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The canonical join complex for biclosed sets

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Abstract. The canonical join complex of a semidistributive lattice is a simplicial complex whose faces are canonical join representations of elements of the semidistributive lattice. We give a combinatorial classification of the faces of the canonical join complex of the lattice of biclosed sets of segments supported by a tree, as introduced by the third author and McConville. We also use our classification to describe the elements of the shard intersection order of the lattice of biclosed sets. As a consequence, we prove that this shard intersection order is a lattice.

Résumé. Le complexe sup-canonique d'un treillis semi-distributif est un complexe simplicial dont les faces sont des représentations sup-canoniques d'élé-ments du treillis semi-distributif. Nous donnons une classification combinatoire des faces du complexe sup-canonique du treillis des ensembles bi-fermés de segments sur un arbre, qui ont été introduits par le troisième auteur et McConville. Nous utilisons notre classification pour décrire les éléments de l'ordre d'intersection des tessons du treillis des ensembles bi-fermés. En conséquence, nous prouvons que cet ordre d'intersection des tessons est un treillis.

Keywords: lattice, weak order, canonical join representation, shard intersection order

1 Introduction

In [11], McConville introduced a lattice of biclosed sets as a tool for studying the lattice structure of Grid-Tamari orders. The class of lattices of biclosed sets includes the weak order on permutations. As the weak order on permutations appears in many mathematical contexts, including (and certainly not limited to) geometric combinatorics [15, 10] and representation theory of preprojective algebras [12, 17], it is natural to study the lattice-theoretic aspects of biclosed sets.

In subsequent work by McConville and the third author [7, 8, 6], biclosed sets were used to understand lattice properties of Grid-Tamari orders and oriented flip graphs.

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Futhermore, in [8, 6], they describe the shard intersection order, in the sense of [9], of Grid-Tamari orders and of oriented flip graphs. Neither one of the classes of lattices of biclosed sets used in these two settings contains the other, but both contain the weak order on permutations. However, both classes of lattices belong to the class of lattices of biclosed sets studied in [14]. The goal of this paper is to gain a combinatorial description of this shard intersection order of the lattice of biclosed sets appearing in [8].

The lattice of biclosed sets is a congruence-uniform lattice and therefore a semidistributive lattice. As a semidistributive lattice, the lattice of biclosed sets has a well-defined canonical join complex. The canonical join complex of a semidistributive lattice L is the simplicial complex whose faces are canonical join representations of elements of L. As a congruence-uniform lattice, the lattice of biclosed sets L has a well-defined shard intersection order, denoted $\Psi(L)$. The poset $\Psi(L)$ is an alternative partial order on the elements of L that is constructed using the data of canonical join representations of elements of L. If a finite lattice L is not congruence-uniform, then $\Psi(L)$ may not be a partial order (see [9, Exercise 9.73]). We are not aware of a characterization of lattices whose shard intersection order is a partial order.

Our approach is to, first, describe the join-irreducible biclosed sets (see Proposition 5.1). After that, we use this description to classify the faces of the canonical join complex of biclosed sets (see Theorem 5.3). We are then in a position to describe the elements of the shard intersection order (see Theorem 6.1) and prove that the shard intersection order is a lattice (see Theorem 6.3). The latter was conjectured by the first and second author in [2, Conjecture 6.3]. The full version of our paper can be found at [3].

Generally speaking, it is an open problem in [9, Problem 9.5] to determine which congruence-uniform lattices L have the property that $\Psi(L)$ is a lattice. Theorem 6.3 provides many new examples of congruence-uniform lattices with this property. When L is the weak order on permutations, it was already shown by Reading in [16] that $\Psi(L)$ is a lattice. As the weak order on permutations is an example of a lattice of biclosed sets, our Theorem 6.3 recovers this result. We also remark that in [13, Theorem 1.1], Mühle found a necessary condition on L in order for $\Psi(L)$ to be a lattice: he showed that the Möbius function on L, denoted $\mu_L(-,-)$, must satisfy $\mu_L(\hat{0},\hat{1}) \neq 0$ if $\Psi(L)$ is a lattice.

The paper is organized as follows. We remind the reader of the lattice theory that we will use throughout the paper in Section 2. We describe the lattices of biclosed sets we will work with in Section 3. In Section 4, we construct a special labeling of the covering relations in lattices of biclosed sets. In Section 5, we use this labeling to describe the join- and meet-irreducible biclosed sets and the faces of the canonical join complex of biclosed sets. We study the shard intersection order of biclosed sets in Section 6.

2 Lattices

Let $(L, \leq_L)$ be a finite lattice. For $x, y \in L$, if $x < y$ and there does not exist $z \in L$ such that $x < z < y$, we write $x \lessdot y$. Let $\mathrm{Cov}(L) := \{(x,y) \in L^2 \mid x \lessdot y\}$ be the set of **covering relations** of L. We let $\hat{0}, \hat{1} \in L$ denote the unique minimal and unique maximal elements of L, respectively.

A set map $\lambda : \mathrm{Cov}(L) \to Q$, where $(Q, \leq_Q)$ is some poset is called an **edge labeling**. We review the concepts of join- and meet-irreducibility in order to discuss an important type of edge labeling.

We say that an element $j \in L$ is **join-irreducible** if $j \neq \hat{0}$ and whenever $j = x \vee y$, one has that $j = x$ or $j = y$. **Meet-irreducible** elements $m \in L$ are defined dually. We denote the subset of join-irreducible (resp., meet-irreducible) elements by $\mathrm{JI}(L)$ (resp., $\mathrm{MI}(L)$). For j (resp., m) in $\mathrm{JI}(L)$ (resp., $\mathrm{MI}(L)$), we let j_* (resp., m^*) denote the unique element of L covered by (resp., that covers) j (resp., m).

For $A \subseteq L$, the expression $\bigvee A := \bigvee_{a \in A} a$ is **irredundant** if there does not exist a proper subset $A' \subsetneq A$ such that $\bigvee A' = \bigvee A$. Given $A, B \subseteq \mathrm{JI}(L)$ such that $\bigvee A$ and $\bigvee B$ are irredundant and $\bigvee A = \bigvee B$, we set $A \preceq B$ if for any $a \in A$ there exists $b \in B$ with $a \leq b$. If $x \in L$ and $A \subseteq \mathrm{JI}(L)$ such that $x = \bigvee A$ is irredundant, we say $\bigvee A$ is a **canonical join representation** of x if $A \preceq B$ for any other irredundant join representation $x = \bigvee B,\ B \subseteq \mathrm{JI}(L)$. Dually, one defines **canonical meet representations**.

We define the **canonical join complex** of L, denoted $\Delta^{CJ}(L)$, to be the abstract simplicial complex whose vertex set is $\mathrm{JI}(L)$ and whose faces are sets of join-irreducibles whose join is a canonical join representation of some element of L.

Now we assume that L is a **semidistributive** lattice. This means that for any three elements $x, y, z \in L$, the following properties hold:

- if $x \wedge z = y \wedge z$, then $(x \vee y) \wedge z = x \wedge z$, and
- if $x \vee z = y \vee z$, then $(x \wedge y) \vee z = x \vee z$.

It is known that a lattice L is semidistributive if and only if each element of L has a canonical join representation and a canonical meet representation [5, Theorem 2.24]. In this case, there is a canonical bijection $L \to \Delta^{CJ}(L)$ sending $x \mapsto A$ where $\bigvee A$ is the canonical join representation of x.

With these notions in hand, we arrive at the notions of **CN-** and **CU-labeling**, the latter of which plays a prominent role in this paper.

Definition 2.1. *A labeling $\lambda : Cov(L) \to Q$ is a **CN-labeling** if L and its dual L^* satisfy the following: given $x, y, z \in L$ with $(z, x), (z, y) \in Cov(L)$ and maximal chains C_1 and C_2 in $[z, x \vee y]$ with $x \in C_1$ and $y \in C_2$,*

(CN1) the elements $x' \in C_1, y' \in C_2$ such that $(x', x \vee y), (y', x \vee y) \in Cov(L)$ satisfy

$$\lambda(x', x \vee y) = \lambda(z, y), \lambda(y', x \vee y) = \lambda(z, x);$$

(CN2) if $(u,v) \in Cov(C_1)$ with $z < u, v < x \vee y$, then $\lambda(z,x), \lambda(z,y) <_Q \lambda(u,v)$;
(CN3) the labels on $Cov(C_1)$ are pairwise distinct.
*We say that λ is a **CU-labeling** if, in addition, it satisfies*
(CU1) $\lambda(j_*, j) \neq \lambda(j'_*, j')$ for $j, j' \in JI(L)$, $j \neq j'$, and
(CU2) $\lambda(m, m^*) \neq \lambda(m', m'^*)$ for $m, m' \in MI(L)$, $m \neq m'$.
*If L admits a CU-labeling, it is said to be **congruence-uniform**.*

It was proved by Day in [4] that any finite congruence-uniform lattice is also semidistributive.

We now mention some general properties of CU-labelings and the definition of the shard intersection order of L. Given an edge labeling $\lambda : \mathrm{Cov}(L) \to Q$, one defines

$$\lambda_\downarrow(x) := \{\lambda(y,x) : \ y \lessdot x\}, \quad \lambda^\uparrow(x) := \{\lambda(x,z) : \ x \lessdot z\}.$$

Lemma 2.2 ([8, Lemma 2.6]). *Let L be a congruence-uniform lattice with CU-labeling $\lambda : Cov(L) \to P$. For any $s \in P$, there is a unique join-irreducible $j \in JI(L)$ (resp., meet-irreducible $m \in MI(L)$) such that $\lambda(j_*, j) = s$ (resp., $\lambda(m, m^*) = s$). Moreover, this join-irreducible j (resp., meet-irreducible m) is the minimal (resp., maximal) element of L such that $s \in \lambda_\downarrow(j)$ (resp., $s \in \lambda^\uparrow(m)$).*

Later, in Propositions 5.1 and 5.2, we use Lemma 2.2 to characterize join- and meet-irreducible elements of $\mathrm{Bic}(T)$, the lattice of biclosed sets defined in the next section.

One can also use CU-labelings to determine canonical join representations and canonical meet representations of elements of a congruence-uniform lattice. We state this precisely as follows.

Lemma 2.3 ([8, Proposition 2.9]). *Let L be a congruence-uniform lattice with CU-labeling λ. For any $x \in L$, the canonical join representation of x is $\bigvee D$, where $D = \{j \in JI(L) : \lambda(j_*, j) \in \lambda_\downarrow(x)\}$. Dually, for any $x \in L$, the canonical meet representation of x is $\bigwedge U$, where $U = \{m \in MI(L) : \ \lambda(m, m^*) \in \lambda^\uparrow(x)\}$.*

Given a lattice L with a CU-labeling, one can define a new partial order on the elements of L known as the **shard intersection order** of L. Reading introduced this concept in [9].

Definition 2.4. *Let L be a congruence-uniform lattice with CU-labeling $\lambda : Cov(L) \to P$. Let $x \in L$ and let $y_1, \ldots y_k$ be the elements of L satisfying $(y_i, x) \in Cov(L)$. Define the **shard intersection order** of L, denoted $\Psi(L)$, to be the collection of sets of the form*

$$\psi(x) := \{\lambda(w,z) \mid \bigwedge_i^k y_i \leq w < z \leq x, (w,z) \in Cov(L)\}$$

*partially ordered by inclusion. We may refer to the interval $[\bigwedge_i^k y_i, x]$ as a **facial interval**.*

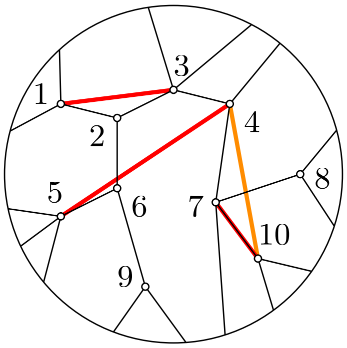


Figure 1: Acyclic paths $[1,3], [5,4]$, $[7,10]$ are segments, but $[4,10]$ is not.

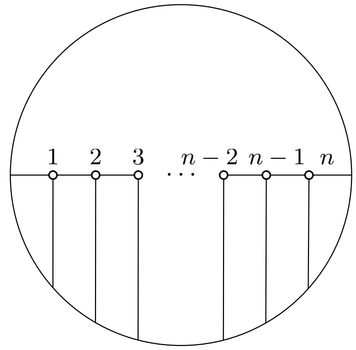


Figure 2: A tree whose biclosed sets are permutations in $\mathfrak{S}_n$.

Remark 2.5. *The shard intersection order was originally defined by Reading in [16] given the data of Pos$(\mathcal{A}, B)$, the poset of regions of a simplicial hyperplane arrangement $\mathcal{A}$ with base region B. Such posets are congruence-uniform lattices, as shown by Reading in [9, Proposition 9-3.33 and Corollary 9-7.22]. It follows from [9, Proposition 9-7.13] that Reading's original definition of the shard intersection order of Pos$(\mathcal{A}, B)$ coincides with the definition of $\Psi(Pos(\mathcal{A}, B))$. The definition of $\Psi(L)$ where L is any congruence-uniform lattice is therefore a generalization of Reading's original definition of the shard intersection order.*

3 Biclosed sets

A **tree** is a finite connected acyclic graph. The degree-one vertices of a tree are called **leaves**. We can always embed a tree T into the disk D^2 so that exactly the leaves lie on the boundary. Unless stated otherwise, a tree is assumed to be equipped with such an embedding. Non-leaf vertices of T are thus in the interior of D^2, and we call these **interior vertices**. We also assume that the interior vertices of T have degree at least 3.

An **acyclic path** is a sequence of pairwise distinct vertices $(v_{i_1}, \ldots, v_{i_n})$ of T such that there is an edge connecting v_{i_j} and $v_{i_{j+1}}$ for all $1 \leq j \leq n-1$. Since an acyclic path is uniquely determined by its endpoints, we can denote $(v_{i_1}, \ldots, v_{i_n})$ by $[v_{i_1}, v_{i_n}]$.

Observe that the embedding of T in D^2 determines a cyclic ordering of the edges of T that are incident to a given vertex. An acyclic path $(v_{i_1}, \ldots, v_{i_n})$ is called a **segment** if, for each $1 \leq j \leq n-2$, $[v_{i_{j+1}}, v_{i_{j+2}}]$ is immediately clockwise or counterclockwise from $[v_{i_j}, v_{i_{j+1}}]$ with respect to the cyclic ordering on the edges incident to $v_{i_{j+1}}$. The set of all segments supported by a tree T is denoted by Seg(T). Figure 1 shows some examples and non-examples of segments.

Given two segments $s_1 = (v_{i_1}, v_{i_2}, \ldots, v_{i_k}), s_2 = (v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_n}) \in$ Seg(T) that share an endpoint v_{i_k} but differ at all other vertices, we define their **composition** to be the acyclic path $s_1 \circ s_2 := (v_{i_1}, \ldots, v_{i_k}, \ldots, v_{i_n})$. We say that two segments s_1 and s_2 are **composable** if $s_1 \circ s_2 \in \text{Seg}(T)$. A subset $B \subset \text{Seg}(T)$ is **closed** if for all composable $s_1, s_2 \in B, s_1 \circ s_2 \in B$. The set B is **biclosed** if both B and its complement, $B^c := \text{Seg}(T) \backslash B$, are closed. Additionally, if $B \subset \text{Seg}(T)$, we let $\overline{B}$ denote the smallest closed set containing B. We let $\text{Bic}(T)$ denote the poset of biclosed subsets of $\text{Seg}(T)$ ordered by inclusion.

We remark that although the poset $\text{Bic}(T)$ depends on the embedding of T, each tree T is equipped with an embedding in D^2 so it makes sense to write $\text{Bic}(T)$. The poset structure of $\text{Bic}(T)$ is studied in [8], where the following result is proved:

Theorem 3.1 ([8, Theorem 4.1])**.** *The poset $Bic(T)$ is a congruence-uniform lattice. Moreover, the poset $Bic(T)$ has the following properties:*

1. for any $X, Y \in Bic(T)$, if $X < Y$, there is a segment $y \in Y$ such that $\overline{X \sqcup \{y\}} \in Bic(T)$;

2. for any $W, X, Y \in Bic(T)$ with $W \leq X \cap Y$, the set $W \cup \overline{(X \cup Y) \backslash W}$ is biclosed;

3. the edge-labeling $\lambda : Cov(Bic(T)) \to Seg(T)$ defined by $\lambda(X, Y) = s$ if $Y \backslash X = \{s\}$ is a CN-labeling.

Theorem 3.1 implies that the map $(-)^c \colon \text{Bic}(T) \to \text{Bic}(T)$, $B \mapsto B^c$, gives rise to a bijection $\text{JI}(\text{Bic}(T)) \to \text{MI}(\text{Bic}(T))$. We frequently use the next lemma, which follows from property (2) in Theorem 3.1 with $X = B_1$, $Y = B_2$, and $W = \emptyset$.

Lemma 3.2. *For any $B_1, B_2 \in Bic(T)$, we have $B_1 \vee B_2 = \overline{B_1 \cup B_2}$.*

Example 3.3. *Let T be the tree shown in Figure 2 with the indicated labeling of the interior vertices. Define a map that sends a segment s to $(i, j) \in \mathbb{N}^2$ with $i < j$ where i and j are the vertex labels of the endpoints of s. This induces a map on biclosed sets that sends each biclosed set to the inversion set of a permutation in $\mathfrak{S}_n$. Moreover, it induces a poset isomorphism $Bic(T) \to Weak(\mathfrak{S}_n)$ where the latter denotes the weak order on permutations.*

Additionally, it follows from [9, Theorem 10-3.1] that $Weak(\mathfrak{S}_n)$ is isomorphic to $Pos(\mathcal{A}, B)$ where $\mathcal{A}$ is Coxeter arrangement of $\mathfrak{S}_n$ and B is the region of the Coxeter arrangement containing the identity permutation. Now it follows from Remark 2.5 that the class of lattices of the form $\Psi(Bic(T))$ includes the shard intersection orders of type A Coxeter arrangements.

4 A CU-labeling of $\text{Bic}(T)$

In [8], the authors prove that $\text{Bic}(T)$ is congruence-uniform, and thus it admits a CU-labeling. In this section, we explicitly construct such a labeling.

We say a segment $s \in \text{Seg}(T)$ is a **split** of a segment t if s is a proper subsegment of t, and s and t share an endpoint. A **break** of a segment $[a, c]$ is a pair of splits of $[a, c]$, denoted $\{[a, b], [b, c]\}$, for some vertex b of segment $[a, c]$ where $b \neq a$ and $b \neq c$. We say that b is the **faultline** of the break $\{[a, b], [b, c]\}$.

Define a poset $\mathcal{S}_T$ whose elements are of the form $(s, \{s_1, s_2, \ldots, s_m\}) \in \text{Seg}(T) \times 2^{\text{Seg}(T)}$ with the following properties:

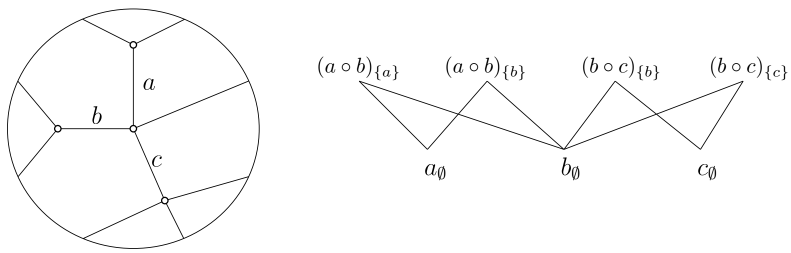
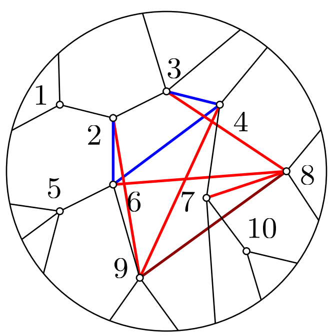


Figure 3: A tree T and the corresponding poset of labels $\mathcal{S}_T$. The shortest segments of T are labeled a, b, and c.



$$\begin{aligned} s &= [9,8] \\ \mathcal{D} &= \{[9,2],[9,4],[6,8],[3,8],[7,8]\} \\ S([9,2]) &= \{[6,2]\} \\ S([9,4]) &= \{[6,4],[3,4]\} \\ S([6,8]) &= \{[6,2],[6,4]\} \\ S([3,8]) &= \{[3,4]\} \\ S([7,8]) &= \emptyset. \end{aligned}$$

Figure 4: The join irreducible biclosed set $B = J([9,8]_{\{[9,2],[9,4],[6,8],[3,8],[7,8]\}})$. One checks that $\widetilde{\lambda}(B\backslash\{[9,8]\}, B) = [9,8]_{\{[9,2],[9,4],[6,8],[3,8],[7,8]\}}$

- $s = (v_0, v_1, \ldots, v_{m+1})$ has m breaks,
- each s_i is a split of s, and
- two distinct splits s_i and s_j do not appear in the same break of s.

We will typically denote $(s, \{s_1, s_2, \ldots, s_m\}) \in \mathcal{S}_T$ by $s_{\mathcal{D}} = s_{\{s_1,s_2,\ldots,s_m\}}$. The elements of $\mathcal{S}_T$ are partially ordered as follows: given $s_{\{s_1,s_2,\ldots,s_k\}}, s'_{\{t_1,t_2,\ldots,t_l\}} \in \mathcal{S}_T$, one has $s'_{\{t_1,t_2,\ldots,t_l\}} \leq s_{\{s_1,s_2,\ldots,s_k\}}$ if s' is a proper subsegment of s. At times, we will also write $s' \subseteq s$ (resp., $s' \not\subseteq s$) to indicate that s' is a subsegment (resp., is not a subsegment) of s. We will refer to elements of $\mathcal{S}_T$ as **labels**.

For an example of this poset of labels, let T be the tree shown in Figure 3. There we also show the poset of labels $\mathcal{S}_T$.

Definition 4.1. *Define a map $\widetilde{\lambda} : Cov(Bic(T)) \to \mathcal{S}_T$ by $\widetilde{\lambda}(B, B \sqcup \{s\}) = s_{\{s_1,s_2,\ldots,s_k\}}$ where $s_1, s_2, \ldots, s_k$ are the splits of s which are contained in B. It is clear that $\widetilde{\lambda}$ is an edge-labeling of $Bic(T)$. If we let $\lambda : Cov(Bic(T)) \to Seg(T)$ denote the first coordinate function of $\widetilde{\lambda}$, we have that λ is the CN-labeling of $Bic(T)$ from Theorem 3.1 (3) (see Figure 4).*

Theorem 4.2. *The edge-labeling $\widetilde{\lambda} : Cov(Bic(T)) \to \mathcal{S}_T$ is a CU-labeling of $Bic(T)$.*

5 Canonical join complex of Bic(T)

In this section, we classify the join- and meet-irreducible biclosed sets. We do this by choosing a label $s_{\mathcal{D}} \in \mathcal{S}_T$ and constructing the minimal biclosed set B where $\widetilde{\lambda}_{\downarrow}(B) = \{s_{\mathcal{D}}\}$ and by constructing the maximal biclosed set where $\widetilde{\lambda}^{\uparrow}(B) = \{s_{\mathcal{D}}\}$.

Given $s_{\mathcal{D}} \in \mathcal{S}_T$, define

$$J(s_{\mathcal{D}}) := \{s\} \sqcup \mathcal{D} \sqcup \bigcup_{t \in \mathcal{D}} S(t)$$

where $S(t) = S(t, \mathcal{D}) \subset \mathrm{Seg}(T)$ is defined to be the set of all splits s' of t satisfying the following:

i) segment s' is not a split of s, and

ii) segment s' is not composable with any segment in $\mathcal{D}$ (see Figure 4).

Proposition 5.1. *The set $J(s_{\mathcal{D}})$ satisfies $\widetilde{\lambda}_{\downarrow}(J(s_{\mathcal{D}})) = \{s_{\mathcal{D}}\}$. Moreover, any biclosed set B with $s_{\mathcal{D}} \in \widetilde{\lambda}_{\downarrow}(B)$ satisfies $J(s_{\mathcal{D}}) \leq B$, and the reverse containment holds if and only if $\widetilde{\lambda}_{\downarrow}(B) = \{s_{\mathcal{D}}\}$. Consequently, the map $J : \mathcal{S}_T \to JI(Bic(T))$ is a bijection.*

Next, we classify the meet-irreducible biclosed sets. Given $s_{\mathcal{D}} \in \mathcal{S}_T$, define

$$M(s_{\mathcal{D}}) := J(s_{\mathcal{D}}) \backslash \{s\} \sqcup \{t \in \mathrm{Seg}(T) : \ t \not\subseteq s\} \sqcup \bigcup_{t \in \mathcal{D}} R(t)$$

where $R(t) = R(t, \mathcal{D}) \subset \mathrm{Seg}(T)$ is defined to be the set of all splits s' of t satisfying the following:

i) segment s' is not a split of s, and

ii) segment s' is composable with some element of $\mathcal{D}$.

Observe that if $s' \in R(t)$ for some $t \in \mathcal{D}$, then there is necessarily a unique element $t' \in \mathcal{D}$ where $t' \subseteq t$, t' is composable with s', and $s' \circ t' = t$.

Proposition 5.2. *The set $M(s_{\mathcal{D}})$ satisfies $\widetilde{\lambda}^{\uparrow}(M(s_{\mathcal{D}})) = \{s_{\mathcal{D}}\}$. Moreover, any biclosed set B with $s_{\mathcal{D}} \in \widetilde{\lambda}^{\uparrow}(B)$ satisfies $B \leq M(s_{\mathcal{D}})$, and the reverse containment holds if and only if $\widetilde{\lambda}^{\uparrow}(B) = \{s_{\mathcal{D}}\}$. Consequently, the map $M : \mathcal{S}_T \to MI(Bic(T))$ is a bijection.*

We now describe the faces of the canonical join complex of Bic(T). In [1, Theorem 1.1], it is shown that the canonical join complex of a finite semidistributive lattice L is a **flag complex**. That is, the minimal nonfaces of $\Delta^{CJ}(L)$ have size two. Thus, it is enough to determine the pairs of elements of JI(Bic(T)) that join canonically.

Theorem 5.3. *A collection $\{J(s^1_{\mathcal{D}^1}), \ldots, J(s^k_{\mathcal{D}^k})\} \subset JI(Bic(T))$ is a face of $\Delta^{CJ}(Bic(T))$ if and only if labels $s^i_{\mathcal{D}^i}$ and $s^j_{\mathcal{D}^j}$ satisfy the following:*

1) segments s^i and s^j are distinct,

2) neither s^i nor s^j is a composition of at least two segments in $J(s^i_{\mathcal{D}^i}) \cup J(s^j_{\mathcal{D}^j})$, and

3) neither $J(s^i_{\mathcal{D}^i}) \leq J(s^j_{\mathcal{D}^j})$ nor $J(s^j_{\mathcal{D}^j}) \leq J(s^i_{\mathcal{D}^i})$.

for any distinct $i, j \in \{1, \ldots, k\}$.

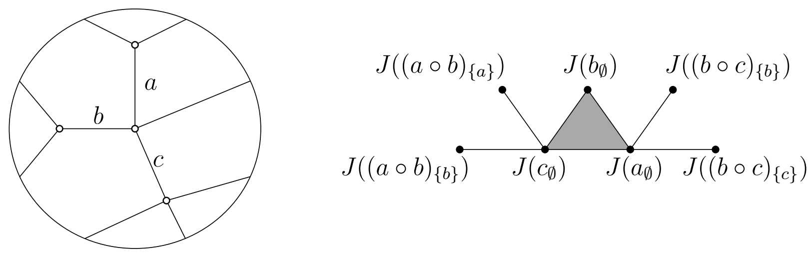


Figure 5: A tree T and the canonical join complex $\Delta^{CJ}(\mathrm{Bic}(T))$. The shortest segments of T are labeled a, b, and c.

Let T be the tree shown in Figure 5. In this same figure, we show the canonical join complex of the biclosed sets of T. Each join-irreducible of $\mathrm{Bic}(T)$ is written next to its corresponding vertex of $\Delta^{CJ}(\mathrm{Bic}(T))$.

6 The shard intersection order of Bic(T)

Let $B \in \mathrm{Bic}(T)$ be a biclosed set that covers exactly the following biclosed sets: $B_1, B_2, \ldots, B_k$. Let $s_{i\mathcal{D}_i} = \widetilde{\lambda}(B_i, B)$ for $i = 1, \ldots, k$ and $\lambda_\downarrow(B) = \{s_1, s_2, \ldots, s_k\}$ where $s_i = \lambda(B_i, B)$ for $i = 1, \ldots, k$. Now fix a segment $s \in \overline{\lambda_\downarrow(B)}$ expressed as $s = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_\ell}$ with each $s_{i_j} \in \lambda_\downarrow(B)$. If $t \in \mathrm{Seg}(T)$ is a split of s that can be expressed as either $t = s_{i_1} \circ \cdots \circ s_{i_j}$ for some $j = 1, \ldots, \ell - 1$ or $t = s_{i_j} \circ \cdots \circ s_{i_\ell}$ for some $j = 2, \ldots, \ell$, we say that t is a **faultline split** of s. Otherwise, we say that t is a **non-faultline split** of t.

Theorem 6.1. *Given a biclosed set $B \in Bic(T)$, we have that $\psi(B)$ is the set of all labels of the form $(s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_\ell})_{\mathcal{D}}$ with $s_{i_j} \in \lambda_\downarrow(B)$ where $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_\ell}$ is any element of $\overline{\lambda_\downarrow(B)}$ and where $\mathcal{D}$ is any set of segments that satisfies the following properties:*

(i) $|\mathcal{D}| = |\{$breaks of $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_\ell}\}|$,

(ii) each segment $t \in \mathcal{D}$ is a split of $s_{i_1} \circ \cdots \circ s_{i_\ell}$,

(iii) no two distinct splits $t_1, t_2 \in \mathcal{D}$ appear in the same break of $s_{i_1} \circ \cdots \circ s_{i_\ell}$, and

(iv) whenever $t \in \mathcal{D}$ is a non-faultline split of $s_{i_1} \circ \cdots \circ s_{i_\ell}$, we have that $t = s_{i_1} \circ \cdots \circ s_{i_{j-1}} \circ t_j$ for some $j = 1, \ldots, \ell$ and some $t_j \in \mathcal{D}_{i_j}$ or $t = t_j \circ s_{i_{j+1}} \circ \cdots \circ s_{i_\ell}$ for some $j = 1, \ldots, \ell$ and some $t_j \in \mathcal{D}_{i_j}$. In the former case if $j = 1$, we mean $t = t_1$, and in the latter case, if $j = \ell$, we mean $t = t_\ell$.

Example 6.2. *Let T be the tree on the left in Figure 6, and let T' be the tree on the right. In Figure 7, we show the shard intersection order of Bic(T). The atoms in this lattice are the 9 labels in $\mathcal{S}_T$. The presence of a dashed segment s indicates that both labels $s_{\mathcal{D}}$ and $s_{\mathcal{D}'}$ belong to the corresponding set $\psi(B)$. This indicates that given $B \in Bic(T)$, one has*

$$|\psi(B)| = |\{\textit{dark red segments in } \psi(B)\}| + 2|\{\textit{dashed segments in } \psi(B)\}|.$$

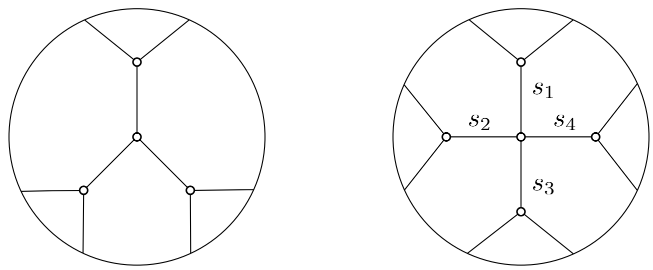


Figure 6: The trees from Example 6.2.

The shard intersection order of biclosed sets fails to be graded in general, although $\Psi(Bic(T))$ *is graded of rank 3. For more information, we refer the reader to [3, Example 5.2]. The trees T and* T' *in this example belong to the one parameter family of trees that are completely determined by the choice of degree on their central vertex. In [2, Conjecture 6.13], the first and second authors conjectured that for this one parameter family of trees* $\Psi(Bic(T))$ *is graded if and only if n is odd.*

We arrive at our main theorem.

Theorem 6.3. *The shard intersection order* $\Psi(Bic(T))$ *is a lattice.*

Since $\Psi(\mathrm{Bic}(T))$ is a finite poset whose unique maximal element is $\psi(\mathrm{Seg}(T)) = \mathcal{S}_T$, it is enough to show that $\psi(B) \cap \psi(B') \in \Psi(\mathrm{Bic}(T))$ for any $B, B' \in \mathrm{Bic}(T)$. We sketch our approach below.

Let $\psi(B), \psi(B') \in \Psi(\mathrm{Bic}(T))$ and let $B_1, \ldots, B_k \in \mathrm{Bic}(T)$ (resp., $B'_1, \ldots, B'_l \in \mathrm{Bic}(T)$) be all of the biclosed sets covered by B (resp., B'). As above, set $s_j := \lambda(B_j, B)$ for $j = 1, \ldots, k$ and $t_j := \lambda(B'_j, B')$ for $j = 1, \ldots, l$. To show that $\psi(B) \cap \psi(B') \in \Psi(\mathrm{Bic}(T))$ we construct a biclosed set B'' that satisfies $\psi(B'') = \psi(B) \cap \psi(B')$. With this goal in mind, we let $\{s^{(i)}_{\mathcal{D}^{(i)}}\}_{i=1}^{\ell}$ denote the elements of $\psi(B) \cap \psi(B')$ where $s^{(i)}$ appears in exactly one label in $\psi(B) \cap \psi(B')$. We can prove that such a collection of labels exists, and thus it is unique. After that, we show that $\bigvee_{i=1}^{\ell} J(s^{(i)}_{\mathcal{D}^{(i)}})$ is a canonical join representation using Theorem 5.3. We then use Theorem 6.1 to prove that $\psi\left(\bigvee_{i=1}^{\ell} J(s^{(i)}_{\mathcal{D}^{(i)}})\right) = \psi(B) \cap \psi(B')$.

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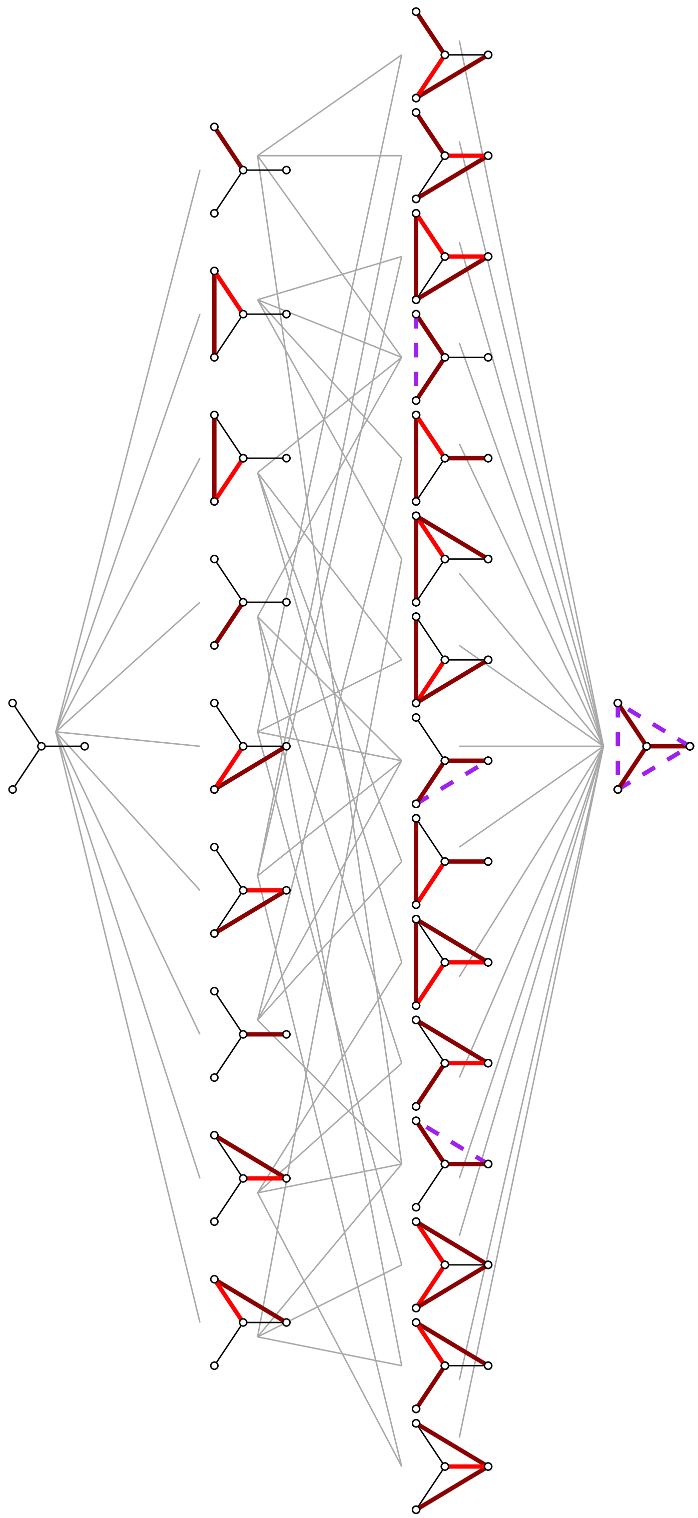

Figure 7: The shard intersection order $\Psi(\mathrm{Bic}(T))$ when T is the tree on the left in Figure 6.

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