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A bijective proof and generalization of Siladić's Theorem

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Abstract. In a recent paper, Dousse introduced a refinement of Siladić's theorem on partitions, where parts occur in two primary and three secondary colors. Her proof used the method of weighted words and q-difference equations. The purpose of this extended abstract is to sketch a bijective proof of Dousse's theorem and show how it can be generalized from two primary colors to an arbitrary number of primary colors.

Keywords: Siladić's theorem, partitions with difference conditions, bijection

1 Introduction

In this paper, we mostly denote by $\lambda_1 + \cdots + \lambda_s$ a partition of a non-negative integer n. For any $x, q \in \mathbb{C}$ with $|q| < 1$, and $n \in \mathbb{N}$, we define

$$(x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \ ,$$

with the convention $(x;q)_0 = 1$, and

$$(x;q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k) \cdot$$

Recall the Rogers–Ramanujan identities [8], which state that for $a \in \{0, 1\}$,

$$\sum_{n \geq 0} \frac{q^{n(n+a)}}{(q;q)_n} = \frac{1}{(q^{1+a};q^5)_\infty (q^{4-a};q^5)_\infty} \cdot \tag{1.1}$$

These identities give an equality between the cardinalities of two sets of partitions: the set of partitions of n with parts differing by at least two and greater than a, and the set of partitions of n with parts congruent to $1+a, 4-a \mod 5$. In the spirit of these identities, a q-series or combinatorial identity is said to be of Rogers–Ramanujan type if it links some sets of partitions with certain difference conditions to others with certain

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congruence conditions. Another well-known example is Schur's partition theorem [9], which states that the number of partitions of n into parts congruent to $\pm 1 \mod 6$ is equal to the number of partitions of n where parts differ by at least three and multiples of 3 differ by at least six.
A rich source of such identities is the representation theory of Lie algebras. This has its origins in work of Lepowsky and Wilson [6], who proved the Rogers–Ramanujan identities by using representations of the affine Lie algebra $\mathfrak{sl}_2(\mathbb{C})^{\sim}$.
Our motivation in this paper is one such identity given by Siladić [10] in his study of representations of the twisted affine Lie algebra $A_2^{(2)}$.

Theorem 1.1 (Siladić). *The number of partitions $\lambda_1 + \cdots + \lambda_s$ of an integer n into parts different from 2 such that $\lambda_i - \lambda_{i+1} \geq 5$ and*

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \equiv \pm 3 \mod 16\,,$$
$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i + \lambda_{i+1} \equiv 0, \pm 4, 8 \mod 16\,,$$
$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i + \lambda_{i+1} \equiv \pm 1, \pm 5, \pm 7 \mod 16\,,$$
$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i + \lambda_{i+1} \equiv 0, \pm 2, \pm 6, 8 \mod 16\,,$$

is equal to the number of partitions of n into distinct odd parts.

The above theorem has recently been refined by Dousse [5] (see Theorem 1.2). She was inspired by the original method of weighted words, first introduced by Alladi and Gordon [2] to generalize Schur's partition theorem, but proceeded in a different way. Her framework is as follows: we consider that parts are colored by two primary colors a, b and secondary colors a^2, b^2, ab, where the colored parts are ordered as

$$1_{ab} <_c 1_a <_c 1_{b^2} <_c 1_b <_c 2_{ab} <_c 2_a <_c 3_{a^2} <_c 2_b <_c 3_{ab} <_c 3_a <_c 3_{b^2} <_c 3_b <_c \cdots \tag{1.2}$$

Note that only odd parts can be colored by a^2, b^2. In terms of q-series, the part k_c with length k and color c will be cq^k. The dilation in the q-series

$$q \to q^4\,, a \to aq^{-3}\,, b \to bq^{-1}\,,$$

leads to the natural order on $\mathbb{N}$

$$0_{ab} < 1_a < 2_{b^2} < 3_b < 4_{ab} < 5_a < 6_{a^2} < 7_b < 8_{ab} < 9_a < 10_{b^2} < 11_b < \cdots \tag{1.3}$$

In fact, the dilation gives a bijection the colored parts and the natural numbers, since we have the following mapping:

$$\begin{array}{cccccccc} 8k & 8k+1 & 8k+2 & 8k+3 & 8k+4 & 8k+5 & 8k+6 & 8k+7 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ (2k+1)_{ab} & (2k+1)_a & (2k+1)_{b^2} & (2k+1)_b & (2k+2)_{ab} & (2k+2)_a & (2k+3)_{a^2} & (2k+2)_b \end{array}. \tag{1.4}$$

Then the minimal differences (minimal values of $\lambda_i - \lambda_{i+1}$) in Siladič theorem, where parts are natural numbers, can be expressed in terms of colored parts by the table

$\lambda_i \backslash \lambda_{i+1}$	a^2_{odd}	a_{odd}	a_{even}	b^2_{odd}	b_{odd}	b_{even}	ab_{odd}	ab_{even}
a^2_{odd}	4	4	3	4	4	3	4	3
a_{odd}	2	2	3	2	2	3	2	1
a_{even}	3	3	2	3	3	2	3	2
b^2_{odd}	2	2	3	4	4	3	2	3
b_{odd}	2	2	1	2	2	3	2	1
b_{even}	1	1	2	3	3	2	1	2
ab_{odd}	2	2	3	4	4	3	2	3
ab_{even}	3	3	2	3	3	2	3	2

. (1.5)

We set a relation $\gg_c$ on colored parts, defined by $k_p \gg_c l_r$ if and only if $k - l$ is at least the minimal difference in (1.5). One can check that $\gg_c$ is transitive, and since $k_p \not\gg_c k_p$, it defines a strict order on the set of colored parts. We are now ready to state Dousse's refinement of Siladič theorem.

Theorem 1.2 (Dousse)**.** *Let $(u,v,n) \in \mathbb{N}^3$. Denote by $\mathcal{D}(u,v,n)$ the set of all the partitions of n, such that no part is equal to $1_{ab}, 1_{a^2}$ or 1_{b^2}, with difference between two consecutive parts following the minimal conditions in (1.5), and with u equal to the number of parts with color a or ab plus twice the number of parts colored by a^2, and v equal to the number of parts with color b or ab plus twice the number of parts colored by b^2. Denote by $\mathcal{C}(u,v,n)$ the set of all the partitions of n with u distinct parts colored by a and v distinct parts colored by b. We then have $\sharp\mathcal{D}(u,v,n) = \sharp\mathcal{C}(u,v,n)$.*

In terms of q-series, we get the equation

$$\sum_{u,v,n\geq 0} \sharp\mathcal{D}(u,v,n)a^u b^v q^n = \sum_{u,v,n\geq 0} \sharp\mathcal{C}(u,v,n)a^u b^v q^n = (-aq;q)_\infty(-bq;q)_\infty \cdot \tag{1.6}$$

Our purpose here is to build an actual bijection between sets $\mathcal{D}(u,v,n)$ and $\mathcal{C}(u,v,n)$. In fact, we build a bijection between two bigger sets that contain $\mathcal{C}(u,v,n)$ and $\mathcal{D}(u,v,n)$, in such a way that it will imply Theorem 1.2.

The bijective proof will allow us to generalize Dousse's theorem, with an arbitrary number of primary colors. In fact, we consider a set of m primary colors $a_1 < \cdots < a_m$. And we order the parts colored by primary colors in the usual way, first according to size and then according to color (see in (2.1)). We also set m^2 secondary colors $a_i a_j$ with $i,j \in \{1,\ldots,m\}$, in such a way $a_i a_j$ only colors parts with the same parity as $\chi(a_i \leq a_j)$, where $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if not.
We then define a certain strict order $\gg_c$ on parts colored with primary and secondary colors which corresponds to minimal difference conditions between the parts (see Section 2.2). This leads to the following theorem.

Theorem 1.3. *Let $\mathcal{C}(u_1,\dots,u_m,n)$ denote the set of partitions of n with u_k distinct parts with color a_k. Let $\mathcal{D}(u_1,\dots,u_m,n)$ denote the set of partitions of n such that parts are ordered by $\gg_c$, with no part equal to $1_{a_ia_j}$, and with u_i equal to the number of parts colored by a_i, a_ia_j or a_ja_i with $i \neq j$, plus twice the number of parts colored by a_i^2. We have then*

$$\sharp\mathcal{C}(u_1,\dots,u_m,n) = \sharp\mathcal{D}(u_1,\dots,u_m,n)\cdot \tag{1.7}$$

In terms of the q-series, we get the equation

$$\begin{aligned}\sum_{u_1,\dots,u_m,n\geq 0}\sharp\mathcal{D}(u_1,\dots,u_m,n)a_1^{u_1}\cdots a_m^{u_m}q^n &= \sum_{u_1,\dots,u_m,n\geq 0}\sharp\mathcal{C}(u_1,\dots,u_m,n)a_1^{u_1}\cdots a_m^{u_m}q^n\\ &= (-a_1q;q)_\infty\cdots(-a_mq;q)_\infty\cdot\end{aligned} \tag{1.8}$$

This may be compared with work of Corteel and Lovejoy [4] who gave interpretations of the same infinite products above but using 2^m-1 colors instead of m^2+m colors as we do here. As an example, we choose $m=3$ and use a,b,d instead of a_1,a_2,a_3. The table which sums up the minimal differences is

$\lambda_i \backslash \lambda_{i+1}$	a	b	d	a^2	ab	ad	b^2	bd	d^2	ba	da	db
a	2	2	2	2	2	2	2	2	2	1	1	1
b	1	2	2	1	1	1	2	2	2	1	1	1
d	1	1	2	1	1	1	1	1	2	0	1	1
a^2	3	3	3	4	4	4	4	4	4	3	3	3
ab	2	3	3	2	2	2	4	4	4	3	3	3
b^2	2	3	3	2	2	2	4	4	4	3	3	3
ad	2	2	3	2	2	2	2	2	4	1	3	3
bd	2	2	3	2	2	2	2	2	4	1	3	3
d^2	2	2	3	2	2	2	2	2	4	1	3	3
ba	2	2	2	3	3	3	3	3	3	2	2	2
da	2	2	2	3	3	3	3	3	3	2	2	2
db	1	2	2	1	1	1	3	3	3	2	2	2

.

We take the dilation

$$\begin{cases} q\mapsto q^{10}\\ a\mapsto aq^{-6}\\ b\mapsto bq^{-4}\\ d\mapsto dq^{-1}\end{cases},$$

and we get the order

$$\begin{aligned}&1_{ab}<_c 1_{b^2}<_c 1_{ad}<_c 1_a<_c 1_{bd}<_c 1_b<_c 1_{d^2}<_c 1_d<_c\\ &2_{ba}<_c 2_{da}<_c 2_a<_c 2_{db}<_c 2_b<_c 3_{a^2}<_c 2_d<_c 3_{ab}<_c\cdots\end{aligned} \tag{1.9}$$

since the dilation gives in the natural ordering in $\mathbb{N}$

$$\begin{aligned} 0_{ab} < 2_{b^2} < 3_{ad} < 4_a < 5_{bd} < 6_b < 8_{d^2} < 9_d < 10_{ba} < \\ 13_{da} < 14_a < 15_{db} < 14_b < 18_{a^2} < 19_d < 20_{ab} \cdots \end{aligned} \tag{1.10}$$

We notice that $>_c$ is always defined and fixed when we restrict it to parts with primary colors. In fact, it is the total usual order (2.1). But we can see with the previous dilation that there exists some extension of $>_c$ over the parts colored with primary and secondary colors, depending on the dilation. However, the strict order $\gg_c$, that we will explicitly define over the colored parts with primary or secondary colors, is partial and fixed. For the last dilation, we have a corollary in the spirit of Siladić's theorem:

Corollary 1.4. *Let u, v, w, n be non-negative integers. Let $A(u, v, w, n)$ denote the number of partitions of n with respectively u, v, w parts congruent to $4, 6, 9 \mod 10$. Let $B(u, v, w, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n, with*

- *no part equal to $2, 3, 5, 8$ or congruent to $1, 7, 11, 12, 17 \mod 20$,*
- *u equal to the number of parts congruent to $0, 3, 4 \mod 10$ plus twice the number of parts congruent to $18 \mod 20$,*
- *v equal to the number of parts congruent to $0, 5, 6 \mod 10$ plus twice the number of parts congruent to $2 \mod 20$,*
- *w equal to the number of parts congruent to $3, 5, 9 \mod 10$ plus twice the number of parts congruent to $8 \mod 20$,*
- *two consecutive parts differing by at least 9 with the additional conditions for $9 \leq \lambda_i - \lambda_{i+1} \leq 20$ according to the table below:*

$\lambda_i - \lambda_{i+1}$	$\lambda_i \mod 20$	$\lambda_i - \lambda_{i+1}$	$\lambda_i \mod 20$
9	4, 19	15	4, 5, 9, 10, 14, 15, 19
10	$\emptyset$	16	0, 4, 6, 9, 10, 15, 16, 19
11	4, 6, 10, 15	17	0, 3, 6, 10, 13, 15, 16, 19
12	6, 15, 16	18	2, 3, 4, 6, 8, 13, 14, 16
13	3, 6, 9, 16, 19	19	2, 3, 4, 5, 9, 13, 14, 15, 18, 19
14	4, 9, 10, 13, 19	20	0, 3, 4, 5, 6, 9, 10, 13, 14, 15, 16, 19

. (1.11)

Then $A(u, v, w, n) = B(u, v, w, n)$.

The remainder of the paper is organised as follows. In the next section, we discuss the existence of a new color ba different from ab, that will lead to an enumeration of explicit relations for the minimal difference conditions in (1.5). In section 3, we will build our bijection, and after that in the next section, we will sketch the main ideas of its well-definedness. Finally in section 5, we will briefly indicate how to generalize Dousse's theorem to obtain Theorem 1.3.

2 Some fundamental remarks

2.1 A new color $ba \neq ab$

First, we set the order $a < b$ for the primary colors. Then for any $(k,l,p,q) \in \mathbb{N}^2 \times \{a,b\}^2$, the usual order $>_c$ is set by the equivalence

$$k_p >_c l_q \Leftrightarrow k - l \geq \chi(p \leq q) \cdot \tag{2.1}$$

At the same time, we have a strict order $\gg_c$ defined by (1.5), and it follows the equivalence

$$k_p \gg_c l_q \Leftrightarrow k - l \geq 1 + \chi(p \leq q) \tag{2.2}$$

for any $(k,l,p,q) \in \mathbb{N}^2 \times \{a,b\}^2$. We set $\delta_{pq} = \chi(p \leq q)$ for notational convenience and denote by $\mathcal{O} = \mathbb{Z} \times \{a,b\}$ which contains the set of parts colored by a,b (we extend this to $\mathbb{Z}$ for the purpose of our bijection). One can observe that the parts colored by a^2, b^2, ab can be uniquely divided into two parts $k_p, l_q \in \mathcal{O}$ such that $k_p >_c l_q$ and $k_p \not\gg_c l_q$, i.e $k - l = \delta_{pq}$. Specifically, we have

$$\begin{aligned} (2k)_{ab} &= k_b + k_a \\ (2k+1)_{ab} &= (k+1)_a + k_b \\ (2k+1)_{a^2} &= (k+1)_a + k_a \\ (2k+1)_{b^2} &= (k+1)_b + k_b \end{aligned} \; .$$

It is then convenient to set another color ba so that for any $p,q \in \{a,b\}$, pq only colors parts with the same parity as δ_{pq}. Furthermore, the following equality holds:

$$(2k + \delta_{pq})_{pq} = (k + \delta_{pq})_p + k_q \cdot \tag{2.3}$$

This means that $ab \neq ba$, since ab colors only odd parts and ba only even parts.

The above notation allows us to introduce the set $\mathcal{E} = \mathbb{Z} \times \{a,b\}^2$ so that the part $(2k + \delta_{pq})_{pq}$ is uniquely represented by (k,p,q). In fact, we have

$$\begin{aligned} (2k)_{ba} &\leftrightarrow (k,b,a) \\ (2k+1)_{ab} &\leftrightarrow (k,a,b) \\ (2k+1)_{a^2} &\leftrightarrow (k,a,a) \\ (2k+1)_{b^2} &\leftrightarrow (k,b,b) \end{aligned} \; . \tag{2.4}$$

It is reasonable then to set γ, μ the functions which map a part of $\mathcal{E}$ to the unique parts in $\mathcal{O}$ as

$$\gamma(k,p,q) = (k + \delta_{pq}, p) \;, \quad \mu(k,p,q) = (k,q) \cdot \tag{2.5}$$

We call $\gamma(k,p,q)$ and $\mu(k,p,q)$ respectively the **upper** and the **lower** halves of (k,p,q). As an example, the part 40_{ab} considered by Dousse will be in fact the part 40_{ba}, which we

denote $(20, b, a)$ and which is the sum of the unique parts $20_b = (20, b)$ and $20_a = (20, a)$ respectively as its upper half γ and its lower half μ.
With this notation, we notice that a part $(k, p) \in \mathcal{O}$ has an actual length k, while a part $(l, q, r) \in \mathcal{E}$ has $2l + \delta_{qr}$ as actual length.

2.2 Explicit relations for the minimal difference conditions

We just saw in the previous section the necessary and sufficient condition (2.2) to have the minimal difference between two parts in $\mathcal{O}$. We state it as a lemma.

Lemma 2.1. *For any $(k, p), (l, q) \in \mathcal{O}^2$, we have*

$$(k, p) \gg_c (l, q) \Leftrightarrow k - l \geq 1 + \delta_{pq} \cdot \tag{2.6}$$

Now we are going to give analogous conditions for any pair of parts in $\mathcal{O} \times \mathcal{E}, \mathcal{E} \times \mathcal{E}, \mathcal{E} \times \mathcal{O}$, by giving some explicit expressions of the minimal difference conditions given in (1.5) according to the colors involved in the comparison.

Lemma 2.2. *For any $(k, p), (l, q, r) \in \mathcal{O} \times \mathcal{E}$, we have*

$$(k, p) \gg_c (l, q, r) \Leftrightarrow k - (2l + \delta_{qr}) \geq \delta_{pq} + \delta_{qr} \Leftrightarrow (k, p) >_c (2(l + \delta_{qr}), q) \cdot \tag{2.7}$$

Lemma 2.3. *For any $(k, p, q), (l, r) \in \mathcal{E} \times \mathcal{O}$, we have*

$$(k, p, q) \gg_c (l, r) \Leftrightarrow (2k + \delta_{pq}) - l \geq 1 + \delta_{pq} + \delta_{qr} \Leftrightarrow (2k, q) \gg_c (l, r) \cdot \tag{2.8}$$

Lemma 2.4. *For any $(k, p, q), (l, r, s) \in \mathcal{E}^2$, we have*

$$(k, p, q) \gg_c (l, r, s) \Leftrightarrow (2k + \delta_{pq}) - (2l + \delta_{rs}) \geq \delta_{pq} + 2\delta_{qr} + \delta_{rs} \cdot \tag{2.9}$$

Furthermore, the last equality is equivalent to

$$(k, p, q) \gg_c (l, r, s) \Leftrightarrow k - (l + \delta_{rs}) \geq \delta_{qr} \Leftrightarrow \mu(k, p, q) >_c \gamma(l, r, s) \cdot \tag{2.10}$$

Condition (2.10) is the most important in our construction. This comes from the fact that comparing two parts in $\mathcal{E}$ in terms of $\gg_c$ is the same as comparing the lower half of the first part and the upper half of the second part using $>_c$.

3 How do we build the bijection?

We build our bijection in the spirit of the bijective proof of the partition theorem of K. Alladi [1] given by Padmavathamma, R. Raghavendra, and B. M. Chandrashekara [7]. The idea was introduced by Bressoud [3] in his bijective proof of Schur's theorem.
Denote by $\mathcal{C}$ the set of partitions with parts in primary colors, i.e in $\mathcal{O}' = \mathbb{N}^* \times \{a, b\}$. Also denote by $\mathcal{D}$ the set of partitions with parts colored by primary or secondary colors, with no part equal to $1_{a^2}, 1_{b^2}, 1_{ab}$, i.e with parts in $\mathcal{E}' = \mathbb{N}^* \times \{a, b\}^2$ or in $\mathcal{O}'$, such that the colored parts are ordered by $\gg_c$.

3.1 The bijection's key operation Λ

Let us set the function Λ as

$$\begin{array}{cccc}\Lambda & : & \mathcal{O}\times\mathcal{E} & \longrightarrow & \mathcal{E}\times\mathcal{O} \\ & & (k,p),(l,q,r) & \longmapsto & (l+\delta_{qr},p,q),(k-\delta_{pq}-\delta_{qr},r)\end{array}. \tag{3.1}$$

The function Λ is invertible with

$$\begin{array}{cccc}\Lambda^{-1} & : & \mathcal{E}\times\mathcal{O} & \longrightarrow & \mathcal{O}\times\mathcal{E} \\ & & (k,p,q),(l,r) & \longmapsto & (l+\delta_{pq}+\delta_{qr},p),(k-\delta_{qr},q,r)\end{array}. \tag{3.2}$$

We explicitly have the following table

$_{(k,p)}\times^{(l,q,r)}$	(l,a,a)	(l,a,b)	(l,b,a)	(l,b,b)
(k,a)	$(l+1,a,a),(k-2,a)$	$(l+1,a,a),(k-2,b)$	$(l,a,b),(k-1,a)$	$(l+1,a,b),(k-2,b)$
(k,b)	$(l+1,b,a),(k-1,a)$	$(l+1,b,a),(k-1,b)$	$(l,b,b),(k-1,a)$	$(l+1,b,b),(k-2,b)$

or with the actual lengths

$k_p\times^{(2l+\delta_{qr})_{qr}}$	$(2l+1)_{a^2}$	$(2l+1)_{ab}$	$(2l)_{ba}$	$(2l+1)_{b^2}$
k_a	$(2l+3)_{a^2},(k-2)_a$	$(2l+3)_{a^2},(k-2)_b$	$(2l+1)_{ab},(k-1)_a$	$(2l+3)_{ab},(k-2)_b$
k_b	$(2l+2)_{ba},(k-1)_a$	$(2l+2)_{ba},(k-1)_b$	$(2l+1)_{b^2},(k-1)_a$	$(2l+1)_{b^2},(k-2)_b$

. (3.3)

3.2 Mapping $\mathcal{C}$ and $\mathcal{D}$

We denote here by Φ the mapping from $\mathcal{C}$ to $\mathcal{D}$ and by Ψ the mapping from $\mathcal{D}$ to $\mathcal{C}$.

3.2.1 How to get $\Phi:\mathcal{C}\to\mathcal{D}$

Let us take any $\lambda=\lambda_1+\cdots+\lambda_s$ in $\mathcal{C}$, with $\lambda_1>_c\cdots>_c\lambda_s$. We have then for any $i\in\{1,\ldots,s\}$ that $\lambda_i\in\mathcal{O}'$. As an example, we take

$$\lambda=24_a+17_b+11_b+10_a+9_b+8_b+6_a+5_a+4_b+4_a\,\cdot \tag{3.4}$$

i/ First, we identify the **consecutive** troublesome pairs of parts, i.e $(\lambda_i,\lambda_{i+1})$ such that $\lambda_i\not\gg_c\lambda_{i+1}$, by taking consecutively the greatest pairs in terms of length, in such a way they are disjoint. With the example we have

$$\lambda=24_a+17_b+11_b+\underline{10_a+9_b}+8_b+\underline{6_a+5_a}+\underline{4_b+4_a}\,\cdot \tag{3.5}$$

Then we simply replace them by the corresponding parts in $\mathcal{E}'$ using (2.5). We denote that very partition $\lambda'=\lambda'_1+\cdots+\lambda'_{s'}$ with parts with the exact order by just replacing the pairs (parts are no longer ordered here). With the example we get

$$\lambda'=24_a+17_b+11_b+19_{ab}+8_b+11_{a^2}+8_{ba}\,\cdot \tag{3.6}$$

ii/ As long as there exists $i \in \{1, \dots, s'-1\}$ such that

$$\lambda'_i, \lambda'_{i+1} \in \mathcal{O} \times \mathcal{E}$$

and $\lambda'_i \not\gg_c \lambda'_{i+1}$, we just replace them by

$$\Lambda(\lambda'_i, \lambda'_{i+1}) \in \mathcal{E} \times \mathcal{O}$$

in this order. With the example, if we proceed by choosing the smallest i at each step, we have

$$\begin{array}{rc} i=3: & 24_a + 17_b + \underline{11_b + 19_{ab}} + 8_b + 11_{a^2} + 8_{ba} \\ & \downarrow \\ i=2: & 24_a + \underline{17_b + 20_{ba}} + 10_b + 8_b + 11_{a^2} + 8_{ba} \\ & \downarrow \\ i=5: & 24_a + 21_{b^2} + 16_a + 10_b + \underline{8_b + 11_{a^2}} + 8_{ba} \\ & \downarrow \\ i=4: & 24_a + 21_{b^2} + 16_a + \underline{10_b + 12_{ba}} + 7_a + 8_{ba} \\ & \downarrow \\ i=6: & 24_a + 21_{b^2} + 16_a + 13_{b^2} + 9_a + \underline{7_a + 8_{ba}} \\ & \downarrow \\ i=5: & 24_a + 21_{b^2} + 16_a + 13_{b^2} + \underline{9_a + 9_{ab}} + 6_a \\ & \downarrow \\ & 24_a + 21_{b^2} + 16_a + 13_{b^2} + 11_{a^2} + 7_b + 6_a \end{array} \cdot \tag{3.7}$$

We denote by λ'' the final result, which exists since the sum of the indices of parts in $\mathcal{E}$ strictly decreases at each step.

We set then $\Phi(\lambda) = \lambda''$. Our example gives

$$\Phi(24_a + 17_b + 11_b + 10_a + 9_b + 8_b + 6_a + 5_a + 4_b + 4_a) = 24_a + 21_{b^2} + 16_a + 13_{b^2} + 11_{a^2} + 7_b + 6_a \tag{3.8}$$

and we easily check that it belongs to $\mathcal{D}$. We sketch a proof of $\lambda'' \in \mathcal{D}$ in the next section.

3.2.2 How to get $\Psi : \mathcal{D} \to \mathcal{C}$

Let's take $\nu = \nu_1 + \cdots + \nu_s \in \mathcal{D}$ with $\nu_1 \gg_c \cdots \gg_c \nu_s$. We also take the example $\nu = \lambda''$ in the previous part,

$$\nu = 24_a + 21_{b^2} + 16_a + 13_{b^2} + 11_{a^2} + 7_b + 6_a \cdot \tag{3.9}$$

i/ As long as there exists $i \in \{1, \dots, s-1\}$ such that

$$\nu_i, \nu_{i+1} \in \mathcal{E} \times \mathcal{O}$$

and

$$\mu(\nu_i) \not\succ_c \nu_{i+1}\,,$$

we replace ν_i, ν_{i+1} by

$$\Lambda^{-1}(\nu_i, \nu_{i+1}) \in \mathcal{O} \times \mathcal{E}$$

in that order. We denote the final result ν', which exists since the sum of the indices of the parts in $\mathcal{O}$ strictly decreases at each step. One can easily check that if we proceed by taking the greatest i at each step, we have the exact reverse steps as we did in (3.7). And then

$$\nu' = 24_a + 17_b + 11_b + 19_{ab} + 8_b + 11_{a^2} + 8_{ba} = \lambda' \cdot \tag{3.10}$$

ii/ We finish by dividing all parts in $\mathcal{E}$ in their upper and lower halves and keeping the order. We get finally ν''. In the example, we get exactly

$$\nu'' = 24_a + 17_b + 11_b + 10_a + 9_b + 8_b + 6_a + 5_a + 4_b + 4_a = \lambda \cdot \tag{3.11}$$

We have then $\Psi(\nu) = \nu''$. With our example we get

$$\Psi(24_a + 21_{b^2} + 16_a + 13_{b^2} + 11_{a^2} + 7_b + 6_a) = 24_a + 17_b + 11_b + 10_a + 9_b + 8_b + 6_a + 5_a + 4_b + 4_a \cdot \tag{3.12}$$

4 Main ideas of the proof of the well-definedness of the mappings Φ and Ψ

Recall

$$\begin{array}{cccc} \Lambda & : & \mathcal{O} \times \mathcal{E} & \longrightarrow & \mathcal{E} \times \mathcal{O} \\ & & (k,p),(l,q,r) & \longmapsto & (l+\delta_{qr}, p, q), (k-\delta_{pq}-\delta_{qr}, r) \end{array} \cdot$$

Firstly, we can state that the sum of the lengths is conserved by the function Λ (and so Λ^{-1}), as

$$k + (2l + \delta_{qr}) = (2(l + \delta_{qr}) + \delta_{pq}) + (k - \delta_{pq} - \delta_{qr}) \cdot \tag{4.1}$$

Secondly, the total set of primary colors is conserved as well as their arrangement, since we considered that any part in $\mathcal{E}$ is twice colored by primary colors, and we have by Λ the colors $p, (q, r)$ become $(p, q), r$. Then, the final result has the same total length and the sequence of color as the original partition, in such a way that if Φ and Ψ are reciprocal bijections, then they imply bijections between $\mathcal{D}(u, v, n)$ and $\mathcal{C}(u, v, n)$.
Now let's provide some properties of Λ that are important in the construction of the bijection.

Proposition 4.1. *For any $(k,p),(l,q,r) \in \mathcal{O}\times\mathcal{E}$, one and only one of these statements is true:*

$$(k,p) \gg_c (l,q,r)\,. \tag{4.2}$$

$$(l+\delta_{qr},p,q) \gg_c (k-\delta_{pq}-\delta_{qr},r)\,. \tag{4.3}$$

Proposition 4.2. *For any $(k,p,q),(l,r) \in \mathcal{E}\times\mathcal{O}$, one and only one of these statements is true:*

$$\mu(k,p,q) >_c (l,r)\,. \tag{4.4}$$

$$(l+\delta_{pq}+\delta_{qr},p) \gg_c \gamma(k-\delta_{qr},q,r)\,. \tag{4.5}$$

The first proposition means that $\lambda_i,\lambda_{i+1} \in \mathcal{O}\times\mathcal{E}$ are such that $\lambda_i \not\gg_c \lambda_{i+1}$ if only if by applying

$$\Lambda(\lambda_i,\lambda_{i+1}) = (\lambda_i',\lambda_{i+1}') \in \mathcal{E}\times\mathcal{O}$$

we have that $\lambda_i' \gg_c \lambda_{i+1}'$. It is the key proposition that ensures that the final state belongs to $\mathcal{D}$. The second proposition allows us to come back to $\mathcal{C}$ in the process of Ψ by using Λ^{-1}. But the fact that we stay in set $\mathcal{O}'$ and $\mathcal{E}'$ after applying Λ and the uniqueness of the final result depend on many other facts. A detailed proof is given in the full article.

5 Generalization of Dousse's theorem

Theorem 1.3 just comes by using in our process

$$\mathcal{O} = \mathbb{Z}\times\{a_1,\dots,a_m\}\,,\ \mathcal{O}' = \mathbb{N}^*\times\{a_1,\dots,a_m\} \tag{5.1}$$

for the parts with primary colors and

$$\mathcal{E} = \mathbb{Z}\times\{a_1,\dots,a_m\}^2\,,\ \mathcal{E}' = \mathbb{N}^*\times\{a_1,\dots,a_m\}^2 \tag{5.2}$$

for the parts with secondary colors. We can apply the usual order $>_c$ of (2.1) after ordering $a_1 < \cdots < a_m$, and use the lemmas of Section 2.2 as definitions of $\gg_c$. And the mappings Φ and Ψ in Section 3 are defined exactly the same as in the case of two primary colors.

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References

[1] K. Alladi. "A variation on a theme of Sylvester - a smoother road to Göllnitz's (Big) theorem". *Discrete Math.* **196**.1-3 (1999), pp. 1–11. DOI: 10.1016/S0012-365X(98)00193-9.

[2] K. Alladi and B. Gordon. "Generalization of Schur's partition theorem". *Manuscripta Math.* **79**.2 (1993), pp. 113–126. DOI: 10.1007/BF02568332.

[3] D. Bressoud. "A combinatorial proof of Schur's 1926 partition theorem". *Proc. Amer. Math. Soc.* **79**.2 (1980), pp. 338–340. DOI: 10.2307/2043263.

[4] S. Corteel and J. Lovejoy. "An iterative-bijective approach to generalizations of Schur's theorem". *European J. Combin.* **27**.4 (2006), pp. 496–512. DOI: 10.1016/j.ejc.2005.01.005.

[5] J. Dousse. "Siladić's theorem: weighted words, refinement and companion". *Proc. Amer. Math. Soc.* 5 (2017), pp. 1997–2009. DOI: 10.1090/proc/13376.

[6] J. Lepowsky and R. Wilson. "The structure of standard modules, I: Universal algebras and the Rogers–Ramanujan identities". *Invent. Math.* **77**.2 (1984), pp. 199–290. URL.

[7] Padmavathamma, R. Raghavendra, and B.M. Chandrashekara. "A new bijective proof of a partition theorem of K. Alladi". *Discrete Math.* **287**.1-3 (2004), pp. 125–128. URL.

[8] L.J. Rogers and S. Ramanujan. "Proof of certain identities in combinatory analysis". *Cambr. Phil. Soc. Proc.* **19** (1919), pp. 211–216.

[9] I.J. Schur. "Zur additiven zahlentheorie". *Sitzungsberichte der Preussischen Aka. der Wissenschaften* (1926), pp. 488–495.

[10] I. Siladić. "Twisted $\mathfrak{sl}(3,\mathbb{C})^{\sim}$-modules and combinatorial identities". *Glas. Mat. Ser. III* **52(72)** (2017), pp. 53–77. DOI: 10.3336/gm.52.1.05.