B, \\ x_i &= x_j && \text{if } i, j \in S_1 \text{ or } i, j \in S_2, \\ x_i &= x_j + 1 && \text{if } i \in S_1 \text{ and } j \in S_2. \end{aligned}$$

By contrast, the intersection poset of the standard arrangement of G (Section 3, item (i)) is simply the poset of bonds of G [14].

There is a similar model for the poset of toric *faces* of $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}A_{n-1})$, which we do not discuss in this note. When G is the complete graph, this recovers the model given in [1, Section 4.3]. The combinatorial model for the toric flats of $\overline{\mathcal{A}}(\Phi_{K_3}, \mathbb{Z}A_2)$ is shown in Figure 4 and may be compared to [1, Figure 19].

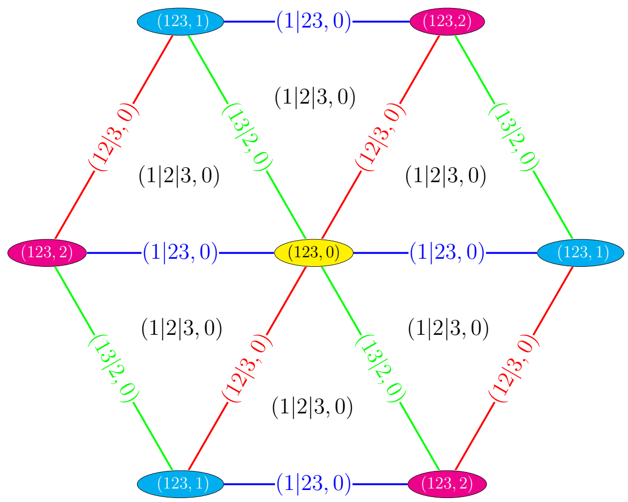

Figure 4: The flats of the arrangement $\overline{\mathcal{A}}(\Phi_{K_3}, \mathbb{Z}A_2)$ with labels from $\overline{\Pi}(G)$.

5 Characteristic and Tutte polynomials

Fix L and Φ as in Section 2. The *characteristic polynomial* of $\overline{\mathcal{A}}(\Phi, L)$ is

$$\overline{\chi}(\Phi, L; t) := \sum_{X \in \overline{\Pi}(\Phi, L)} \mu(X, T)\, t^{\dim X},$$

where μ is the Möbius function of the intersection poset, T is the torus V/L, and $\dim X$ is the dimension of the hypertorus X. Assume Φ spans V. A result of Zaslavsky [17, Theorem 1.2] implies that $(-1)^{\dim T}\overline{\chi}(\Phi, L; 0)$ is the number of *chambers* in $\overline{\mathcal{A}}(\Phi, L)$; see also [8, 5, 12, 14].

The *arithmetic Tutte polynomial* of $\overline{\mathcal{A}}(\Phi, L)$, as defined by Moci [12], is

$$\overline{M}(\Phi, L; x, y) := \sum_{A \subseteq \Phi} m(A)(x-1)^{\dim(\Phi) - \dim(A)}(y-1)^{|A| - \dim(A)},$$

where $m(A)$ is the number of toric flats of $\overline{\mathcal{A}}(\Phi, L)$ given as an intersection of the form $\bigcap_{\alpha \in A} \overline{H}_{\alpha, k_\alpha}$ ($k_\alpha \in \mathbb{Z}$), and $\dim(A)$ is the dimension of $\mathbb{R}$-span of A. The characteristic polynomial is a specialization of the arithmetic Tutte polynomial [12, Theorem 5.6]:

$$\overline{\chi}(\Phi, L; t) = (-1)^{\dim(\Phi)} t^{\dim(T) - \dim(\Phi)} \overline{M}(\Phi, L; 1-t, 0).$$

Consider now a weighted graph $\mathscr{G} = (G, \mathrm{wt})$ and the corresponding toric arrangement $\overline{\mathcal{A}}(\Phi_{\mathscr{G}}, L_{\mathscr{G}})$, as in Section 3. We turn to a combinatorial description of the characteristic polynomial of this arrangement, which we denote by $\overline{\chi}(\mathscr{G}; t)$.

A *proper divisible coloring* of $\mathscr{G}$ with q colors is a function $f : V(G) \to \mathbb{Z}_q$ such that

- $f(i) \neq f(j)$ whenever i and j are adjacent in G;

- $\sum_{i \in V(G)} \mathrm{wt}(i) f(i) \equiv 0 \mod q$.

Write $\mathrm{wt}(\mathscr{G}) := \mathrm{wt}(1) + \ldots + \mathrm{wt}(n)$ and $\gcd(\mathrm{wt}) := \gcd\{\mathrm{wt}(i) \mid i \in V(\mathscr{G})\}$.

Theorem 2. *Let $\mathscr{G}$ be a weighted graph with $\mathrm{wt}(\mathscr{G}) = m$. For any positive multiple q of m,*

$$\overline{\chi}(\mathscr{G}; q) = \frac{|\{\text{proper divisible coloring of } \mathscr{G} \text{ with } q \text{ colors}\}|}{\gcd(\mathrm{wt})}.$$

This result may be derived with the aid of the *toric finite field method* [8, Theorem 3.7].

Let $\overline{\chi}(G; t)$ denote the characteristic polynomial of $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}A_{n-1})$. Specializing Theorem 2 to the case when all weights are 1, we see that $\overline{\chi}(G; nt)$ counts proper divisible colorings of G. The following explicit expressions are worth-mentioning: for the path P_n and the star graph $K_{1,n-1}$ (both on n vertices),

$$\overline{\chi}(P_n; t) = (-1)^{n-1} \sum_{d|n} \varphi(d)\,(1-t)^{\frac{n}{d}-1}, \quad \overline{\chi}(K_{1,n-1}; t) = (t-1)^{n-1} + (-1)^{n-1}(n-1),$$

where φ is *Euler's totient function*. Recall, by contrast, that P_n and $K_{1,n-1}$, as well as any other tree on n vertices, have the same ordinary Tutte polynomial.

Let K_n be the complete graph on n vertices. The following formula, known from [2, Theorem 1.19], may also be derived from Theorem 2:

$$\overline{\chi}(K_n; t) = (-1)^{n-1}(n-1)! \sum_{d|n} (-1)^{\frac{n}{d}-1} \varphi(d) \binom{\frac{t}{d}-1}{\frac{n}{d}-1}.$$

Evaluating at 0 we see that the number of chambers in the Steinberg torus $\overline{\mathcal{A}}(\Phi_{K_n}, \mathbb{Z}A_{n-1})$ is $n!$, as known from [1, 7].

We turn to the arithmetic Tutte polynomial of $\overline{\mathcal{A}}(\Phi_{\mathscr{G}}, L_{\mathscr{G}})$, denoted $\overline{M}(\mathscr{G}; x, y)$.

Given $d \in \mathbb{Z}_{>0}$ and a subgraph H of G, we say that H is *d-divisible* if the sum of the weights of the vertices in each connected component of H is divisible by d. We define the *d-divisible Tutte polynomial* of $\mathscr{G}$ as

$$T_d(\mathscr{G}; x, y) := \sum_{H\ d\text{-divisible}} (x-1)^{k(H)-k(G)} (y-1)^{|H|+k(H)-|V(G)|}.$$

The sum is over the d-divisible subgraphs of G and $k(H)$ is the number of connected components of H. When d and all weights are 1, T_d is the ordinary Tutte polynomial of G.

Theorem 3. *Let $\mathscr{G}$ be a weighted graph with $\mathrm{wt}(\mathscr{G}) = m$. Then*

$$\overline{M}(\mathscr{G}; x, y) = \frac{1}{\gcd(\mathrm{wt})} \sum_{d|m} \varphi(d)\, T_d(\mathscr{G}; x, y).$$

By contrast, the arithmetic Tutte polynomial of the standard arrangement of G (Section 3, item (i)) is simply the Tutte polynomial of G [8].

In the special case when all weights are 1, Theorem 3 may be derived from either Theorem 1 or [2, Lemma 4.12].

Recall that the ordinary Tutte polynomial admits an expression in terms of *activities*. There exists an analogous result for the arithmetic Tutte polynomial due to D'Adderio and Moci [4]. We turn to a similar statement for the polynomial $T_d(\mathscr{G};\, x, y)$.

Fix a positive integer d and a total order on $E(G)$. Let F be a spanning forest of G. An edge $e \in E(G) \setminus F$ is *externally active* with respect to F if e is the smallest edge in the unique cycle of $F \cup \{e\}$. (This is the standard notion.) We say that an edge $e \in F$ is *d-internally active* with respect to F if $F \setminus \{e\}$ is d-divisible and e is the smallest edge in the cut defined by e. (When $d = 1$, this is the standard notion of internal activity.) The *external activity* of F, denoted by $\mathrm{ex}(F)$, is the number of externally active edges with respect to F. The *d-internal activity* of F, denoted by $\mathrm{in}_d(F)$, is the number of d-internally active edges with respect to F. Let $\mathrm{SF}_d(\mathscr{G})$ denotes the set of spanning forests of $\mathscr{G}$ that are d-divisible.

Theorem 4. *Let $\mathscr{G}$ be a weighted graph and let d be a positive integer. Then*

$$T_d(\mathscr{G};\, x, y) = \sum_{F \in SF_d(\mathscr{G})} x^{in_d(F)} y^{ex(F)}.$$

Our proof of Theorem 4 uses a *deletion-contraction recurrence* for divisible Tutte polynomials, which we do not describe in this note.

The following formulas may be derived as applications of Theorems 3 and 4 (P_n is the path, $K_{1,n-1}$ the star, C_n the cycle, and all weights are 1).

$$\begin{aligned}
\overline{M}(P_n;\, x, y) &= \sum_{d|n} \varphi(d)\, x^{\frac{n}{d}-1}, \\
\overline{M}(K_{1,n-1};\, x, y) &= x^{n-1} + n - 1, \\
\overline{M}(C_n;\, x, y) &= \sum_{d|n} d\, \varphi(d) \left(\frac{x^{\frac{n}{d}} - 1}{x - 1} + \frac{y - 1}{d} \right).
\end{aligned}$$

6 Voronoi equivalence

Let Φ and L be as in Section 2. We show that the chambers of certain toric arrangements $\overline{\mathcal{A}}(\Phi, L)$ may be seen as equivalence classes for a canonical equivalence relation on the set of chambers of the linear arrangement $\mathcal{A}(\Phi)$.

The *Voronoi cell* (at the origin) of L is the polytope

$$\mathrm{Vor}(L) := \{x \in V \mid |x| \le |x - \lambda| \text{ for all } \lambda \in L\}.$$

It is a fundamental domain for the action of L on V by translations, so the torus V/L may be realized as a quotient of $\mathrm{Vor}(L)$ in which boundary points are glued modulo L.

All hyperplanes in the linear arrangement $\mathcal{A}(\Phi)$ intersect the interior of $\mathrm{Vor}(L)$. We say that L is *confined by* Φ if no other hyperplane in the affine arrangement $\widetilde{\mathcal{A}}(\Phi)$ intersects the interior of $\mathrm{Vor}(L)$.

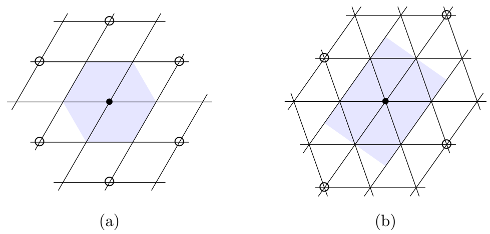

Figure 5: The lattice L (circles), its Voronoi cell $\mathrm{Vor}(L)$ (in blue), and the affine arrangement $\widetilde{\mathcal{A}}(\Phi)$ (lines) for two choices of L and Φ, one confined, the other not. (a) $\Phi = \Phi_{P_3}$ and $L = \mathbb{Z}A_2$. (b) $\Phi = \Phi_{\mathscr{G}}$ for $\mathscr{G}$ the complete graph K_3 with weights $\mathrm{wt}(1) = \mathrm{wt}(2) = 1$, $\mathrm{wt}(3) = 2$, and $L = L_{\mathscr{G}}$.

Assume that L is confined by Φ. Note that $\mathrm{Vor}(L)$ is divided into regions by $\mathcal{A}(\Phi)$, one for each chamber of $\mathcal{A}(\Phi)$. As L is confined by Φ and $\mathrm{Vor}(L)$ is a fundamental domain of the torus V/L, each of this region belongs to a toric chamber of $\overline{\mathcal{A}}(\Phi, L)$. We say that two linear chambers are *Voronoi equivalent* if the corresponding regions in $\mathrm{Vor}(L)$ belong to the same toric chamber. This defines an equivalence relation on the set of linear chambers of $\mathcal{A}(\Phi)$, which we call *Voronoi equivalence*. By construction, Voronoi equivalence classes correspond to chambers of $\overline{\mathcal{A}}(\Phi, L)$. Figure 6 illustrates the situation for the toric arrangement $\overline{\mathcal{A}}(\Phi_{P_3}, \mathbb{Z}A_2)$ of the path P_3.

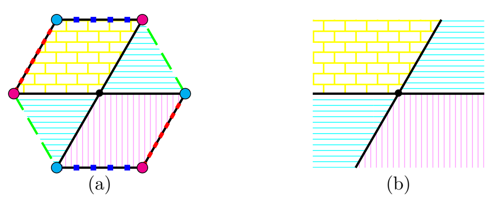

Figure 6: (a) Toric arrangement and Voronoi cell. (b) Linear arrangement. The two chambers with the same pattern are Voronoi equivalent.

Proposition 5. *When Φ is a crystallographic root system, both the coroot lattice $\mathbb{Z}\Phi^\vee$ and the coweight lattice $\widehat{\mathbb{Z}\Phi}$ are confined by Φ.*

The proof relies on the observation that the Voronoi cell of $\mathbb{Z}\Phi^\vee$ is the largest confined neighborhood of the origin.

7 Voronoi equivalence for toric graphic arrangements

We turn to the study of Voronoi equivalence for the toric arrangements $\overline{\mathcal{A}}(\Phi_G, \widehat{\mathbb{Z}A_{n-1}})$ and $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}A_{n-1})$ of Section 3. Confinement of its Voronoi cell follows from Proposition 5. In each case, Voronoi equivalence is a relation on the set $\mathrm{Cham}(G)$ of chambers of the linear arrangement $\mathcal{A}(\Phi_G)$.

We assume throughout this section that G is connected. This guarantees that Φ_G spans and that $\overline{\mathcal{A}}(\Phi_G)$ is essential. With minor adjustments, the results that follow can be extended to all simple graphs.

The set $\mathrm{Cham}(G)$ is in bijection with the set $\mathrm{Acyc}(G)$ of acyclic orientations of G by the following correspondence [10]: an orientation $\mathcal{O} \in \mathrm{Acyc}(G)$ is associated to the chamber $C = \{x \in V \mid x_i < x_j \text{ if } i \to j \text{ in } \mathcal{O}\}$. Through this correspondence, Voronoi equivalence is a relation on the set $\mathrm{Acyc}(G)$.

For the coweight lattice, Voronoi equivalence turns out to be a familiar relation. A *source-to-sink flip* is an operation that turns an acyclic orientation $\mathcal{O}$ into another $\mathcal{O}'$ by choosing a source vertex i in $\mathcal{O}$, and then changing the orientation in $\mathcal{O}$ of all edges incident to i. Then i is a sink vertex in the new orientation $\mathcal{O}'$. See Figure 7 (a).

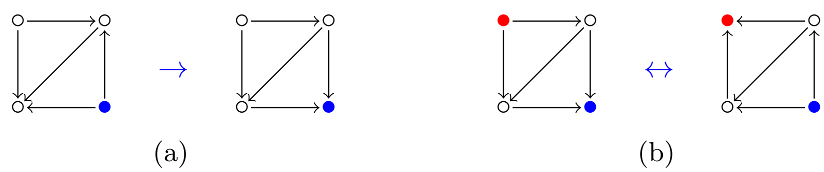

Figure 7: (a) A source-to-sink flip. (b) A source-sink exchange.

Theorem 6. *Let $L = \widehat{\mathbb{Z}A_{n-1}}$. Two orientations in* $\mathrm{Acyc}(G)$ *are Voronoi equivalent if and only if they are related by a sequence of source-to sink flips.*

This relation arose in [13] and [15], and has been studied extensively for its connection to chip-firing and conjugacy of Coxeter elements [9, 16]. The fact that equivalence classes correspond to toric chambers of the standard arrangement $\overline{\mathcal{A}}(\Phi_G, \mathbb{Z}^n)$ is due to [6, Theorem 1.4]. (Recall $\overline{\mathcal{A}}(\Phi_G, \widehat{\mathbb{Z}A_{n-1}})$ is its essentialization.)

For the coroot lattice, Voronoi equivalence appears to be a new and perhaps equally interesting relation. It possesses several combinatorial descriptions which we discuss next.

A *source-sink exchange* is an operation that transforms an acyclic orientation $\mathcal{O}$ into another $\mathcal{O}'$ by choosing a source vertex i and a non-adjacent sink vertex j of $\mathcal{O}$, and then changing the orientation of all edges in $\mathcal{O}$ that are incident to either i or j. Then i is a sink vertex and j is a source vertex in the new orientation $\mathcal{O}'$. See Figure 7 (b).

Theorem 7. *Let $L = \mathbb{Z}A_{n-1}$. For $\mathcal{O}_1, \mathcal{O}_2 \in$* $\mathrm{Acyc}(G)$*, the following are equivalent:*

(i) *$\mathcal{O}_1$ and $\mathcal{O}_2$ are Voronoi equivalent.*

(ii) *$\mathcal{O}_1$ and $\mathcal{O}_2$ are related by a sequence of source-sink exchanges.*

(iii) *$\mathcal{O}_1$ and $\mathcal{O}_2$ are related by a sequence of source-to-sink flips of length divisible by n.*

(iv) *There exists $\mathcal{O}_3 \in \mathrm{Acyc}(G)$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are related to $\mathcal{O}_3$ by a sequence of source-to-sink flips of the same length.*

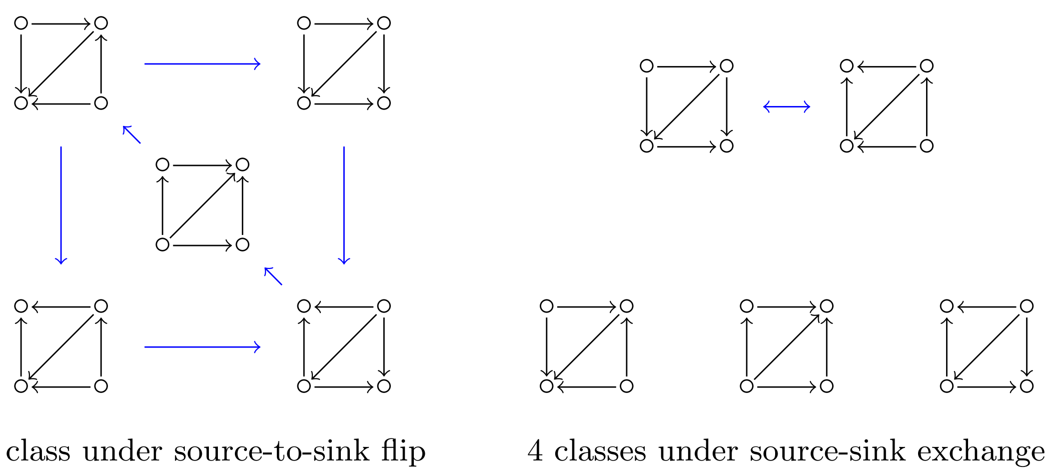

Figure 8: Comparison between two Voronoi equivalences.

There is also an interesting description of this relation in terms of *modular flows* on G. As a consequence, one has the following explicit calculations. The path with n vertices has n coroot Voronoi equivalence classes, and the i-th class has cardinality

$$\frac{1}{2}|\{S \subseteq [n] \mid \sum_{x\in S} x \equiv i \mod n\}|.$$

The cycle with n vertices has $n(n-1)$ coroot Voronoi equivalence classes, and the (i,j)-th class has cardinality

$$|\{S \subseteq [n] \mid |S| = i, \sum_{x\in S} x \equiv j \mod n\}|.$$

References

[1] M. Aguiar and T. K. Petersen. "The Steinberg torus of a Weyl group as a module over the Coxeter complex". *J. Algebraic Combin.* **42** (2015), pp. 1135–1175. DOI.

[2] F. Ardila, F. Castillo, and M. Henley. "The arithmetic Tutte polynomials of the classical root systems". *Int. Math. Res. Not.* **2015** (2015), pp. 3830–3877. DOI.

[3] P. Brändén and L. Moci. “The multivariate arithmetic Tutte polynomial”. *Trans. Amer. Math. Soc.* **366** (2014), pp. 5523–5540. DOI.

[4] M. D’Adderio and L. Moci. “Arithmetic matroids, the Tutte polynomial and toric arrangements”. *Adv. Math.* **232** (2013), pp. 335–367. DOI.

[5] P. Deshpande. “On a generalization of Zaslavsky’s theorem for hyperplane arrangements”. *Ann. Combin.* **18** (2014), pp. 35–55. DOI.

[6] M. Develin, M. Macauley, and V. Reiner. “Toric partial orders”. *Trans. Amer. Math. Soc.* **368** (2016), pp. 2263–2287. DOI.

[7] K. Dilks, T. K. Petersen, and J. R. Stembridge. “Affine descents and the Steinberg torus”. *Adv. Appl. Math.* **42** (2009), pp. 423–444. DOI.

[8] R. Ehrenborg, M. Readdy, and M. Slone. “Affine and toric hyperplane arrangements”. *Discrete Comput. Geom.* **41** (2009), pp. 481–512. DOI.

[9] H. Eriksson and K. Eriksson. “Conjugacy of Coxeter elements”. *Electron. J. Combin.* **16**.2 (2009), Art. #R4. URL.

[10] C. Greene and T. Zaslavsky. “On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs”. *Trans. Amer. Math. Soc.* **280** (1983), pp. 97–126. DOI.

[11] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics, Vol. 29. Cambridge University Press, 1990.

[12] L. Moci. “A Tutte polynomial for toric arrangements”. *Trans. Amer. Math. Soc.* **364** (2012), pp. 1067–1088. DOI.

[13] K. M. Mosesjan. “Strongly basable graphs”. *Akad. Nauk Armjan. SSR Dokl.* **54** (1972), pp. 134–138.

[14] I. Novik, A. Postnikov, and B. Sturmfels. “Syzygies of oriented matroids”. *Duke Math. J.* **111** (2002), pp. 287–317. DOI.

[15] O. Pretzel. “On reorienting graphs by pushing down maximal vertices”. *Order* **3** (1986), pp. 135–153. DOI.

[16] D. E. Speyer. “Powers of Coxeter elements in infinite groups are reduced”. *Proc. Amer. Math. Soc.* **137** (2009), pp. 1295–1302. DOI.

[17] T. Zaslavsky. “A combinatorial analysis of topological dissections”. *Adv. Math.* **25** (1977), pp. 267–285. DOI.