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On the Lattice Property of Shard Orders

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Abstract. Let $\mathcal{L}$ be a congruence-uniform lattice. In this article, we investigate the shard order on $\mathcal{L}$ that was introduced by N. Reading. When $\mathcal{L}$ can be realized as a poset of regions of a hyperplane arrangement the shard order is always a lattice. For general $\mathcal{L}$, however, this fails. We provide a necessary condition for the shard order to be a lattice, and we show how to construct a congruence-uniform lattice $\mathcal{L}'$ from $\mathcal{L}$ such that the shard order on $\mathcal{L}'$ fails to be a lattice.

Résumé. Soit $\mathcal{L}$ un treillis congruence-uniforme. Dans cet article, nous étudions l'ordre des tessons sur $\mathcal{L}$ introduit par N. Reading. Lorsque $\mathcal{L}$ peut être réalisé comme ensemble ordonné des régions d'un arrangement d'hyperplans, l'ordre des tessons est toujours un treillis. C'est faux en revanche pour $\mathcal{L}$ quelconque. Nous donnons une condition nécessaire pour que l'ordre des tessons soit un treillis, et nous montrons comment construire un treillis congruence-uniforme $\mathcal{L}'$ à partir de $\mathcal{L}$ de sorte que l'ordre des tessons sur $\mathcal{L}'$ ne soit pas un treillis.

Keywords: congruence-uniform lattices, interval doubling, semidistributive lattices, crosscut theorem, Möbius function, biclosed sets

1 Introduction

A (real) hyperplane arrangement $\mathcal{A}$ is a collection of hyperplanes in $\mathbb{R}^n$, and the connected components of $\mathbb{R}^n \setminus \mathcal{A}$ are called the regions of $\mathcal{A}$. P. Edelman defined a partial order on the set of regions of $\mathcal{A}$ with respect to a fixed base region: two regions are comparable in this order whenever we can go from the one region to the other by crossing one hyperplane at a time and never decreasing the number of hyperplanes between the current region and the base region [7].

It was shown in [5] that this poset of regions is a lattice whenever $\mathcal{A}$ is simplicial. Subsequently, N. Reading thoroughly studied the structure of the poset of regions [18, 23], see also [19]. One of the main results in his study is a characterization of those hyperplane arrangements that have posets of regions which are semidistributive or congruence-uniform lattices, see Theorem 9-3.8 and Corollary 9-7.22 in [19]. See also [13, Theorem 3].

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A key tool for understanding lattice congruences in the lattice of regions (and therefore congruence-uniformity) are so-called shards of hyperplanes. The terminology suggests that these can be understood as pieces of hyperplanes that are broken off by intersections with other (in some sense stronger) hyperplanes. Proposition 3.3 in [21] states that the shards of $\mathcal{A}$ are in bijection with the join-irreducible elements of the lattice of regions (and thus, if this lattice is congruence-uniform, with the join-irreducible lattice congruences). The shards give rise to an alternate partial order on the regions of $\mathcal{A}$: the *shard intersection order*. It turns out that this order is always a lattice [21, Section 4]. Perhaps the most prominent example of a shard intersection order is the lattice of noncrossing partitions associated with a finite Coxeter group, which arises from certain quotient lattices of the poset of regions of the corresponding Coxeter arrangement [21, Theorem 8.5]. These quotient lattices are known as Cambrian lattices; see [17, 22] for more background. The shard intersection order of the lattice of regions of a Coxeter arrangement was also studied in [1, 2, 16].

N. Reading suggested a generalization of the shard intersection order to arbitrary congruence-uniform lattices [19, Section 9-7.4], which essentially associates a set of join-irreducible congruences with each lattice element, and where these sets are ordered by containment. It turns out that at this level of generality the lattice property for this alternate partial order is no longer guaranteed. If $\mathcal{L}$ is a finite congruence-uniform lattice, then [19, Problem 9.5] asks for conditions on $\mathcal{L}$ such that the corresponding shard order is again a lattice. The first main result of this article is a necessary condition stating that if the shard order on $\mathcal{L}$ is a lattice, then $\mathcal{L}$ is *spherical*, i.e. the Möbius function on $\mathcal{L}$ does not vanish between least and greatest element.

Theorem 1.1. *Let $\mathcal{L}$ be a finite congruence-uniform lattice. If the shard order* $\mathrm{Shard}(\mathcal{L})$ *is a lattice, then $\mathcal{L}$ is spherical.*

The proof of Theorem 1.1 essentially follows from the semidistributivity of $\mathcal{L}$ and G.-C. Rota's Crosscut Theorem. This condition is, however, not sufficient. We can explicitly construct spherical congruence-uniform lattices whose shard order is not a lattice.

Theorem 1.2. *Let $\mathcal{L}$ be a finite spherical congruence-uniform lattice with at least three atoms. There exists a spherical congruence-uniform lattice $\mathcal{L}'$ with $|\mathcal{L}'| = |\mathcal{L}| + 1$ such that* $\mathrm{Shard}(\mathcal{L}')$ *is not a lattice.*

It is quickly verified that the smallest congruence-uniform lattice with three atoms is the Boolean lattice of size eight. It follows that the smallest spherical congruence-uniform lattice whose shard order is not a lattice has nine elements. (It can be checked that sphericity is sufficient for the lattice property of the shard order of a congruence-uniform lattice of size at most eight.) We prove Theorems 1.1 and 1.2 in Section 3, after we have recalled the necessary lattice-theoretic notions in Section 2.

We have omitted some easy proofs; these can be found in [15], together with some further explorations on the lattice property of shard orders.

2 Background

2.1 Lattices and Congruences

Let $\mathcal{L} = (L, \leq)$ be a finite *lattice*, i.e. a partially ordered set in which every two elements $x, y \in L$ have a greatest lower bound (their *meet*; written $x \wedge y$), and a least upper bound (their *join*; written $x \vee y$). It follows that $\mathcal{L}$ has a least element $\hat{0}$ and a greatest element $\hat{1}$. Two elements $x, y \in L$ form a *cover relation* in $\mathcal{L}$ if $x < y$ and there is no $z \in L$ with $x < z < y$. We usually write $x \lessdot y$, and say that x is a *lower cover* of y; or equivalently that y is an *upper cover* of x.

The *dual* of $\mathcal{L}$ is the lattice $\mathcal{L}^* = (L, \preceq)$, where $x \preceq y$ if and only if $y \leq x$ for all $x, y \in L$. If $\mathcal{L} \cong \mathcal{L}^*$, then $\mathcal{L}$ is *self-dual*.

An element $j \in L \setminus \{\hat{0}\}$ is *join-irreducible* if whenever $j = x \vee y$ for $x, y \in L$, then $j \in \{x, y\}$. Since $\mathcal{L}$ is finite, it follows that every join-irreducible element j has a unique lower cover j_*. Let us denote the set of join-irreducible elements of $\mathcal{L}$ by $\mathcal{J}(\mathcal{L})$. Dually, $m \in L \setminus \{\hat{1}\}$ is *meet-irreducible* if whenever $m = x \wedge y$ for $x, y \in L$, then $m \in \{x, y\}$; and we conclude that m has a unique upper cover m^*. We denote the set of meet-irreducible elements of $\mathcal{L}$ by $\mathcal{M}(\mathcal{L})$.

A *lattice congruence* is an equivalence relation Θ on L such that $[x]_\Theta = [y]_\Theta$ and $[u]_\Theta = [v]_\Theta$ imply $[x \wedge u]_\Theta = [y \wedge v]_\Theta$ and $[x \vee u]_\Theta = [y \vee v]_\Theta$ for all $u, v, x, y \in L$. The set $\mathrm{Con}(\mathcal{L})$ of all lattice congruences of $\mathcal{L}$ ordered by set inclusion is again a lattice [9], and we therefore refer to it as the *congruence lattice* of $\mathcal{L}$. For $x, y \in L$ with $x \lessdot y$, let $\mathrm{cg}(x, y)$ denote the smallest lattice congruence of $\mathcal{L}$ in which x and y are equivalent. If $y \in \mathcal{J}(\mathcal{L})$, then we write $\mathrm{cg}(y)$ instead of $\mathrm{cg}(y_*, y)$.

We have the following characterization of join-irreducible lattices congruences; see [11, Section 2.14] for the equivalence of (i) and (ii) and [8, Theorem 3.20] for the equivalence of (i) and (iii).

Theorem 2.1. *Let $\mathcal{L}$ be a finite lattice, and let $\Theta \in \mathrm{Con}(\mathcal{L})$. The following are equivalent.*

(i) *Θ is join-irreducible in $\mathrm{Con}(\mathcal{L})$.*

(ii) *$\Theta = \mathrm{cg}(x, y)$ for some $x \lessdot y$.*

(iii) *$\Theta = \mathrm{cg}(j)$ for some $j \in \mathcal{J}(\mathcal{L})$.*

The map $j \mapsto \mathrm{cg}(j)$ is surjective by Theorem 2.1, but in general it may fail to be injective. A finite lattice is *congruence-uniform* if this map is a bijection for both $\mathcal{L}$ and $\mathcal{L}^*$. Congruence-uniform lattices sometimes appear in the literature (mainly in universal algebra and lattice theory publications) under the name "bounded lattices", which has its origins in [14] and refers to the fact that these are precisely the bounded-homomorphic images of a free lattice. This notation, however, clashes with the term "bounded poset",

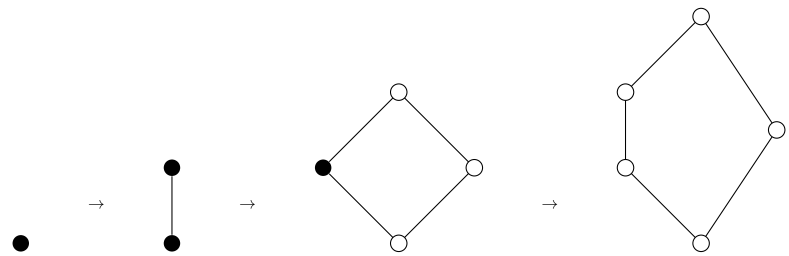


Figure 1: The pentagon lattice can be obtained by a sequence of doublings.

which refers simply to the fact that a poset has a least and a greatest element, and is widely used in combinatorics.

2.2 Doubling by Intervals

It follows from a result of A. Day that the congruence-uniform lattices can be characterized by means of the following doubling construction.

Let $\mathcal{P} = (P, \leq)$ be an arbitrary partially ordered set, and let $I \subseteq P$. Define $P_{\leq I} = \{x \in P \mid x \leq y \text{ for some } y \in I\}$. Let $\mathbf{2} = \{0, 1\}$ be the 2-element lattice defined by $0 < 1$. The *doubling* of $\mathcal{P}$ by I is the subposet of the direct product $\mathcal{P} \times \mathbf{2}$ given by the ground set

$$\Big(P_{\leq I} \times \{0\}\Big) \uplus \Big(\big((P \setminus P_{\leq I}) \cup I\big) \times \{1\}\Big),$$

where "$\uplus$" denotes disjoint set union. We denote the resulting poset by $\mathcal{P}[I]$, and if $I = \{i\}$ we write $\mathcal{P}[i]$ instead of $\mathcal{P}[\{i\}]$.

Theorem 2.2 ([6, Theorem 5.1])**.** *A finite lattice is congruence-uniform if and only if it can be obtained from the singleton lattice by a sequence of doublings by intervals.*

Figure 1 shows an instance of this doubling procedure. The intervals at which we double are marked by solid dots.

2.3 Semidistributive Lattices

A lattice $\mathcal{L} = (L, \leq)$ is *join-semidistributive* if for every $x, y, z \in L$ with $x \vee y = x \vee z$ we have $x \vee (y \wedge z) = x \vee y$. It is *meet-semidistributive* if $\mathcal{L}^*$ is join-semidistributive. We say that $\mathcal{L}$ is *semidistributive* if it is both join- and meet-semidistributive.

Proposition 2.3 ([6, Lemma 4.2 and Theorem 5.1])**.** *Every congruence-uniform lattice is semidistributive.*

Join-semidistributive lattices have another characterizing property. A set $X \subseteq L$ is a *join-representation* of $x \in L$ if $\bigvee X = x$. A join-representation X of x is *irredundant* if there is no $X' \subsetneq X$ with $x = \bigvee X'$. If X and X' are two irredundant join-representations of x, then X *refines* X' if for every $z \in X$ there exists $z' \in X'$ with $z \leq z'$. A join-representation of x is *canonical* if it is irredundant and refines every other irredundant join-representation of x. We define *(canonical) meet-representations* dually.

Theorem 2.4 ([8, Theorem 2.24]). *Every element of a join-semidistributive lattice has a canonical join-representation.*

The next result states that the canonical join-representations in fact form a simplicial complex; see also [3, 4].

Proposition 2.5 ([20, Proposition 2.2]). *Let $\mathcal{L} = (L, \leq)$ be a finite lattice, and let $X \subseteq L$. If $\bigvee X$ is a canonical join-representation, and $X' \subseteq X$, then $\bigvee X'$ is also a canonical join-representation.*

Now suppose that $\mathcal{L}$ is congruence-uniform, and pick $x, y \in L$ with $x \lessdot y$. Theorem 2.1 and the fact that $j \mapsto \mathrm{cg}(j)$ is a bijection imply that there is a unique $j \in \mathcal{J}(\mathcal{L})$ with $\mathrm{cg}(j) = \mathrm{cg}(x, y)$; we usually write $j_{\mathrm{cg}(x,y)}$ to denote this element. From this we can explicitly describe canonical join-representations in $\mathcal{L}$.

Proposition 2.6 ([10, Proposition 2.10]). *Let $\mathcal{L} = (L, \leq)$ be a finite congruence-uniform lattice. The canonical join-representation of $x \in L$ is $\{j_{\mathrm{cg}(y,x)} \mid y \lessdot x\}$.*

2.4 Möbius Function and Crosscuts

Let $\mathcal{P} = (P, \leq)$ be a finite partially ordered set. The *Möbius function* of $\mathcal{P}$ is the function $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined recursively by

$$\mu_{\mathcal{P}}(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum\limits_{x \leq z < y} \mu_{\mathcal{P}}(x, z), & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

An *antichain* of $\mathcal{P}$ is a subset of P consisting of pairwise incomparable elements. A *chain* of $\mathcal{P}$ is a subset of P in which every two elements are comparable. A chain is *maximal* if it is maximal under inclusion.

There is a nice way to compute the Möbius function in a finite lattice $\mathcal{L} = (L, \leq)$. A *crosscut* of $\mathcal{L}$ is an antichain $C \subseteq P$ which contains neither $\hat{0}$ nor $\hat{1}$ and such that every maximal chain of $\mathcal{L}$ intersects C exactly once. Examples for crosscuts are the sets of *atoms* (i.e. elements covering $\hat{0}$) or *coatoms* (i.e. elements covered by $\hat{1}$). A subset $X \subseteq L$ is *spanning* if $\bigwedge X = \hat{0}$ and $\bigvee X = \hat{1}$. The following result is known as the Crosscut Theorem.

Theorem 2.7 ([24, Theorem 3])**.** *Let $\mathcal{L} = (L, \leq)$ be a finite lattice and let $C \subseteq L$ be a crosscut. We have*
$$\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = \sum_{X \subseteq C \text{ spanning}} (-1)^{|X|}.$$

Proposition 2.8. *Let $\mathcal{L}$ be a finite semidistributive lattice. If there exists a set X of atoms with $\bigvee X = \hat{1}$, then X must contain all atoms.*

Proof. Let A denote the set of all atoms of $\mathcal{L}$. Let $X \subsetneq A$ with $\bigvee X = \hat{1}$, and let $a \in A \setminus X$. For any $x \in X$ we have $a \wedge x = \hat{0}$, since a and x are atoms. The meet-semidistributivity of $\mathcal{L}$ then implies that $\hat{0} = a \wedge \left(\bigvee X\right) = a \wedge \hat{1} = a$. This contradicts the assumption that a is an atom, and we conclude $X = A$. □

Clearly, the dual of Proposition 2.8 also holds in a semidistributive lattice. As a consequence, we obtain the following result, which may also be concluded from [12, Theorems 5.1.3 and 5.4.1].

Theorem 2.9. *If $\mathcal{L}$ is a finite congruence-uniform lattice, then $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \in \{-1, 0, 1\}$.*

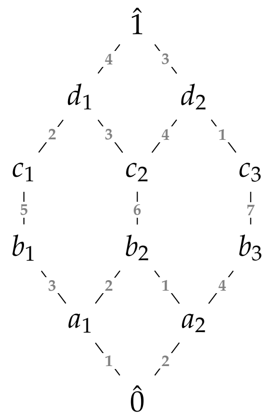
If $\mathcal{L}$ is a finite congruence-uniform lattice with the property that $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$, then we call $\mathcal{L}$ *spherical*.

3 The Shard Order of a Finite Congruence-Uniform Lattice

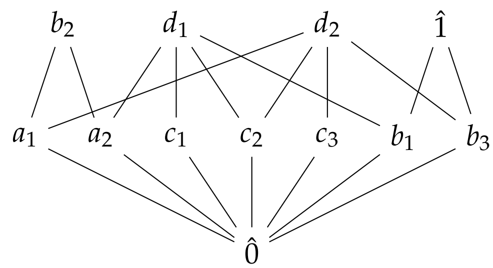
Let $\mathcal{L} = (L, \leq)$ be a finite congruence-uniform lattice. N. Reading defined in [19, Section 9-7.4] an alternate partial order on L as follows. For $x \in L$ set $x_{\downarrow} = \bigwedge_{y \in L : y \lessdot x} y$; the *shard set* of x is then
$$\Psi(x) = \left\{ \operatorname{cg}(u, v) \mid x_{\downarrow} \leq u \lessdot v \leq x \right\}.$$
We set $x \sqsubseteq y$ if and only if $\Psi(x) \subseteq \Psi(y)$. Let us call the poset $(L, \sqsubseteq)$ the *shard order* of $\mathcal{L}$ and denote it by $\operatorname{Shard}(\mathcal{L})$. The main motivation for this definition (and also the terminology) comes from the poset of regions in a hyperplane arrangement, where we have the following result.

Theorem 3.1 ([19, Section 9-7.4])**.** *Let $\mathcal{L}$ be a poset of regions of a hyperplane arrangement. If $\mathcal{L}$ is a finite congruence-uniform lattice, then* $\operatorname{Shard}(\mathcal{L})$ *is a lattice.*

The hyperplane arrangements that have posets of regions which are congruence-uniform lattices are characterized in [19, Corollary 9-7.22]. In the case described in Theorem 3.1, the poset $\operatorname{Shard}(\mathcal{L})$ is usually referred to as the *shard intersection order*. However, in the general case, the set of shard sets of $\mathcal{L}$ is not necessarily closed under



(a) A congruence-uniform lattice.



(b) The shard order of the lattice in Figure 2a.

Figure 2: A congruence-uniform lattice whose shard order is not a lattice.

intersections. In order to avoid confusion we decided to drop the term "intersection" from the name[1].

If $\mathcal{L}$ is a finite congruence-uniform lattice that does not arise as a poset of regions of some hyperplane arrangement, then the shard order need not be a lattice. Consider for instance the lattice $\mathcal{L}$ in Figure 2a. There we have labeled the cover relations by integers according to the following rule: the cover relation $u \lessdot v$ is labeled by the integer i if and only if $\mathrm{cg}(u,v)$ is the i^{th} congruence in the sequence

$$\mathrm{cg}(a_1), \mathrm{cg}(a_2), \mathrm{cg}(b_1), \mathrm{cg}(b_3), \mathrm{cg}(c_1), \mathrm{cg}(c_2), \mathrm{cg}(c_3)$$

of the join-irreducible congruences of $\mathcal{L}$. The shard sets of $\mathcal{L}$ can then be read off this labeling:

$$\begin{array}{llll} \Psi(\hat{0}) = \emptyset, & \Psi(a_1) = \{1\}, & \Psi(a_2) = \{2\}, & \Psi(b_1) = \{3\}, \\ \Psi(b_2) = \{1,2\}, & \Psi(b_3) = \{4\}, & \Psi(c_1) = \{5\}, & \Psi(c_2) = \{6\}, \\ \Psi(c_3) = \{7\}, & \Psi(d_1) = \{2,3,5,6\}, & \Psi(d_2) = \{1,4,6,7\}, & \Psi(\hat{1}) = \{3,4\}. \end{array}$$

The corresponding poset of shard sets is shown in Figure 2b. We observe that it is a meet-semilattice, i.e. any two elements have a meet, but it is not a lattice since it does not have a greatest element. We observe further that $\mathcal{L}$ is not spherical.

Our main result, Theorem 1.1, which we are going to prove in the remainder of this section, establishes that the shard order of $\mathcal{L}$ is a lattice only if $\mathcal{L}$ is spherical.

Lemma 3.2. *Let $\mathcal{L} = (L, \leq)$ be a finite congruence-uniform lattice. Let $j \in \mathcal{J}(\mathcal{L})$ and $x, y \in L$ with $x \lessdot y$. If $\mathrm{cg}(x,y) = \mathrm{cg}(j)$, then $j \vee x = y$.*

[1] The term "shard" is not intuitive outside the realm of hyperplane arrangements. Unfortunately, we currently lack a better name for this alternate order. Suggestions are very welcome.

Lemma 3.3. *If $\Psi(x) \subseteq \Psi(y)$, then $x \leq y$.*

Corollary 3.4. *$\hat{1}$ is a maximal element in* $\mathsf{Shard}(\mathcal{L})$*.*

Proposition 3.5. *Let $\mathcal{L}$ be a finite congruence-uniform lattice, and let C denote its set of coatoms. We have $\bigwedge C = \hat{0}$ if and only if $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$.*

Proof. Let e (resp. o) denote the number of spanning subsets of C of even (resp. odd) size. The dual of Proposition 2.8 implies that $e + o \leq 1$.

If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = 0$, then Theorem 2.7 implies that $e = o$, which forces $e = o = 0$. Hence $\hat{0} < \bigwedge C$.

Conversely if $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$, then Theorem 2.9 implies $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = \pm 1$. Hence we have either $e = 1$ and $o = 0$, or $e = 0$ and $o = 1$. The dual of Proposition 2.8 implies that $\bigwedge C = \hat{0}$. □

Of course, the dual of Proposition 3.5 is also true.

Corollary 3.6. *We have $\Psi(\hat{1}) = \mathcal{J}\big(\mathrm{Con}(\mathcal{L})\big)$ if and only if $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$.*

Proof. Let C denote the set of coatoms of $\mathcal{L}$.

If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$, then Proposition 3.5 implies that $\bigwedge C = \hat{0}$, so that by definition $\Psi(\hat{1})$ contains all join-irreducible congruences of $\mathcal{L}$.

If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = 0$, then Proposition 3.5 implies that $\bigwedge C = x > \hat{0}$. In particular, there is some atom a of $\mathcal{L}$ with $a \leq x$. If $\mathrm{cg}(a) \in \Psi(\hat{1})$, then there exist $u, v \in L$ with $x \leq u \lessdot v$ such that $\mathrm{cg}(u, v) = \mathrm{cg}(a)$. Lemma 3.2 implies $a \vee u = v$. However, $a \leq x \leq u$ implies $a \vee u = u$, which is a contradiction. We conclude $\mathrm{cg}(a) \notin \Psi(\hat{1})$. □

Corollary 3.7. *There exists a greatest element in* $\mathsf{Shard}(\mathcal{L})$ *if and only if $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$.*

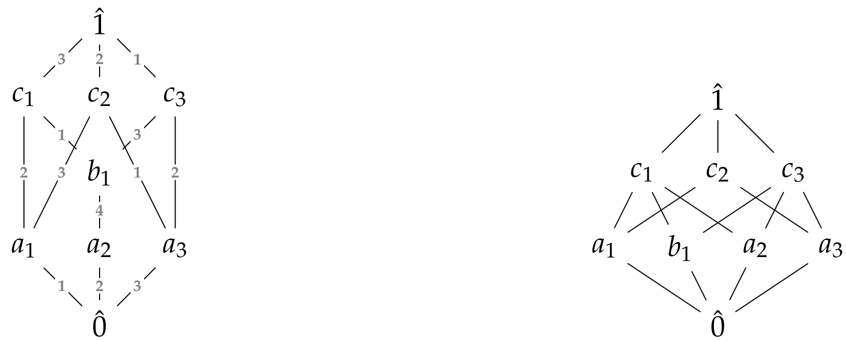
Proof. Let C denote the set of coatoms of $\mathcal{L}$.

If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) \neq 0$, then Corollary 3.6 implies $\Psi(\hat{1}) = \mathcal{J}\big(\mathrm{Con}(\mathcal{L})\big)$. It follows that for any $x \in L$ we have $\Psi(x) \subseteq \Psi(\hat{1})$, which implies that $\hat{1}$ is the unique maximal element of $\mathsf{Shard}(\mathcal{L})$.

If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = 0$, then Corollary 3.6 implies that there is $\Theta \in \mathcal{J}\big(\mathrm{Con}(\mathcal{L})\big)$ with $\Theta \notin \Psi(\hat{1})$. Theorem 2.1 implies that there is $j \in \mathcal{J}(\mathcal{L})$ with $\Theta = \mathrm{cg}(j)$. Corollary 3.4 implies that $\hat{1}$ is maximal in $\mathsf{Shard}(\mathcal{L})$, and we conclude that it is incomparable to j in $\mathsf{Shard}(\mathcal{L})$. The maximality of $\hat{1}$ implies further that there is no upper bound for $\hat{1}$ and j in $\mathsf{Shard}(\mathcal{L})$, which therefore does not have a greatest element. □

We can now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. If $\mu_{\mathcal{L}}(\hat{0}, \hat{1}) = 0$, then Corollary 3.7 implies that $\mathsf{Shard}(\mathcal{L})$ does not have a greatest element. Since $\mathcal{L}$ is finite, $\mathsf{Shard}(\mathcal{L})$ can therefore not be a lattice. □



(a) A spherical congruence-uniform lattice.

(b) The shard order of the lattice in Figure 3a.

Figure 3: A spherical congruence-uniform lattice whose shard order is not a lattice.

The example in Figure 3 illustrates that there exist spherical congruence-uniform lattices whose shard order is not a lattice. The labels correspond to positions in the sequence $\mathrm{cg}(a_1), \mathrm{cg}(a_2), \mathrm{cg}(a_3), \mathrm{cg}(b_1)$ of join-irreducible congruences.

Observe that Figure 3a is a Boolean lattice of size 8 doubled by an atom, and it is exactly this doubling that kills the lattice property of the shard order. We conclude the following result.

Proposition 3.8. *Let $x, y \in L$, and suppose that there exists some $j \in \mathcal{J}(\mathcal{L})$ with $j \in [x_\downarrow, x] \cap [y_\downarrow, y]$. If $\Psi(j) \subseteq \Psi(x) \cap \Psi(y)$, then the shard order of $\mathcal{L}[j]$ is not a lattice.*

Proof. Let Ψ' denote shard sets in $\mathcal{L}[j]$. Observe that $\mathcal{L}[j]$ has exactly one additional element j', which is an upper cover of j. Since $j \in \mathcal{J}(\mathcal{L})$, we conclude that $j, j' \in \mathcal{J}(\mathcal{L}[j])$.

Corollary 3.3 implies that j is a lower bound of x and y in $\mathcal{L}$, and it follows by construction that j' is a lower bound of x and y in $\mathcal{L}[j]$, and we thus have $\Psi'(x) = \Psi(x) \cup \{j'\}$ and $\Psi'(y) = \Psi(y) \cup \{j'\}$. By assumption we have $\{j\} = \Psi'(j) \subseteq \Psi'(x) \cap \Psi'(y)$ and by construction follows $\{j'\} = \Psi'(j') \subseteq \Psi'(x) \cap \Psi'(y)$. We conclude that $\mathrm{Shard}(\mathcal{L})$ is not a lattice. □

We certainly cannot leave out the extra condition on j in Proposition 3.8, since we need to double at an interval contained in $[x_\downarrow, x] \cap [y_\downarrow, y]$ in order to change the shard sets of x and y. We may now prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{L}$ be a finite spherical congruence-uniform lattice with at least three atoms a, b, c. Since $\mathcal{L}$ is spherical we conclude from the dual of Proposition 3.5 that $\hat{1}$ is the join of all atoms, and Proposition 2.5 implies that there are elements $x = a \vee b$ and $y = b \vee c$ (where these are canonical join-representations).

Proposition 2.6 implies that there are exactly two lower covers of x, say r_1 and r_2, and let $r = r_1 \wedge r_2$. Since $r < x = a \vee b$, we conclude that $a \not\leq r$. It then follows

n	1	2	3	4	5	6	7	8	9	10	11	12	13
l_n	1	1	1	2	5	15	53	222	1078	5994	37622	262776	2018305
c_n	1	1	1	2	4	9	22	60	174	534	1720	5767	20013
s_n	1	1	0	1	1	2	3	8	17	45	123	367	1148
S_n	1	1	0	1	1	2	3	8	16	41	107	304	891

Table 1: Numerology of congruence-uniform lattices.

from the fact that a is an atom that $a \wedge r = \hat{0}$. Analogously we obtain $b \wedge r = \hat{0}$. The meet-semidistributivity of $\mathcal{L}$ implies that $\hat{0} = (a \vee b) \wedge r = x \wedge r = r$. We conclude that $\Psi(b) \subseteq \Psi(x)$. By symmetry we obtain $\Psi(b) \subseteq \Psi(y)$.

Since we have just seen that $x_{\downarrow} = \hat{0} = y_{\downarrow}$, and since $b \leq x$ and $b \leq y$ by construction, we conclude that $b \in [\hat{0}, x] \cap [\hat{0}, y]$. We can therefore apply Proposition 3.8, which proves the claim. Observe that $\mathcal{L}[b]$ is spherical, since $\mathcal{L}$ was. □

The example in Figure 3 is thus the smallest spherical congruence-uniform lattice whose shard order is not a lattice. (Note that any congruence-uniform lattice of size ≤ 7 has at most two atoms.) Table 1 lists the number of congruence-uniform lattices of size ≤ 13, and the number of such lattices that are spherical and have a shard order that is a lattice. These numbers were obtained with the help of `Sage-Combinat` [25, 26]. In this table we use the following abbreviations:

- l_n denotes the number of all lattices of size n,
- c_n denotes the number of all congruence-uniform lattices of size n,
- s_n denotes the number of spherical congruence-uniform lattices of size n, and
- S_n denotes the number of all congruence-uniform lattices of size n whose shard order is a lattice.

We conclude this abstract with an open problem concerning the meet operation in $\mathrm{Shard}(\mathcal{L})$ whenever it exists. To that end, let $\mathcal{L} = (L, \leq)$ be a finite congruence-uniform lattice. We say that $\mathcal{L}$ has the *shard intersection property* (SIP) if for all $x, y \in L$ there exists some $z \in L$ with $\Psi(x) \cap \Psi(y) = \Psi(z)$.

Proposition 3.9. *If $\mathcal{L}$ is a finite spherical congruence-uniform lattice which has the SIP, then* $\mathrm{Shard}(\mathcal{L})$ *is a lattice.*

Proof. If $\mathcal{L}$ has the SIP, then $\{\Psi(x) \mid x \in L\}$ is closed under intersections, which means that $\mathrm{Shard}(\mathcal{L})$ is a meet-semilattice. If $\mathcal{L}$ is spherical, then $\mathrm{Shard}(\mathcal{L})$ has a greatest element by Corollary 3.7. The claim now follows by a standard lattice-theoretic argument, see for instance [19, Proposition 9-2.1]. □

Of course, we have just shifted the question when $\mathcal{L}$ has a shard lattice to the question when $\mathcal{L}$ has the SIP. Moreover, it may well be true that there exists a finite congruence-uniform lattice without the SIP such that $\mathrm{Shard}(\mathcal{L})$ is a lattice nonetheless.

Problem 3.10. *Find a spherical congruence-uniform lattice without the SIP such that* $\mathrm{Shard}(\mathcal{L})$ *is a lattice.*

Computer experiments have shown that any congruence-uniform lattice of size ≤ 12 whose shard order is a meet-semilattice has the SIP.

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