Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #1, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

Promotion on Generalized Oscillating Tableaux and Web Rotation

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Abstract. We introduce the notion of a generalized oscillating tableau and define a promotion operation on such tableaux that generalizes the classical promotion operation on standard Young tableaux. As our main application, we show that this promotion corresponds to rotation of the irreducible A_2-webs of G. Kuperberg.

Résumé. Nous introduisons la notion de tableau oscillant généralisé et définissons une opération de promotion sur ces tableaux qui généralise l'opération de promotion classique sur les tableaux de Young. Notre principale application est de montrer que cette promotion correspond à la rotation des A_2-toiles irréductible de G. Kuperberg.

Keywords: tableaux, promotion, webs

1 Introduction

Recall that a *partition* is a finite, nonincreasing list of positive integers $\lambda = (\lambda_1, \ldots, \lambda_t)$ and that any partition can be identified with the corresponding *Young diagram*—a left-justified array of boxes with λ_i boxes in the ith row from the top. An *oscillating tableau* of length k is a sequence of $k+1$ partitions $(\lambda^0 = \varnothing, \ldots, \lambda^k)$, where $\lambda^0 = \varnothing$ and λ^i is obtained from λ^{i-1} by either adding or deleting one box. In this paper, we generalize these notions.

We define a *generalized partition with n parts* $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ to be a nonincreasing list of n (not necessarily positive) integers. We introduce the notion of a *generalized oscillating tableau of length k with n parts*: a sequence of $k+1$ generalized partitions $(\varnothing, \lambda^1, \ldots, \lambda^k)$ such that each λ^i has n parts, $\lambda^0 = \varnothing = (0, \ldots, 0)$, and λ^{i+1} can be obtained from λ^i by either adding or subtracting 1 from one of $\lambda^i_1, \ldots, \lambda^i_n$. We visualize generalized partitions using a generalization of Young diagrams, where we allow negative row sizes and indicate negative rows by coloring the corresponding boxes red. We may then associate a set-valued tableau T to each generalized oscillating tableau, where the set of boxes of T is the union of boxes in $\lambda^1, \ldots, \lambda^k$ and we add entry i (resp. i') to the subset of primed and unprimed positive integers in a box if λ^i is obtained from λ^{i-1} by

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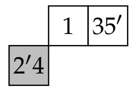
adding (resp. deleting) the corresponding box. For example, the generalized oscillating tableau of length 5 with 2 parts $((0,0),(1,0,),(1,-1),(2,-1),(2,0),(1,0))$ corresponds to the set-valued filling below.



Let $\mathrm{GOT}(k,n)$ denote the set of generalized oscillating tableaux of length k with n parts. We define a promotion operation $p : \mathrm{GOT}(k,n) \to \mathrm{GOT}(k,n)$ that generalizes classical tableau promotion. We define this promotion operation using both growth rules and growth diagrams and using tableau rules. Figure 1 shows an example of generalized oscillating promotion. A reader familiar with promotion on standard Young tableaux will recognize the similarities.

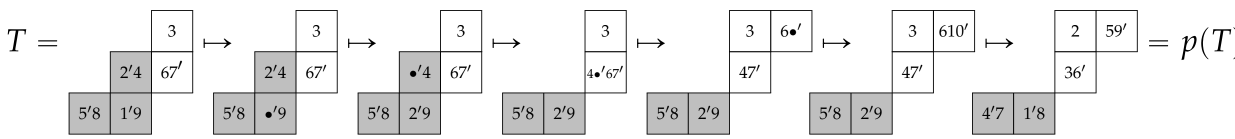


Figure 1: We start with generalized oscillating tableau T and construct its image under generalized oscillating promotion, $p(T)$.

As our main application, we relate generalized oscillating promotion on $\mathrm{GOT}(k,3)$ to rotation of irreducible A_2-webs. An *irreducible A_2-web* can be defined as a bipartite graph with fixed coloring embedded in a disk such that each vertex on the boundary of the disk has degree 1, each interior vertex has degree 3, and all internal faces have at least 6 sides. Webs were defined by G. Kuperberg motivated by the study of multilinear invariant theory [6]. In his paper, Kuperberg introduces combinatorial rank 2 spiders, which are a diagrammatic presentation of the space $\mathrm{Inv}(V_1 \otimes \cdots \otimes V_n)$, i.e., the invariant space of a tensor product of irreducible representations V_i of a rank 2 Lie algebra $\mathfrak{g}$. Webs are a basis for the invariant space in this diagrammatic presentation.

Webs have since been studied by G. Kuperberg and M. Khovanov [4]; T.K. Peterson, P. Pylyavskyy, and B. Rhoades [8]; S. Fomin and P. Pylyavskyy [2]; and many others. In particular, Khovanov and Kuperberg describe a bijection between webs and *signature and state strings*: a vector of pairs, where each pair $(j_i, s_i) \in \{\bullet, \circ\} \times \{1, 0, \bar{1}\}$. Using this correspondence between webs and signature and state strings, it is easy to associate to each web with all black boundary vertices a three-row standard Young tableau of rectangular shape. In their paper, Peterson, Pylyavskyy, and Rhoades describe how to interpret the action of tableau promotion on these rectangular tableaux as web rotation.

Using the signature and state strings of Khovanov and Kuperberg [4], we associate to each web with fixed first/leftmost vertex and k boundary vertices a generalized oscillating tableau of length k with 3 parts. Our main result is the following. We refer the reader to [7] for further details and proofs.

Theorem 1.1. *Let D be a web with fixed leftmost vertex. The generalized oscillating tableau associated with counterclockwise rotation of D is given by generalized oscillating promotion of the tableau associated with D itself.*

2 Preliminaries

2.1 Tableaux and Promotion

For partition λ, let $|\lambda| = \lambda_1 + \cdots + \lambda_k$ denote the number of boxes in λ. A *standard Young tableau* of shape λ is a filling of the cells of a Young diagram of shape λ with $1, 2, \ldots, |\lambda|$ such that entries in rows and columns are increasing and each entry appears exactly once. We say partition μ is contained in λ if the Young diagram for μ fits inside that for λ, and in this case, we let λ/μ denote the set of boxes of λ that are not also in μ.

We next describe an action on standard Young tableaux of shape λ called *jeu de taquin promotion*, or simply *promotion*, originally defined as an action on partially ordered sets by Schützenberger [11]. Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps. First, delete the entry 1 from the box in the upper lefthand corner of T and replace it with $\bullet$. Next, for each $i \in \{2, \ldots, k\}$, perform the following swap with i starting at 2 and consecutively increasing after each swap.

1. If the box containing i is directly below or directly right of the box containing $\bullet$, switch the labels of the two boxes.

2. If the box containing i is not directly below and not directly right of the box containing $\bullet$, do nothing.

Finally, delete the $\bullet$, fill its box with $k + 1$, and subtract 1 from each entry.

We can equivalently describe promotion using *promotion growth diagrams* and a set of *promotion growth rules*, as we now explain. We refer the reader to [12] for more details. Suppose we wish to perform promotion on standard Young tableau T with k boxes. First, write T as a sequence of partition shapes $(\lambda^0 = \varnothing, \lambda^1, \ldots, \lambda^k)$ starting with the empty shape, where λ^i is obtained from λ^{i-1} by adding the box with label i in T. Suppose λ^s/μ^{s-1} is one box in row i and λ^{s+1}/λ^s is a box in row j. We inductively create a new sequence $(\mu^0 = \varnothing, \mu^1, \ldots, \mu^k) = p(T)$ using the following rules.

1. If the result of adding a box to μ^{s-1} in row j is a partition, then μ^s is the result of adding this box to μ^{s-1}.

2. If the result of adding a box to μ^{s-1} in row j is not a partition then $\mu^s = \lambda^s$.

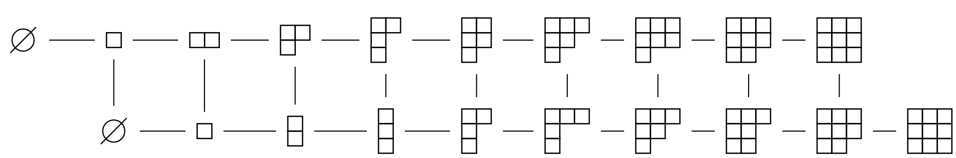
3. $\mu^k = \lambda^k$.

Starting with a tableau T, we illustrate these rules in a growth diagram, as shown in Example 2.1.

Example 2.1. *Starting with standard Young tableau T, we obtain $p(T)$ using the tableau promotion rules.*

$$T = \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5&7\\\hline 4&8&9\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline \bullet&2&6\\\hline 3&5&7\\\hline 4&8&9\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&\bullet&6\\\hline 3&5&7\\\hline 4&8&9\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 3&\bullet&7\\\hline 4&8&9\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 3&7&\bullet\\\hline 4&8&9\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 3&7&9\\\hline 4&8&\bullet\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 3&7&9\\\hline 4&8&10\\\hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&8\\\hline 3&7&9\\\hline\end{array} = p(T)$$

We can instead use the promotion growth rules, where T is represented by the sequence of partitions on the top line, and we construct $p(T)$ on the line below.



2.2 Webs

Webs were introduced by G. Kuperberg in the following way.

Definition 2.2 ([6]). *An A_2-web is a planar, directed graph D with no multiple edges embedded in a disk satisfying the following conditions: D is bipartite, (i.e., each vertex has either all adjacent edges pointing away from it or all adjacent edges pointing toward it), all of the boundary vertices have degree 1, and all internal vertices have degree 3. An A_2-web is non-elliptic if all internal faces of D have at least 6 sides. When all four conditions are satisfied, we call D an irreducible A_2-web.*

In this document, we will refer to irreducible A_2-webs simply as *webs*. In other words, all webs are assumed to be irreducible A_2-webs. We will also omit the directions of the edges of a web D and instead bicolor the vertices of D. A vertex v will be black if all adjacent edges point toward v and will be white if all adjacent edges point away from v. We view webs as combinatorial objects and thus are only concerned with webs up to homeomorphism on the interior of the disk and place boundary vertices canonically. We will often fix a *leftmost* or *starting* boundary vertex for the web we are considering. See Figure 2 for examples of webs.

In [4], M. Khovanov and G. Kuperberg describe a bijection between webs with n boundary vertices and chosen leftmost vertex and certain length n strings, which we now describe. A *signature* of length n is a sequence $S = (s_1, s_2, \ldots, s_n) \in \{\circ, \bullet\}^n$. A *state*

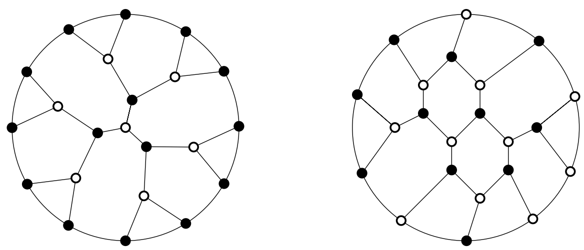


Figure 2: Examples of webs. The web on the left is in $\mathcal{M}_4$.

string is a sequence $J = (j_1, j_2, \ldots, j_n) \in \{\bar{1}, 0, 1\}^n$. A *signature and state string* is a sequence $((j_1, s_1), (j_2, s_2), \ldots, (j_n, s_n))$, where each element is a state paired with either $\circ$ or $\bullet$.

Khovanov and Kuperberg classify certain signature and state string as *dominant* based on a correspondence between these strings and weight lattice paths in a Weyl chamber of sl(3) that begin and end at the origin. For signature and state strings where $s_i = \bullet$ for all i, the dominant condition translates into the familiar Yamanouchi condition for the state string. Given a dominant signature and state string, they build a web with chosen leftmost vertex by giving a series of inductive growth rules, which we will not describe here. This growth algorithm has an inverse for dominant signature and state strings. We thus have a one-to-one correspondence between (irreducible, non-elliptic) webs with a chosen leftmost vertex and dominant signature and state strings. See [4, 7]for examples of this bijection.

3 Rotation of webs in $\mathcal{M}_n$ and promotion

Let $\mathcal{M}_n$ denote the set of webs with $3n$ boundary vertices, all of which are the same color. We will assume that all boundary vertices are black. Note that in this setting, we may ignore the signature of a web in $D \in \mathcal{M}_n$ with chosen leftmost vertex and instead focus on its state string. We consider its state string to be a word $w(D)$ in the alphabet $\{1, 0, \bar{1}\}$, and we refer to the word as w if the corresponding web is clear from context.

Using $w(D) = w_1 \cdots w_{3n}$ for a web $D \in \mathcal{M}_n$, we can easily associate a standard Young tableau $T(w(D))$ of shape (n, n, n) as follows. Fill the top row of $T(w(D))$ with the indices of the 1's in $w(D)$, the second row of $T(w(D))$ with the indices of the 0's in $w(D)$, and the third row with the indices of the $\bar{1}$'s of $w(D)$. For example, the word $w(D) = 110\bar{1}010\bar{1}\bar{1}$ corresponds to the standard Young tableau T in Example 2.1.

In [8], the authors reinterpret the action of promotion on rectangular standard Young tableaux of shape (n, n, n) as counterclockwise rotation of webs in $\mathcal{M}_n$. In the following theorem, if D is a web with chosen leftmost vertex v, let $p(D)$ denote the web with chosen leftmost vertex the next vertex reached traveling clockwise around the boundary from v. In other words, $p(D)$ is the result of rotating D one vertex counterclockwise.

Theorem 3.1 ([8]). *For any web D with fixed leftmost vertex and with black boundary vertices, we have*

$$T(w(p(D))) = p(T(w(D))).$$

That is, the tableau associated with the rotation of D is given by promotion of the tableau associated with D itself.

4 Oscillating Tableaux and Promotion

We define a *generalized partition* to be a finite non-increasing sequence of integers $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$. We say that λ is a *generalized partition with n parts* if it is possible to write λ with n components, where we may add terminal zeros to the partition when doing so preserves that $\lambda_i \geqslant \lambda_{i+1}$. For example, $(2,1)$ is a generalized partition with n parts for any $n \geqslant 2$, as we consider $(2,1)$ to be the same as $(2,1,0)$, etc. We consider the empty partition $\varnothing$ to be the same as $(0,\ldots,0)$, and so $\varnothing$ can be written with any number of parts. However, $(5,5,3,0,-2,-2)$ is a generalized partition with 6 parts and cannot be written with a different number of parts.

To each generalized partition, we associate a *generalized Young diagram* by considering the generalized Young diagram to lie in the lower half plane of $\mathbb{Z}^2$ and allowing boxes to lie in both the third and fourth quadrant. Boxes corresponding to negative parts of the generalized partition are in the third quadrant and boxes corresponding to the positive parts are in the fourth quadrant. See Example 4.2 for a generalized Young diagram of shape $(1,1,-2)$.

Definition 4.1. *A generalized oscillating tableau of length k with n parts is a sequence of $k+1$ generalized partitions $(\lambda^0 = \varnothing, \lambda^1, \ldots, \lambda^k)$ such that λ^i is obtained from λ^{i-1} by either adding or removing one box and such that each λ^i is a generalized partition with n parts. We denote the set of such sequences by $GOT(k,n)$.*

To each element of $\mathrm{GOT}(k,n)$, we can associate a set-valued filling of the union of boxes appearing in $\lambda^1,\ldots,\lambda^k$ by entering i or i' in the box that was added to or removed from, respectively, λ^{i-1} to obtain λ^i. Within each box, we write the subset in increasing order. We identify a generalized oscillating tableau with the corresponding set-valued filling. Note that the shape of the generalized oscillating tableau need not be a generalized partition shape. We again color cells corresponding to negative cells gray.

In addition, note that it is not necessary to use both primed and unprimed entries in this construction as the location of the primed entries can easily be recovered from the analogous construction using only unprimed entries. However, we will use primed and unprimed entries for ease of understanding and notational convenience.

Example 4.2. *The generalized oscillating tableau*
$(\varnothing,(1,0,0),\varnothing,(0,0,-1),(0,0,-2),(1,0,-2),(1,1,-2),(1,1,-1)) \in GOT(7,3)$ *corresponds*

to the filling on the left of shape $(1,1,-2)$*. The sequence* $(\varnothing,(0,0,-1),(0,-1,-1),(1,-1,-1),(1,0,-1),(1,0,-2),(1,1,-2),(1,0,-2),(1,0,-1),(1,0,0)) \in GOT(9,3)$ *corresponds to the filling on the right, which does have generalized partition shape.*

4.1 Webs and generalized oscillating tableaux

Given a web D with chosen leftmost vertex v and k boundary vertices, or equivalently a dominant signature and state string with k components, we may associate a generalized oscillating tableau of length k with three parts as follows. The pairs in the signature and state string will correspond to the following actions on generalized partition λ^i.

$(1,\bullet)$	Add one to the first part of λ^i to obtain λ^{i+1}
$(0,\bullet)$	Add one to the second part of λ^i to obtain λ^{i+1}
$(\bar{1},\bullet)$	Add one to the third part of λ^i to obtain λ^{i+1}
$(1,\circ)$	Subtract one from the first part of λ^i to obtain λ^{i+1}
$(0,\circ)$	Subtract one from the second part of λ^i to obtain λ^{i+1}
$(\bar{1},\circ)$	Subtract one from the third part of λ^i to obtain λ^{i+1}

Build the generalized oscillating tableau by reading the string left to right and using the table above to determine how to obtain the next generalized partition in the generalized oscillating tableau. For example, the tableau on the right in Example 4.2 comes from the signature and state string $((\bar{1},\circ),(0,\circ),(1,\bullet),(0,\bullet),(\bar{1},\circ),(0,\bullet),(0,\circ),(\bar{1},\bullet),(\bar{1},\bullet))$.

It is not hard to argue that this procedure applied to a web will indeed produce a sequence of generalized partitions. Moreover, it comes easily from the definition of a dominant signature and state string that a generalized oscillating tableau $T = (\lambda^0,\dots,\lambda^k) \in \mathrm{GOT}(k,3)$ comes from a web D if and only if $\lambda^k = (m,m,m)$ for some $m \in \mathbb{Z}$.

4.2 Generalized Oscillating Promotion Growth rules

We first define a promotion action $p : \mathrm{GOT}(k,n) \to \mathrm{GOT}(k,n)$ using growth rules. We call this action *generalized oscillating promotion*. It is important to note that in contrast to promotion on standard Young tableaux, $p(T)$ may not have the same shape as generalized oscillating tableau T.

Let $T = (\lambda^0,\lambda^1,\dots,\lambda^k) \in \mathrm{GOT}(k,n)$. We inductively build a new sequence of generalized partition shapes $(\mu^0 = \varnothing,\mu^1,\dots,\mu^k)$ using a set of growth rules as in Section 2.1. The sequence we create gives $p(T)$.

The idea of the growth rules is this: Suppose we perform action 1 to obtain λ^s from μ^{s-1} and action 2 to obtain λ^{s+1} from λ^s. If performing action 2 on μ^{s-1} gives a generalized partition with n parts, μ^s is obtained from μ^{s-1} by performing action 2. Otherwise we either obtain μ^s from μ^{s-1} by performing action 1 or we must otherwise modify. See Figure 3.

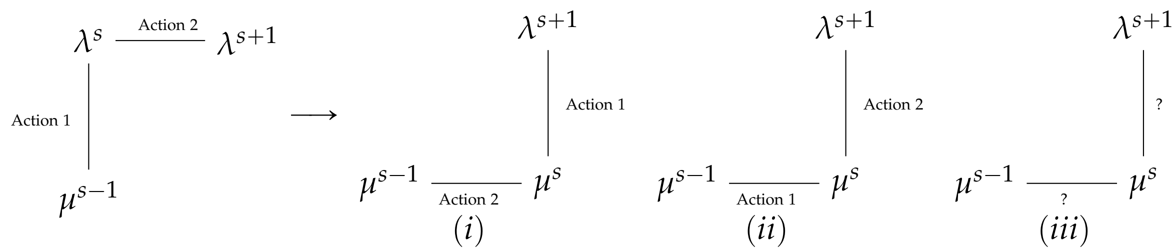


Figure 3: The idea of the generalized oscillating promotion growth rules. If the result of (i) is a generalized partition with the correct number of parts, we choose (i). Otherwise, we choose (ii) in (OP1b). Rule (OP1d) corresponds to (iii).

Definition 4.3 (Generalized oscillating promotion growth rules)**.**

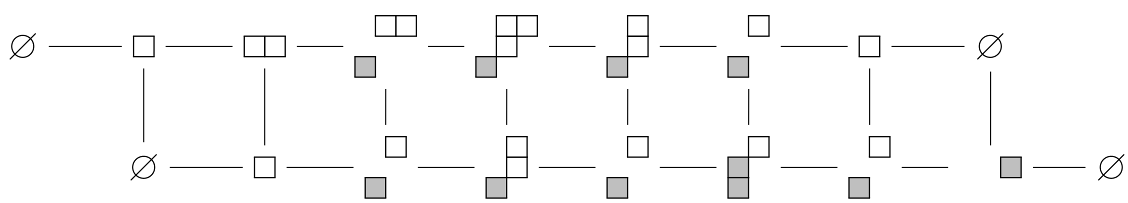
(OP1) Suppose λ^s is obtained from μ^{s-1} by adding (resp. deleting) a box in row i.

- *(a) If λ^{s+1} is obtained from λ^s by adding (resp. deleting) a box in row j and the result of adding (resp. deleting) a box to row j of μ^{s-1} is a generalized partition with n parts, then μ^s is obtained from μ^{s-1} by adding (resp. deleting) a box in row j.*
- *(b) If λ^{s+1} is obtained from λ^s by adding (resp. deleting) a box in row $j \neq i$ and the result of adding (resp. deleting) a box to row j of μ^{s-1} is not a generalized partition with n parts, then μ^s is obtained from μ^{s-1} by adding (resp. deleting) a box in row i.*
- *(c) If λ^{s+1} is obtained from λ^s by adding (resp. deleting) a box in row j and adding (resp. deleting) a box in row j of μ^{s-1} is a generalized partition with n parts, then μ^s is the result of adding (resp. deleting) a box in row j of μ^{s-1}.*
- *(d) If λ^{s+1} is obtained from λ^s by adding (resp. deleting) a box in row i and adding (resp. deleting) a box in row i of μ^{s-1} is not a generalized partition with n parts, then μ^s is the result of adding (resp. deleting) a box of μ^{s-1} in row $i+t$ (resp. row $i-t$) for $t > 0$ as small as possible.*

(OP2) $\mu^k = \lambda^k$.

Note that (OP1b) need only be stated for $j \neq i$ because if $j = i$, the result is always a generalized partition with n parts. Similarly, (OP1d) need only be stated for $j = i$ because the situations described in (OP1c) can fail to be generalized partitions with n parts only when $j = i$.

Example 4.4. *Below is the growth diagram for promotion of the generalized oscillating tableau* $T = (\varnothing, (1,0,0), (2,0,0), (2,0,-1), (2,1,-1), (1,1,-1), (1,0,-1), (1,0,0), \varnothing)$*. To construct* $p(T)$*, we use rules (OP1a) with* $i = j = 1$*, (OP1c) with* $i = 1$ *and* $j = 3$*, (OP1a) with* $i = 1$ *and* $j = 2$*, (OP1d) with* $i = 1$ *and* $t = 1$*, (OP1c) with* $i = j = 2$*, (OP1b) with* $i = 2$ *and* $j = 3$*, (OP1c) with* $i = 3$ *and* $j = 1$*, and finally (OP2).*



By comparing the growth rules, it is easy to see that generalized oscillating promotion restricts to classical promotion when applied to a standard Young tableau T. Also, since $\mu^k = \lambda^k$, we see that if T corresponds to a web then so does $p(T)$.

4.3 Generalized oscillating promotion on tableaux

We now describe the same generalized oscillating promotion action $p : \mathrm{GOT}(k, n) \to \mathrm{GOT}(k, n)$ in terms of tableaux.

Definition 4.5 (Generalized Oscillating Promotion)**.** *Given a generalized oscillating tableau* $T = (\varnothing, \lambda^1, \ldots, \lambda^k) \in GOT(k, n)$*, form* $p(T)$ *using the following steps.*

1. *If the entry 1 exists, delete it and replace it with* $\bullet$*. If instead the entry* $1'$ *exists, delete it and replace it with* $\bullet'$*. At each step in the promotion, we denote the box currently containing* $\bullet$ *or* $\bullet'$ *by* b*, the row containing* b *by* r_b*, and the column containing* b *by* c_b*. Let* $c_b + 1$ *denote the column to the right of* c_b *and* $c_b - 1$ *denote the column to the left of* c_b*.*

2. *Perform jeu de taquin: For each* $i \in \{2, \ldots, n\}$*, perform the following swap with* i *starting at 2 and consecutively increasing after each swap.*

    (a) *If* $\bullet$ *is unprimed:*

    i. *If the box containing* i *is directly below or directly right of* b*, switch the labels within these two boxes.*

    ii. *If* $\bullet$ *is in the same box at* i' *consider* λ^i*.*

    - *If* $\lambda^i_{r_b} \neq \lambda^i_{r_b+1}$ *or* $\lambda^i_{r_b+1}$ *does not exist, delete* $\bullet$ *and* i' *from* b *and add the subset* $\{i', \bullet\}$ *to the box directly to its left.*
    - *If* $\lambda^i_{r_b} = \lambda^i_{r_b+1}$*, delete* $\bullet$ *and* i' *from box* b *and add the subset* $\{i', \bullet\}$ *to the box in column* $c_b - 1$ *and row* r*, where* r *is the bottommost row of* λ^i *of size* $\lambda^i_{r_b}$*.*

    iii. *If neither of the previous two things is true, do nothing.*

(b) *If $\bullet$ is primed:*

i. *If the box containing i' is directly above or directly left of b, switch these labels within the two boxes.*

ii. *If $\bullet'$ is in the same box at i, consider λ^i.*

- *If $\lambda^i_{r_b} \neq \lambda^i_{r_b-1}$ or $\lambda^i_{r_b-1}$ does not exist, delete $\bullet'$ and i from b and add the subset $\{i, \bullet'\}$ to the box directly to its right.*
- *If $\lambda^i_{r_b} = \lambda^i_{r_b-1}$, delete $\bullet'$ and i from box b and add the subset $\{i, \bullet'\}$ to the box in column $c_b + 1$ and row r, where r is the topmost row of λ^i of size $\lambda^i_{r_b}$.*

iii. *If neither of the previous two things is true, do nothing.*

3. *Delete the $\bullet$ or $\bullet'$ and fill its box with $k+1$ or $k+1'$, respectively. Then subtract 1 from each entry.*

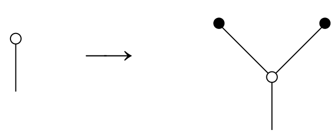
See Figure 1 for an example of generalized oscillating promotion using these tableau rules. Notice that in the example shown, T and $p(T)$ do not have the same shape.

Proposition 4.6. *The growth rules in Section 4.2 describe the generalized oscillating promotion in Definition 4.5.*

5 Rotation corresponds to generalized oscillating promotion

To prove our main results, we heavily use the results of T.K. Peterson, P. Pylyavskyy, and B. Rhoades [8] along with the following idea.

Suppose we have a web D with chosen leftmost vertex v. Without loss of generality, suppose v is black as analogous arguments always hold for v white. We extend D to a larger web D' with all black boundary vertices by replacing each white boundary vertex of D by a fork with two new black boundary vertices as shown below.



Applying the results of Peterson–Pylyavskyy–Rhoades now leads to Theorem 1.1, where $w(D)$ is the word obtained from the state string of D with fixed leftmost boundary vertex. For ease of reading, we restate Theorem 1.1 below.

Theorem. *For any web D with chosen leftmost vertex v, we have*

$$T(w(p(D))) = p(T(w(D))).$$

That is, the generalized oscillating tableau associated with the rotation of D is given by generalized oscillating promotion of the tableau associated with D itself.

6 Future directions

6.1 Enumeration and cyclic sieving

Let X be a finite set, $C = \langle c \rangle$ be a finite cyclic group acting on X, and $X(q) \in \mathbb{Z}[q]$ be a polynomial with integer coefficients. Then the triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon* [9] if for each $d > 0$, $|X^{c^d}| = X(\zeta^d)$, where $\zeta \in \mathbb{C}$ is a $|C|$th root of unity and X^{c^d} is the set of fixed points of the action of c^d.

In [10], B. Rhoades shows that standard Young tableau promotion on rectangular tableaux exhibits the cyclic sieving phenomenon. In [8], the authors reprove this result in the special case that the tableaux have two or three rows using the connection between promotion and webs. We would be interested in knowing if it is possible to extend this result to the generalized oscillating tableaux corresponding to webs.

Theorem 6.1 ([8, 10])**.** *Let $\lambda \vdash N = bn$ be a rectangle with $b = 2$ or 3 rows and let $C = \mathbb{Z}/n\mathbb{Z}$ act on $X = \{$standard Young tableaux of shape $\lambda\}$ by promotion. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q)$ is the q-analogue of the hook length formula:*

$$X(q) = \frac{[n]_q!}{\prod_{(i,j)\in\lambda}[h_{ij}]_q}.$$

6.2 sl(n) webs

The webs described here correspond to sl(3), and webs corresponding to sl(n) for $n > 3$ are much less developed. See, for example, [1, 3, 5]. In particular, when $n > 3$ there is no appropriate notion of an irreducible web and no rotation-invariant basis of webs.

However, given a definition of a signature and state string for sl(n) webs, we think it is possible that our generalized oscillating promotion describes rotation of these webs. Specifically, perhaps the promotion $p : \mathrm{GOT}(k, n) \to \mathrm{GOT}(k, n)$ describes rotation for webs corresponding to sl(n). We state these ideas as conjectures.

Conjecture 6.2. *There is a bijection between generalized oscillating tableaux of length k with n parts such that the last component is $(m, \ldots, m)$ for some $m \in \mathbb{Z}$ and sl(n) webs with k boundary vertices.*

Assuming Conjecture 6.2 holds, we also have the following conjecture. Suppose D is an sl(n) web with chosen leftmost vertex. As before, let $T(D)$ denote the generalized oscillating tableau (conjecturally) associated to D, $p(D)$ denote the result of rotating D one vertex counterclockwise, and $w(D)$ denote the word obtained from the states corresponding to the boundary vertices of D.

Conjecture 6.3. *For any sl(n) web D with chosen leftmost vertex, we have $T(w(p(D))) = p(T(w(D)))$. That is, the generalized oscillating tableau associated with the rotation of D is given by generalized oscillating promotion of the tableau associated with D itself.*

Acknowledgements

The author thanks Hugh Thomas for suggesting this problem and for his support; Chris Fraser, Alexander Garver, and Pasha Pylyavskyy for helpful discussions; and Véronique Bazier-Matte, Guillaume Douville, Alexander Garver, Hugh Thomas, and Emine Yildirim for their collaboration in the LaCIM working group on webs.

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